Fundamental lemma for unitary groups

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This talk is based on a joint work with G. Laumon.

**Unitary group scheme.** Let $k = \mathbb{F}_q$ be a finite field of characteristic $p > 2$, $\overline{k}$ its algebraic closure. Let $X$ be a geometrically connected smooth projective curve over $k$. Let $\pi : X' \to X$ be a double etale covering which is geometrically connected. Let denote by $\tau$ the non-trivial automorphism of $X'$ over $X$.

The double etale covering $X'/X$ determines a unitary group over $X$. Let $\left( \mathcal{O}_{X'}, \Phi_n \right)$ be the standard hermitian bundle where

$$\Phi_n : \tau^* \mathcal{O}_{X'}^n \times \mathcal{O}_{X'}^n \to \mathcal{O}_{X'}$$

is the non-degenerate symmetric form given by the anti-diagonal unit matrix. Let $G/X$ be the group scheme of automorphisms of the pair $(\mathcal{O}_{X'}, \Phi_n)$ that is the functor which associates to any $X$-scheme $S$ the group

$$G(S) = \{ g \in \text{GL}_n(\mathcal{H}^0(S', \mathcal{O}_{S'})) \mid \tau^*(t^g)\Phi_n g = \Phi_n \}$$

where $S' = X' \times_X S$. For all $X$-scheme $S$, giving a $G$-torsor over $S$ is the same as giving a hermitian vector bundle $(\mathcal{E}, \Phi)$ consisting in a vector bundle of rank $n$ on $S' = X' \times_X S$ and a hermitian form $\Phi : \tau^* \mathcal{E} \times \mathcal{E} \to \mathcal{O}_{S'}$. Such a hermitian bundle is locally isomorphic to $(\mathcal{O}_{X'}, \Phi_n)$ locally for the etale topology of $X$.

Let $D$ be an effective divisor of degree $\geq 2g + 2$ where $g$ is the genus of $X$. For all vector bundle $\mathcal{E}$ on $X$, we will write $\mathcal{E}(iD) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(iD)$ for all $i \in \mathbb{Z}$.

The triple $(X, G, D)$ has been fixed, we have the notion of Hitchin pair defined as in the last talk. For instant, a Hitchin pair with coefficient in the base field $k$ is a pair $(\mathcal{T}, \varphi)$ where $\mathcal{T}$ is a $G$-torsor on $X$ and $\varphi \in H^0(X, \text{ad}(\mathcal{T})(D))$. In terms of hermitian bundles, such a Hitchin pair is equivalent to a triple $(\mathcal{E}, \Phi, \theta)$ where $(\mathcal{E}, \Phi)$ is a hermitian bundle, and $\Phi : \mathcal{E} \to \mathcal{E}(D)$ verifies the equation

$$\Phi \circ \varphi + \tau^*(t^\varphi) \circ \Phi = 0.$$
that is nothing but the equation defining the Lie algebra of the unitary group.

We consider the algebraic stack $\mathcal{M}$ which associates a $k$-scheme $S$ the category $\mathcal{M}(S)$ of Hitchin pair parameterized by $S$.

**Hitchin morphism.** The concrete definition of Hitchin morphism for unitary group goes as follows. Let $(\mathcal{E}, \Phi, \varphi) \in \mathcal{M}(k)$. The exterior power of $\varphi: \mathcal{E} \to \mathcal{E}(D)$ is a map $\wedge^i \varphi: \wedge^i \mathcal{E} \to \wedge^i \mathcal{E}(iD)$ whose the trace belongs to $\text{tr}(\wedge^i \varphi) \in H^0(X', \mathcal{O}_{X'}(iD))^{\tau = (-1)^i}$.

Let denote $L$ be the odd part in the decomposition

$$\pi^* \mathcal{O}_{X'} = \mathcal{O}_X \oplus L.$$  

We have $H^0(X', \mathcal{O}_{X'}(iD))^{\tau = (-1)^i} = H^0(X, \mathcal{L}(iD))$. The Hitchin affine space is

$$\mathbb{A} = \bigoplus_{i=1}^n H^0(X, \mathcal{L}(iD))$$

and the Hitchin map $f: \mathcal{M} \to \mathbb{A}$ is the map defined by

$$(\mathcal{E}, \Phi, \varphi) \mapsto a = (a_1, \ldots, a_n) \in \mathbb{A}$$

with $a_i = \text{tr}(\wedge^i \varphi)$.

**Spectral curve.** To all $a \in \mathbb{A}$, we can associate a spectral curve. Let consider the ruled surface $\Sigma = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}(D)^{\otimes -1})$ over $X$ which contains as open subvariety $\Sigma^0 = \mathbb{V}(\mathcal{L}_D^{\otimes -1})$ the total space of the line bundle $\mathcal{L}(D)^{\otimes -1}$. As a Proj associated to a graded $\mathcal{O}_X$-Algebra, $\Sigma$ is equipped with a relatively ample line bundle $\mathcal{O}_\Sigma(1)$ such that

$$p_*(\mathcal{O}_\Sigma(n) \otimes p^* \mathcal{L}(D)^{\otimes n}) = \mathcal{O}_X \oplus \mathcal{L}(D) \oplus \cdots \oplus \mathcal{L}(D)^{\otimes n}.$$  

A point $a = (a_1, \ldots, a_n) \in \mathbb{A}$ defines a global section of

$$(1, a_1, \ldots, a_n) \in H^0(\Sigma, \mathcal{O}_\Sigma(n) \otimes p^* \mathcal{L}(D)^{\otimes n})$$

and hence a divisor $Y_a \subset \Sigma$. Moreover the map $Y_a \to X$ is a finite flat morphism of degree $n$. One can check easily that, $Y_a \subset \Sigma^0$. Let denote

$$Y_a' = X' \times_X Y_a$$

the base change of $Y_a$ by the double etale covering $\pi: X' \to X$.  

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Elliptic part of \( A \). In the last talk, we have defined the open subvarieties \( A^{\text{ell}} \subset A^\circ \)
of the Hitchin affine space, for all reductive group scheme \( G/X \). The concrete definition for unitary group goes as follows.

Let \( a \in A(\mathbb{K}) \) a geometric point of \( A \).

- \( a \in A^\circ(\mathbb{K}) \) if and only if the map \( Y_a \to X \) is generically etale. If we assume that \( p > n \), this is equivalent to requiring the curve \( Y_a \) to be a reduced curve.
- \( a \in A^{\text{ell}}(\mathbb{K}) \) if \( a \in A^\circ(\mathbb{K}) \) and the map \( \text{Irr}(Y'_a) \to \text{Irr}(Y_a) \) from the set of irreducible components of \( Y'_a \) to the set of irreducible components of \( Y_a \), is a bijection.

Abelian description of Hitchin fibers. Let \( a \in A^\circ(\mathbb{K}) \). We will give an abelian description of the Hitchin fiber \( \mathcal{M}_a = f^{-1}(a) \) following [Hitchin] and [Beauville-Narasimhan-Ramanan]. Let \( (\mathcal{E},\Phi,\varphi) \in \mathcal{M}_a(\mathbb{K}) \) be a hermitian bundle \( (\mathcal{E},\Phi) \) equipped with a twisted unitary endomorphism \( \varphi : \mathcal{E}(-D) \to \mathcal{E} \). It follows that \( \mathcal{E} \) is equipped with a structure of \( \text{Sym}_X(\mathcal{O}_{X'}(D)) \)-Module. As coherent sheaf over \( \Sigma^0 = \mathbb{V}(\mathcal{O}_{X'}(D)) \), \( \mathcal{E} \) is supported by the spectral curve \( Y_a \) and therefore it defines a \( \mathcal{O}_{Y'_a} \)-Module \( \mathcal{F} \) which is torsion-free and has generic rank 1. This gives us a bijection between the set of vector bundle \( \mathcal{E} \) on \( X' \) equipped with a twisted endomorphism \( \varphi : \mathcal{E}(-D) \to \mathcal{E} \) whose characteristic is \( a \) and the set of torsion-free \( \mathcal{O}_{Y'_a} \)-Module of generic rank one. By taking into account the hermitian form \( \Phi \), we obtain the following assertion.

**Lemma 1** The fiber \( \mathcal{M}_a(\mathbb{K}) \) is isomorphic the scheme of pairs \( (\mathcal{F},t_{\mathcal{F}}) \) where \( \mathcal{F} \) is torsion-free \( \mathcal{O}_{Y'_a} \)-Module of generic rank one and \( t_{\mathcal{F}} \) is a unitary structure on \( \mathcal{F} \) i.e. \( t_{\mathcal{F}} : \tau^*\mathcal{F} \to \mathcal{F}^\vee \) such that \( \tau^*(t_{\mathcal{F}}) = t_{\mathcal{F}}^\vee \). The dual \( \mathcal{F}^\vee \) is understood with respect to the relative dualizing \( \omega_{Y'_a/X'} \) i.e. \( \mathcal{F}^\vee = \text{Hom}_{\mathcal{O}_{Y'_a}}(\mathcal{F},\omega_{Y'_a/X'}) \).

**Group action.** We will give now the concrete definition of the Picard stack \( P_a \) in the case of unitary group. The Picard category \( P_a(\mathbb{K}) \) is the category of pairs \( (\mathcal{G},t_{\mathcal{G}}) \) where \( \mathcal{G} \) is an invertible \( \mathcal{O}_{Y'_a} \)-Module and \( t_{\mathcal{G}} \) is a unitary structure of \( \mathcal{G} \)

\[
t_{\mathcal{G}} : \tau^*\mathcal{G} \cong \mathcal{G}^{\otimes -1} = \text{Hom}(\mathcal{G},\mathcal{O}_{Y'_a})
\]

such that \( \tau^*(t_{\mathcal{G}}) = t_{\mathcal{G}}^{\otimes -1} \). The Picard category \( P_a \) acts on \( \mathcal{M}_a \) by tensor product

\[
(\mathcal{G},t_{\mathcal{G}})(\mathcal{F},t_{\mathcal{F}}) = (\mathcal{G} \otimes \mathcal{F},t_{\mathcal{G}} \otimes t_{\mathcal{F}}).
\]

The Picard stacks \( P_a \) can be put together into a relative Picard stack \( P \to \mathbb{A} \).
Lemma 2  
1. \( M^\circ := M \times_{A^\circ} A^\circ \) is smooth.
2. \( P^\circ := P \otimes_{A^\circ} A^\circ \) is a smooth Picard stack over \( A^\circ \).
3. \( P^\text{ell} := P \otimes_{A^\ell} A^\ell \) is of finite type.
4. \( M^\text{ell} := M \times_{A^\ell} A^\ell \) is a Deligne-Mumford stack, the map \( M^\text{ell} \to A^\ell \) is proper.

As we have seen in the last talk, there exists a sheaf \( \pi_0(P/A^\circ) \) for the etale topology of \( A^\circ \) such that for all geometric point \( a \in A^\circ(k) \), we have the fiber of \( \pi_0(P/A^\circ) \) over \( a \) is the group of connected components \( \pi_0(P_a) \) of \( P_a \).

Lemma 3  
Let \( a \in A^\text{ell}(\overline{k}) \) i.e. the map \( \text{Irr}(Y'_a) \to \text{Irr}(Y_a) \) is a bijection. Then we have a canonical isomorphism
\[
\pi_0(P_a) = (\mathbb{Z}/2\mathbb{Z})^{\text{Irr}(Y_a)}.
\]

Let denote \( \pi_a : Y'_a \to Y_a \) the double etale covering of \( Y_a \) deduced from \( \pi : X' \to X \) by base change. We have an exact sequence of sheaf for etale topology of \( Y_a \)
\[
0 \to \mathcal{O}_{Y_a}^x \to (\pi_a)_* \mathcal{O}_{Y'_{a}}^x \to ((\pi_a)_* \mathcal{O}_{Y'_{a}}^x)^{\tau=-1} \to 0.
\]
Since \( H^2(Y_a, \mathcal{O}_{Y_a}^x) = 0 \), we have an exact sequence
\[
H^1(Y_a, \mathcal{O}_{Y_a}^x) \to H^1(Y'_a, \mathcal{O}_{Y'_a}^x) \to P_a = H^1(Y_a, ((\pi_a)_* \mathcal{O}_{Y'_a}^x)^{\tau=-1}) \to 0.
\]
It follows that
\[
\pi_0(\text{Pic}(Y_a)) \to \pi_0(\text{Pic}(Y'_a)) \to \pi_0(P_a) \to 0
\]
where \( \pi_0(\text{Pic}(Y_a)) = \mathbb{Z}^{\text{Irr}(Y_a)} \), \( \pi_0(\text{Pic}(Y'_a)) = \mathbb{Z}^{\text{Irr}(Y'_a)} \). Using the bijection \( \text{Irr}(Y_a) \to \text{Irr}(Y'_a) \), the map \( \pi_0(\text{Pic}(Y_a)) \to \pi_0(\text{Pic}(Y'_a)) \) is the multiplication by 2.

Stratification  
We have a natural stratification of the elliptic part by the set of partitions
\[
n = n_1 + \cdots + n_r
\]
where \( n_\bullet = (n_1 \geq \cdots \geq n_r > 0) \) is a decreasing sequence of positive integer numbers. Let \( \mathbb{A}_{n_\bullet} \) be the set of \( a \in A^\text{ell}(\overline{k}) \) such that the spectral curve \( Y_a \) is a union
\[
Y_a = \bigcup_{a=1}^r Y_{n_a}
\]
where $Y_{n_a}$ are reduced irreducible curves of degree $n_a$ over $X$. The elliptic condition on $a$ means that $Y_a' = \bigcup_{\alpha=1}^r Y_{n_a}''$ where $Y_{n_a}''$ are reduced irreducible curves of degree $n_a$ over $X''$.

Let denote $W' = \mathfrak{S}_n \rtimes \{1,\tau\}$ where $\tau$ acts on $\mathfrak{S}_n$ as the conjugation by the longest involution $w_0$. Consider the action of $W'$ on $X' = \mathbb{Z}^n$ defined by the usual action of $\mathfrak{S}_n$ on $\mathbb{Z}^n$ and by

$$\tau(d_1, \ldots, d_n) = (-d_n, \ldots, -d_1).$$

This induces an action of $W'$ on $\hat{T} = \text{Hom}(X', \mathbb{C}^\times)$. A subgroup $\Sigma \subset W'$ is said to be elliptic if there is no $W'$-invariants vectors inside $X'$. The elliptic subgroups $\Sigma \subset W'$ are of the form

$$\Sigma_{n,\bullet} = (\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_d}) \rtimes \mathbb{Z}/2\mathbb{Z}.$$ 

The elliptic strata $\mathbb{A}_{n,\bullet}$ are the strata $\mathbb{A}_{[\Sigma,\bullet]}$ defined in the last talk.

**The $\kappa$-decomposition.** The action of $\pi_0(P/\mathbb{A}^{\text{ell}})$ on $K^j = ^pH^j(f_*\mathbb{Q}_\ell|_{\mathbb{A}^{\text{ell}}})$ decomposes $K^j$ into direct sum

$$K^j = \bigoplus_{[\kappa]} K^j_{[\kappa]}$$

where $[\kappa]$ runs over the set of $W'$-conjugation classes of $\kappa \in \hat{T}$ whose the stabilizer in $W'$ is an elliptic subgroup. One can check that $\kappa$ must conjugate to an element of the form

$$\left(1, \ldots, 1, -1, \ldots, -1\right) \in \hat{T}$$

for some partition of length two $n = n_1 + n_2$ of $n$. So we can rewrite our decomposition

$$K^j = \bigoplus_{[n_1, n_2]} K^j_{[n_1, n_2]}$$

where $K^j_{[n_1, n_2]}$ is supported by the closure of the stratum $\mathbb{A}_{[n_1, n_2]}$.

**Endoscopic group.** Let $n_1, n_2$ be natural integers such that $n = n_1 + n_2$. Let $H = U(n_1) \times U(n_2)$ where $U(n_1)$ and $U(n_2)$ are unitary group scheme over $X$ defined by the double etale covering $X' \to X$. We have the Hitchin fibration for $H$

$$f_H : \mathcal{M}_H \to \mathbb{A}_H.$$
where the affine Hitchin space $\mathbb{A}_H$ of $H$ is $\mathbb{A}_H = \mathbb{A}_{n_1} \times \mathbb{A}_{n_2}$ with

$$\mathbb{A}_{n_\alpha} = \bigoplus_{i=1}^{n_\alpha} H^0(X, \mathcal{L}(D)^{\otimes i})$$

for all $\alpha \in \{1,2\}$. The natural map $\iota_H : \mathbb{A}_H = \mathbb{A}_{n_1} \times \mathbb{A}_{n_2} \to \mathbb{A}$ is defined using the multiplication map

$$H^0(X, \mathcal{L}(D)^{\otimes i}) \times H^0(X, \mathcal{L}(D)^{\otimes j}) \to H^0(X, \mathcal{L}(D)^{\otimes i+j}).$$

**Lemma 4** The map $\iota_H : \mathbb{A}_H \to \mathbb{A}$ has image $\mathbb{A}_{[n_1,n_2]}$. Over the open part $\mathbb{A}_\Sigma$ of $\mathbb{A}_{[n_1,n_2]}$, this map is finite, etale of degree 1 if $n_1 \neq n_2$, and is of degree 2 if $n_1 = n_2$.

**Purity statement** Let denote $\mathbb{A}^\circ_H$ the open part of $\mathbb{A}_H$ containing the pairs $(a_1,a_2)$ such that $Y_{a_\alpha}$ are irreducible. In other words, $\mathbb{A}^\circ_H$ is the pullback of $\mathbb{A}_{[n_1,n_2]} \subset \mathbb{A}_{[n_1,n_2]}$.

**Proposition 5** The restriction of $\iota^*_H K^j_{[n_1,n_2]}$ to the open $\mathbb{A}^\circ_H$ is a pure perverse sheaf.

**The geometric setting of the fundamental lemma.** Let $(a_1,a_2) \in \mathbb{A}^\circ_H(\overline{k})$ and let $a \in \mathbb{A}(\overline{k})$ be its image by $\iota_H : \mathbb{A}_H \to \mathbb{A}$. The spectral curve $Y_a$ is the union of the spectral curves $Y_{a_1}$ and $Y_{a_2}$ in the same ruled surface $\Sigma/X$. Let denote

$$c_{a_1,a_2} : Y_{a_1} \sqcup Y_{a_2} \to Y_a$$

the partial normalization of $Y_a$ which consists in separating the two irreducible components.

The functor $c^*$ defines a homomorphism $\text{Pic}(Y_a) \to \text{Pic}(Y_{a_1}) \times \text{Pic}(Y_{a_2})$ which induces a homomorphism

$$c^*_{a_1,a_2} : P_a \to P_{(a_1,a_2)} = P_{a_1} \times P_{a_2}$$

which is surjective. Let denote $K_a$ the kernel of $c^*_{a_1,a_2}$ that is an affine algebraic group over $\overline{k}$. This homomorphism can be put in family, we have a homomorphism of Picard stack

$$c^* : P \times_{\mathbb{A}} \mathbb{A}_H \to P_H$$

whose the kernel is an affine group scheme $K$ over $\mathbb{A}_H$.

Using Grothendieck duality, the functor $(c^*_{a_1,a_2})_*$ defines a map

$$(c_{(a_1,a_2)})_* : \mathcal{M}_{(a_1,a_2)} = \mathcal{M}_{a_1} \times \mathcal{M}_{a_2} \to \mathcal{M}_a$$
compatible with the action of $P_a$ on $\mathcal{M}_a$ and the action of $P_{(a_1,a_2)}$ on $\mathcal{M}_{(a_1,a_2)}$. This map can also be put into a family

$$c_* : \mathcal{M}_H \to \mathcal{M} \times A A.$$

**Lemma 6** The map $(c_{(a_1,a_2)})_*$ identifies $\mathcal{M}_{(a_1,a_2)}$ with the closed subset of $\mathcal{M}_a$ of fixed points under the action of $K_a$.

This lemma allows one to use the localization theorem of Atiyah-Borel-Segal.

**Localization theorem of Atiyah-Borel-Segal.** Let us recall the localization theorem of Atiyah-Borel-Segal and how Goreski-Kottwitz-MacPherson and Laumon introduced this tool in their approach to the fundamental lemma.

Let $X$ be an algebraic variety equipped with an action of a torus $T$. Let $X^*(T)$ be the group of cocharacters of $T$. The equivariant cohomology of $X$

$$\bigoplus_j H^j_T(X,\mathbb{Q}_\ell)$$

is a graded module over the graded polynomial ring

$$\bigoplus_j H^{2j}(BT) = \bigoplus_j \text{Sym}^j(X^*(T)) = \mathbb{Q}_\ell[X^*(T)].$$

Assuming that $H^j(X)$ is pure of weight $j$ then the degeneration of spectral sequence implies that there exists an isomorphism

$$\bigoplus_j H^j_T(X,\mathbb{Q}_\ell) = \bigoplus_j H^j(X,\mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[X^*(T)]$$

and in particular, $\bigoplus_j H^j_T(X,\mathbb{Q}_\ell)$ is a free $\mathbb{Q}_\ell[X^*(T)]$-module.

The restriction from $X$ to the fixed point locus $X^T$ defines a map

$$\bigoplus_j H^j_T(X,\mathbb{Q}_\ell) \to \bigoplus_j H^j_T(X^T).$$

The content of the Atiyah-Borel-Segal theorem is that the restriction map is **injective** under the assumption that $\bigoplus_j H^j_T(X,\mathbb{Q}_\ell)$ is a free $\mathbb{Q}_\ell[X^*(T)]$-module. In fact, the cohomology of the third vertex of the natural triangle is a torsion $\mathbb{Q}_\ell[X^*(T)]$ and a map from a torsion module to a free module vanishes necessarily.

Under some favorable circumstances for instant (finite number of fixed points, finite number of 1-dimensional orbits), there is a stronger version
of localization theorem, due to Chang and Sklejbred, which gives a precise description of the image of the restriction map. The work of Goresky-Kottwitz-MacPherson is concerned with the case of regular semi-simple element \( \gamma \in G(F_v) \) whose centralizer \( G_{\gamma_0} \) is an unramified torus. In this particular case, and under the purity assumption of the cohomology of Springer fiber, they are able to prove the fundamental lemma. They have introduced the important idea that the Langlands-Shelstad transfer factor should be encoded in an localization map in equivariant cohomology.

Laumon’s work is concerned only with the case of unitary group but the element \( \gamma \) can be in principle an arbitrary element. Under the same assumption of purity of Springer fiber, he proved the fundamental lemma in this case using a deformation argument. He constructed the deformation in some ad hoc way.

The Hitchin fibration realizes a deformation that contains somehow Laumon’s deformation. One advantage of Hitchin fibration is in that framework, we have a general purity statement almost for free.

**Relative situation.** The localization theorem can be easily generalized to a relative condition: let \( f : X \to S \) be a morphism, let \( T \) be a \( S \)-torus acting on \( X \), we also have to replace cohomology by perverse cohomology to have the purity assumption.

The injectivity is in fact much stronger in the framework of perverse sheaves. As we will see, it fits well with the deformation idea.

**Two tori.** Let \( Z_a = Y_{a_1} \cap Y_{a_2} \) and \( Z'_a \) is its base change by the double etale covering \( \pi : X' \to X \). Then we have

\[
K_a = H^0(Z'_a, \mathcal{O}_{Z'_a}^{\tau = -1}).
\]

This group is a torus if and only if all intersection points of \( Y_{a_1} \) and \( Y_{a_2} \) are of multiplicity one, in other words these two curves intersects transversally. Let \( \mathbb{A}_H^2 \) be the open of \( \mathbb{A}_H^2 \) containing the pairs \((a_1, a_2)\) such that \( Y_{a_1} \) and \( Y_{a_2} \) are reduced irreducible curve that may be singular but their intersection is transversal. The restriction of the affine group scheme \( K \) to \( \mathbb{A}_H^2 \) is a torus \( K^2 \). Over the open part \( \mathbb{A}_H^2 \) we can apply directly the Atiyah-Borel-Segal theorem directly to the torus \( K^2 \).

Let \( a \in \mathbb{A}_H^2(k) \) not necessarily in \( \mathbb{A}_H^2 \). The group scheme \( K_a \) may have unipotent part. Its torus part \( T_a \subset K_a \) has rank equal to the number of points of \( Z_a^{\text{red}} \). Let \( S \) be the henselization of \( \mathbb{A}_H \) at \( a \). The torus \( T_a \) can be extended uniquely to a subtorus \( T_S \subset K_S \). Over the henselization \( S \) we can apply the Atiyah-Borel-Segal localization theorem to the torus \( T_S \).
Over $\mathbb{A}^1_H$. Using the localization theorem of Atiyah-Borel-Segal over the transversal part $\mathbb{A}^1_H$ we can prove: there exists a graded isomorphism

$$
\bigoplus_j \ p^H((f_{zh})_*,\mathbb{Q}_\ell) \xrightarrow{\sim} \bigoplus_j p^{H^j}(f_{zh})_*\mathbb{Q}_\ell \otimes L_{Z'/Z/H}(-r)
$$

where the shifting $r$ is the intersection number of $Y_{a_1}$ and $Y_{a_2}$

$$r = \lg(Z_a)
$$

and where $L_{Z'/Z/H}$ is a local system of rank one and of order two. Over $\mathbb{A}^1_H$, it is defined as the determinant of $p^{\xi}_Z$ where $p_Z$ is the finite etale map $Z' \to \mathbb{A}^1_H$. We can check that this local system of rank one can be extended to $\mathbb{A}_H$.

Over the henselian scheme $S$. Over $S$ we have $f_S : \mathcal{M}_S \to S$ the base change of the Hitchin map $f : \mathcal{M} \to \mathbb{A}_1$ to $S$ and $g_S = f_{H,S} : \mathcal{N}_S = \mathcal{M}_{H,S} \to S$ the base change of the map $f_H : \mathcal{M}_H \to \mathbb{A}_H$ to $S$. We also have a closed embedding $\mathcal{N}_S \hookrightarrow \mathcal{M}_S$ that identifies $\mathcal{N}_S$ with the locus of fixed points of $\mathcal{M}_S$ under the action of the torus $T_S$. We have the restriction map between $T_S$-equivariant cohomology of $f_S$ and $g_S$

$$
\bigoplus_j p^H((f_S)_T^*\mathbb{Q}_\ell) \rightarrow \bigoplus_j p^H((g_S)_T^*\mathbb{Q}_\ell) = \bigoplus_j p^H((g_S)_*\mathbb{Q}_\ell) \otimes H^*(BT_S)
$$

which induces a map between the $\kappa$-parts

$$
\bigoplus_j p^H((f_S)_T^*\mathbb{Q}_\ell)_\kappa \rightarrow \bigoplus_j p^H((g_S)_*\mathbb{Q}_\ell)_\kappa \otimes H^*(BT_S).
$$

Since $p^H((f_S)_T^*\mathbb{Q}_\ell)_\kappa$ is pure of weight $j$, this last map is injective.

Let $S^2$ be the base change of $\mathbb{A}^1_H$ to $S$. A perverse sheaf $K$ over $S$ is said to have the property of $\xi$-extension if we have $K = j_{!*}j^*K$. We can prove $p^H((g_S)_*\mathbb{Q}_\ell)_\kappa \otimes H^*(BT_S)$ are pure semi-simple perverse sheaves which have the property of $\xi$-extension. The injectivity implies, this is also true for $\bigoplus_j p^H((f_S)_T^*\mathbb{Q}_\ell)_\kappa$.

The isomorphism we have constructed on $S^2$ extends to all $S$. This implies the global version of fundamental lemma. To prove the local version of the fundamental lemma we use approximation method. At this point, we need for the moment the hypothesis $p > n$.