Today char $k=0$ (or char k positive and big enough). If you prefer, k is algebraically closed

G reductive (connected), $\mathfrak{g} = \mathrm{Lie}(G)$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\chi"] & c := \mathfrak{t}/W \\
\text{open}\ \mathfrak{g}^{reg} \arrow[u, phantom, "\cup"] & \text{open}\ c^{reg,ss} \arrow[u, phantom, "\cup"] \\
\text{open}\ \mathfrak{g}^{reg,ss} \arrow[u, phantom, "\cup"] \arrow[ur] &
\end{tikzcd}$$

$c = \{$"characteristic polynomials"$\}$

Letters c, χ stand for "characteristic"

G acts transitively on each fiber of $\mathfrak{g}^{reg} \xrightarrow{\chi} c$

$\mathfrak{g}^{reg,ss} = \chi^{-1}(c^{reg,ss})$.

How big are $c \setminus c^{reg,ss}$, $\mathfrak{g} \setminus \mathfrak{g}^{reg,ss}$, $\mathfrak{g} \setminus \mathfrak{g}^{reg}$?

Discriminant divisor

(Codimension $3 \geq 2$).

Centralizers and regular centralizers.

$x \in \mathfrak{g} \Rightarrow I_x := \{ g \in G \mid g x g^{-1} = x \}$ - affine alg. group scheme

As x varies, get an affine group $I \to \mathfrak{g}$.

It is not flat: $\dim I_x$ can jump.

Now consider $I^{reg} := I|_{\mathfrak{g}^{reg}}$.

By definition, $x \in \mathfrak{g}^{reg} \Leftrightarrow \dim I_x = r$ (minimal value).

So I^{reg} is flat over $\mathfrak{g}$. Since char $k=0$ each I_x is smooth, so I^{reg} is smooth over $\mathfrak{g}$.

Fact: $x \in \mathfrak{g}^{reg} \Rightarrow I_x$ is abelian.

<u>Corollary</u>. $x \in \mathfrak{g}^{reg} \Rightarrow I_x$ depends only on $\chi(x)$.

More precisely: if $x, x' \in \mathfrak{g}^{reg}$ and $\chi(x') = \chi(x)$ then $x' = g x g^{-1}$. A choice of g defines an isomorphism $\varphi_g : I_x \xrightarrow{\sim} I_{x'}$. I_x abelian $\Rightarrow \varphi_g$ doesn't depend on g.

<u>Def.</u> Let $y \in c$. Then $J_y := I_x$, where $x \in \mathfrak{g}^{reg}$, $\chi(x) = y$.

As $y \in \mathfrak{c}$ varies we get a smooth commutative group scheme J over $\mathfrak{c}$ of relative dimension r.

<u>Def.</u> J is the <u>scheme of regular centralizers</u>.

We have a group scheme $\chi^* J$ over $\mathfrak{g}$. (It will also be called the scheme of regular centralizers.)

<u>Example.</u> $G = GL(n)$, $\mathfrak{g} = \mathfrak{gl}(n)$, $\mathfrak{c} = \{$monic polynomials of degree $n\}$.

$f \in k[x]$ monic, $\deg f = n \Rightarrow J_f = (k[x]/(f))^\times$

($(k[x]/(f))^\times$ is viewed as an algebraic group).

By def., $\chi^* J|_{\mathfrak{g}^{reg}} = I^{reg}$.

<u>Prop.</u> The isomorphism $\chi^* J|_{\mathfrak{g}^{reg}} \xrightarrow{\sim} I^{reg}$ uniquely extends to a homomorphism $\chi^* J \to I$.

<u>Example.</u> $G = GL(n)$, $A \in \mathfrak{gl}(n)$, $f \in k[x]$ the charact. polynomial of A,

$$\begin{array}{ccc} J_A := (k[x]/(f))^*, & & I_A := \text{centralizer of } A \text{ in } GL(n) \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ P & \longmapsto & p(A) \end{array}$$

(A not regular $\Rightarrow$ neither mono nor epi).

<u>Proof.</u> $I' := \chi^* J$, $I'^{reg} := I'|_{\mathfrak{g}^{reg}}$

Extension problem:
$$\begin{array}{ccc} I'^{reg} & = & I^{reg} \\ \cap & & \cap \\ I' & \dashrightarrow & I \end{array}$$

$I' \to \mathfrak{g}$ smooth, so I' is smooth

$\operatorname{codim}(I' \setminus I'^{reg}) \geq 2$

I is affine.

$\forall$ morphism $I'^{reg} \to \{$affine scheme$\}$ extends to I' in a unique way. (The extension is a morphism of group schemes over $\mathfrak{g}$ because this is true over $\mathfrak{g}^{reg} \subset \mathfrak{g}$). ∎

Digression: an important 2-stack.

Consider the following 2-groupoid.

Objects: elements of $\mathfrak{g}$.

Given $x, y \in \mathfrak{g}$, a 1-isomorphism $x \xrightarrow{\sim} y$ is an element $g \in G$ such that $gxg^{-1} = y$.

Given $g_1, g_2 \in G$ such that $g_i x g_i^{-1} = y$, a 2-isomorphism $g_1 \xrightarrow{\sim} g_2$ is an element $z \in J_x$ such that $g_2 = g_1 z$.

Composition: clear.

Note that the full 2-groupoid formed by regular elements of $\mathfrak{g}$ is 2-equivalent to a set, namely to c.

Obvious generalization of this 2-groupoid: for any scheme S replace $\mathfrak{g}$ by $\mathfrak{g}(S)$ and G by $G(S)$. Thus we get a "presheaf of 2-groupoids".

I hope that the associated sheaf of 2-groupoids is an algebraic 2-stack (if not then the definition of algebraic 2-stack should be changed!) This 2-stack plays an important role in Ngo's article (although I am not sure that he mentions this explicitly). So I would denote it by $[\mathfrak{g}/G]_{Ngo}$. Note that this 2-stack contains the scheme $c = [\mathfrak{g}^{reg}/G]_{Ngo}$ as an open 2-substack.

A description of J in terms of T.

$T :=$ the maximal torus of G (not a subtorus), i.e.,

$$T := B/[B,B], \quad B \text{ any Borel.}$$

Given B, B' $\exists\, g \in G : gBg^{-1} = B'$, then $\mathrm{Ad}\, g$ induces $B/[B,B] \xrightarrow{\sim} B'/[B',B']$.
g is defined up to $N(B) = B$, so the isomorphism $B/[B,B] \xrightarrow{\sim} B'/[B',B']$.
There is a way to define $W \subset \mathrm{Aut}\, T$.

J is a group scheme over $c = t/W$.
Although c is defined in terms of T the definition of J is not in terms of T (it involves regular non-semisimple elements of $\mathfrak{g}$).

Donagi-Gaitsgory described J in terms of T:
1. They define a group scheme $\check{J}$ over c, rel. dim $= r \cdot |W|$
2. They define $\tilde{J} \subset \check{J}$ closed, rel. dim $= r$
Steps 1-2 involve only (T, W)
3. They define a morphism $J \to \tilde{J}$, show that it is an open embedding.
4. They describe the image (usually it equals $\tilde{J}$).

Step 1. $t \xrightarrow{\pi} c = t/W$, π finite and flat
$a \in c \Rightarrow |\pi^{-1}(a)| = |W|$, $\pi^{-1}(a)$ not necessarily reduced.

$$\check{J}_a := \underset{\text{scheme of morphisms}}{\mathrm{Mor}}(\pi^{-1}(a), T)$$

(Scientifically: Weil restriction).

Example. $S = \mathrm{Spec}\, k[x]/(x^2 - \varepsilon)$, M any scheme.
$\varepsilon \neq 0 \Rightarrow \mathrm{Mor}(S, M) = M \times M$, $\varepsilon = 0 \Rightarrow \mathrm{Mor}(S, M) =$ tangent bundle of M.
M smooth connected $\Rightarrow \forall \varepsilon > 0$ $\mathrm{Mor}(S, M)$ is smooth connected, dim $= 2 \dim M$.
$\check{J}_a \to c$ is smooth, with connected fibers of dim $= r \cdot |W|$
The generic fiber is a torus. In general, $\exists$ unipotent part.

Step 2. W acts on $\pi^{-1}(a)$, so W acts on $\check{J}_a$ and on $\check{J}$.

$\tilde{J} := \check{J}^W$. Equivalently, $\tilde{J}_a = Mor_W(\pi^{-1}(a), \check{T})$.

$\tilde{J}$ is still a smooth group scheme over C.

Exercise. $G = SL(2) \Rightarrow$ the fiber of $\tilde{J}$ over $0 \in C$ has 2 connected components.

$a \in C^{reg,ss} \Rightarrow \tilde{J}_a \simeq T$ $\boxed{\text{(the isomorphism depends on the choice of } x \in \pi^{-1}(a)\text{)}}$

In fact, $\pi^{-1}(a)$ is a W-torsor and $\tilde{J}_a$ is the twist of T by this W-torsor.

Recall that $J_a \simeq T$ (the isomorphism depends on the choice of $y \in \mathfrak{g}$ with $\chi(y) = a$ and a choice of a Borel $\mathfrak{b} \ni y$).

In fact, we have a canonical isomorphism

$J^{reg,ss} \xrightarrow{\sim} \tilde{J}^{reg,ss}$ (isomorphism of group schemes over C).

(Because a choice of $y, \mathfrak{b}$ as above gives $x \in \pi^{-1}(a)$).

Theorem (Donagi-Gaitsgory). (i) The isomorphism $J^{reg,ss} \xrightarrow{\sim} \tilde{J}^{reg,ss}$ extends to a morphism $f: J \to \tilde{J}$.

(ii) f is an open immersion

(iii) Explicit description of $f(J) \subset \tilde{J}$.

In particular, if $[G,G]$ does not have $PGL(2)$ as a factor then $f(J) = \tilde{J}$, so f is an isomorphism.

Comments. (ii) Once f is constructed it suffices to check that it is an open immersion over generic points of the discriminant hypersurface in C. This reduces to checking the statement for reductive groups of rank 1.

(iii) J_a and $\tilde{J}_a$ may be disconnected. The problem is to describe $\pi_0(J_a)$ as a subgroup of $\pi_0(\tilde{J}_a)$.

<u>Key examples</u>: $G = SL(2)$, $G = PGL(2)$.

$\tilde{J}$ depends only on (T, W), so $\tilde{J}^{SL(2)} \simeq \tilde{J}^{PGL(2)}$.

$|\pi_0(\tilde{J}_0)| = 2$, $\tilde{J}_0 :=$ fiber over 0.

$$J_0^{SL(2)} := \{\text{centralizer of } \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \text{ in } SL(2)\} = \simeq \mathbb{G}_a \times \{-1, 1\}$$

$$J_0^{PGL(2)} \simeq \mathbb{G}_a.$$

<u>Explicit description of $f(J) < \tilde{J}$.</u> Fix $a \in c$. Want to describe $f(J_a) < \tilde{J}_a$. A point of $\tilde{J}_a$ is a W-equivariant morphism $\varphi : \pi^{-1}(a) \to T$. When does this point belong to $f(J_a)$?

<u>Answer</u>: if and only if $\forall$ root $\alpha : T \to \mathbb{G}_m$ if $x \in \pi^{-1}(\alpha)$ is fixed by the reflection $s_\alpha \in W$ then $\alpha(\varphi(x)) = 1$.

<u>Comment</u>. Automatically, $\alpha(\varphi(x)) = -1$.

<u>Proof.</u> By W-equivariance, $s_\alpha(\varphi(x))$, i.e., $\check{\alpha}(\alpha(\varphi(x))) = 1$. (Here $\check{\alpha} : \mathbb{G}_m \to T$). So $\alpha(\varphi(x)) \in \operatorname{Ker} \check{\alpha}$. Since $\alpha \circ \check{\alpha} : \mathbb{G}_m \to \mathbb{G}_m$ is the squaring map $\varphi(x)$ $\operatorname{Ker} \check{\alpha} \subset \{1, -1\}$. ■

<u>Sketch of the proof of statement (i) of the Donagi-Gaitsgory theorem</u> (details can be found in Ngo's article, see the proof of Proposition 2.4.2).

Given an element of J_a how to construct a morphism $\pi^{-1}(a)\to T$? We have the diagram

$$\begin{array}{ccc} & & \mathfrak{t} \\ & & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{t}/W = \mathfrak{c} \end{array}$$

By definition, $J_a = I_x$, where $x\in\mathfrak{g}^{reg}$ is such that $\chi(x)=a$. So the above question can be reformulated as follows: given $x\in\mathfrak{g}^{reg}$ and $z\in I_x$ how to construct a morphism $\pi^{-1}(\chi(x))\to T$?

<u>Key observations</u>. (i) If $x\in\mathfrak{g}^{reg}$ then the theory of the Springer resolution provides an isomorphism

$$\left\{\begin{matrix}\text{Borels } B\subset G \\ \text{such that Lie } B\ni x\end{matrix}\right\} \xrightarrow{\sim} \pi^{-1}(\chi(x))$$

(Note that here the l.h.s. and the r.h.s. are schemes which are not necessarily reduced)

(ii) If $x\in\mathfrak{g}^{reg}$ and $B\subset G$ is a Borel such that $x\in \operatorname{Lie} B$ then $I_x\subset B$ (this is clear if $x\in\mathfrak{g}^{reg,ss}$, and the general case follows by continuity).

Now given $x\in\mathfrak{g}^{reg}$ and a point of $\pi^{-1}(\chi(x))$ we get a Borel $B\subset G$ such that $I_x\subset B$. So a point $z\in I_x$ yields a point of $B/[B,B]=T$.

Auxiliary material related to $\pi_0(J_a)$.

1. Lemma. J_a is connected for all a if and only if $Z(G)$ is connected.

Proof. It suffices to show that if $x \in \mathfrak{g}$ is a regular nilpotent then $\pi_0(I_x) = \pi_0(Z(G))$. Our x is contained in a unique Borel B. Uniqueness implies that $I_x \subset N(B) = B$. It is easy to deduce from this that $I_x = Z(G) \cdot \{\text{unipotent group}\}$.

2. Exercise. $G = SL(2) \Rightarrow$ the fiber of $\tilde{J}$ over $0 \in \mathbb{C}$ has 2 connected components.

Solution. Let T be the maximal torus (so $T = \mathbb{G}_m$, $W = \mathbb{Z}/2\mathbb{Z}$, and the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{G}_m$ is non-trivial). We have $\tilde{J}_0 = T^W \times \mathrm{Hom}_W^{\text{linear}}(\mathfrak{t}, \mathfrak{t}) = \{\pm 1\} \times \mathbb{G}_a$ (note that any linear map $\mathfrak{t} \to \mathfrak{t}$ is W-equivariant).

3. The diagonalizable part of $\tilde{J}_0$ equals T^W, while the diagonalizable part of J_0 equals $Z(G)$. Clearly $Z(G) \subset T^W$. Donagi and Gaitsgory say that if $[G,G]$ does not have $PGL(2)$ as a factor then $Z(G) = T^W$.
Proof for $PGL(n)$, $n > 2$: if $\mathrm{diag}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots \lambda_n)$ is proportional to $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ for all i then the proportionality coefficients equal 1 (because $n > 2$), so $\lambda_1 = \ldots = \lambda_n$.