Let G be a connected split reductive group over a field k, We assume that char k is not a divisor of the order of the Weyl group W (e.g., char $k = 0$). We fix a framing ("épinglage") of G, i.e., a split maximal torus $T \subset G$, a Borel $B \supset T$, and a nonzero element $x_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Delta$. Here Δ denotes the set of simple roots of T w.r.t. B.

For each $\alpha \in \Delta$ define $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ so that $x_\alpha, x_{-\alpha}$, and $h_\alpha := [x_\alpha, x_{-\alpha}]$ form an $sl(2)$-triple. Let $x_- := \sum\limits_{\alpha\in\Delta} x_{-\alpha}$, $x_+ := \sum\limits_{\alpha\in\Delta} x_\alpha$. Let $\mathfrak{g}^{x_+}$ be the centralizer of x_+. Let $\mathfrak{c}$ be the geometric quotient of $\mathfrak{g}$ by the adjoint action of G; $\mathfrak{c}$ is also the geometric quotient $\mathfrak{t}/W$.

<u>Kostant's Theorem</u> (if char $k \neq 0$ it is due to Veldkamp).

(i) The map $x_- + \mathfrak{g}^{x_+} \to \mathfrak{c}$ is an isomorphism,

(ii) All elements of $x_- + \mathfrak{g}^{x_+}$ are regular.

<u>Corollary</u>. Every regular element of $\mathfrak{g}$ is geometrically conjugate to a unique element of $x_- + \mathfrak{g}^{x_+}$.

<u>Remark</u>. One shows that $\mathfrak{g}^{x_+} \subset \mathfrak{n}_+ + \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

A proof in characteristic 0 can be found in Kostant's article or on p. 28 of e-print 0501398 by Beilinson and me (in the e-print we work not with x_+ but with the element $x'_+ \in \sum\limits_{\alpha\in\Delta} \mathfrak{g}_\alpha$ such that x'_+, x_-, and $[x'_+, x_-]$ form an $sl(2)$-triple).

<u>Variant</u>: instead of $x_- + \mathfrak{g}^{x_+}$ consider $x_- + \mathfrak{b}$, $\mathfrak{b} := \mathrm{Lie}(B)$. Note that $x_- + \mathfrak{b} \subset \mathfrak{g}$ is stable w.r.t. N-conjugation.

<u>Variant of Kostant's theorem.</u> The action of N on $x_- + \mathfrak{b}$ is free and the maps

$$x_- + \mathfrak{g}^{x_+} \longrightarrow (x_- + \mathfrak{b})/N \longrightarrow \mathfrak{c} = \mathfrak{h}/W$$

are isomorphisms of algebraic varieties.

Moreover, if $\mathfrak{g}$ is equipped with the $\mathbb{Z}$-grading such that $\deg \mathfrak{f} = 0$ and $\deg e_{\pm\alpha} = \pm 1$ for $\alpha \in \Delta$ then for every graded subspace $V \subset \mathfrak{g}$ such that $\mathfrak{b} = [e_-, \mathfrak{n}] \oplus V$ the maps

$$x_- + V \to (x_- + \mathfrak{b})/N \to \mathfrak{c} = \mathfrak{f}/W$$

are isomorphisms of algebraic varieties. All elements of $x_- + \mathfrak{b}$ are regular.

Remarks. (i) One shows that the condition on V is satisfied for $V = \mathfrak{g}^{x_+}$.

(ii) According to my experience, the isomorphism $(x_- + \mathfrak{b})/N \xrightarrow{\sim} \mathfrak{c}$ is more useful than the isomorphism $x_- + V \xrightarrow{\sim} \mathfrak{c}$ for any particular choice of V (including the choice $V = \mathfrak{g}^{n_+}$).

Example: $G = GL(n)$. An element $A \in \mathfrak{gl}(n)$ is regular if and only if the operator A has a cyclic vector (or equivalently, iff for every eigenvalue λ the Jordan normal form of A has only one Jordan block corresponding to λ). For a suitable framing of $G = GL(n)$, the set $x_- + \mathfrak{b}$ consists of matrices (a_{ij}) such that $a_{i+1,i} = 1$ and $a_{ij} = 0$ for $i > j+1$, while $x_- + \mathfrak{g}^{e_+}$ consists of matrices (a_{ij}) from $x_- + \mathfrak{b}$ such that a_{ij} depends only on $i - j$.

In this example the fact that all elements of $x_- + \mathfrak{b}$ are regular is clear (the first vector of the basis is cyclic). It is also easy to prove that the map $x_- + \mathfrak{b} \to \mathfrak{c}$ is surjective, i.e., each monic polynomial $f \in k[t]$ of degree n is the characteristic polynomial of a matrix from $x_- + \mathfrak{b}$. Namely, consider the matrix of multiplication by t in $k[t]/(f)$ w.r.t. the basis $\{1, t, \dots, t^{n-1}\}$. In fact, this argument gives an isomorphism of algebraic varieties $x_- + V \xrightarrow{\sim} \mathfrak{c}$, where $V \subset \mathfrak{b}$ is the subspace of matrices (a_{ij}) with $a_{ij} = 0$ for $j \neq n$. (Matrices from $x_- + V$ are sometimes called *companion matrices*).