Let X be a smooth connected projective curve over an algebraically closed field k of characteristic 0 (or big enough positive characteristic). Let G be an algebraic group over k.

Some notation and conventions.

Notation: if $E \to X$ is a (principal) G-bundle (a.k.a. G-torsor) and if M is a variety with G-action then the E-twist of M is denoted by M_E; so $M_E := (E \times M)/G$. If M is a vector space and the action is linear we get a vector bundle. E.g., we can take $M = \mathfrak{g}$ and get the adjoint bundle $\mathfrak{g}_E$ (also denoted by $\operatorname{ad} E$).

Usually we identify $GL(n)$-torsors with rank n vector bundles. In particular, a $\mathbb{G}_m$-torsor is the same as a line bundle.

If $G = GL(n)$ and E is a G-bundle then $\mathfrak{g}_E = \operatorname{End} E$ (i.e., $\mathfrak{g}_E$ is the bundle of endomorphisms of E viewed as a vector bundle).

Higgs bundles.

Let G be reductive (and connected).

We fix a line bundle $\mathcal{D}$ on X of large degree. "Large" means that $\deg \mathcal{D} > 2g$, where g is the genus. We assume that $\mathcal{D} = \mathcal{D}'^{\otimes 2}$ for some line bundle $\mathcal{D}'$, and we fix a representation of $\mathcal{D}$ as $\mathcal{D}'^{\otimes 2}$ (We will use it on p. 10)

Definition. A Higgs bundle on X is a pair (E, ϕ), where E is a G-bundle on X and $\phi \in \Gamma(X, \mathfrak{g}_E \otimes \mathcal{D})$. ϕ is called "Higgs field".

Remark. In fact, the words "Higgs bundle" or "Higgs field" should be in quotation marks: in the true Higgs bundle one should have $\mathcal{D}=\Omega_X$. This also applies to the definition of Hitchin fibration below. The true Hitchin fibration (for $\mathcal{D}=\Omega_X$) is symplectic, and the fibers of the true Hitchin fibration are Lagrangian. But this is not the case if $\mathcal{D}$ is arbitrary.

Let $\mathcal{M}$ denote the stack of Higgs bundles on X. This is an algebraic stack (rather than a space) because Higgs bundles may have nontrivial automorphisms. In practice we will work only with Higgs bundles whose automorphism group is finite modulo the center of G. In other words, we will "live" on an open substack of $\mathcal{M}$ which is "almost a space".

The Hitchin fibration $f: \mathcal{M} \to \mathcal{A}$.

We have to define $\mathcal{A}$ and f.

Example. $G = GL(n)$. A Higgs bundle is a pair (E, ϕ), $\phi \in \Gamma(X, (\mathcal{E}nd\, \mathcal{E}) \otimes \mathcal{D})$. Hitchin's idea was to associate to (E, ϕ) the coefficients of the characteristic polynomial of ϕ. Let us number them so that the first coefficient is the trace. Then the i-th coefficient is in $\Gamma(X, \mathcal{D}^{\otimes i})$. Thus we get $f: \mathcal{M} \to \mathcal{A} := \Gamma(X, \bigoplus_i \mathcal{D}^{\otimes i})$.

Now let G be any reductive group.

A slightly noncanonical approach. The coordinate ring of $\mathfrak{c} := \mathfrak{t}/W$ is isomorphic (as a graded algebra) to $k[u_1, \dots, u_r]$, $\deg u_i = k_i$. Choose such an isomorphism and define $\mathcal{A}$ to be $\Gamma(X, \bigoplus_{i=1}^{r} \mathcal{D}^{\otimes k_i})$. Then we can define $f: \mathcal{M} \to \mathcal{A}$ just as in the case of $GL(n)$.

The above approach is noncanonical because there is no God-given choice of $u_1, \dots, u_r$. Equivalently, $\mathfrak{c}$ is not really a vector space but just an algebraic variety with

$\mathbb{G}_m$-action. This $\mathbb{G}_m$-variety admits a linearization, but there is no God-given linearization.

Canonical definition of $\mathcal{A}$: $\mathcal{A} := \Gamma(X, c_{\mathcal{D}})$

($c_{\mathcal{D}}$ is not a vector bundle but just a bundle, and $\mathcal{A}$ is the space of its sections; so $\mathcal{A}$ is not a vector space but a $\mathbb{G}_m$-variety isomorphic to a vector space).

Definition of $f: \mathcal{M} \to \mathcal{A}$. The morphism $\chi: \mathfrak{g} \to c$ is $\mathbb{G}_m$-equivariant and G-equivariant (the action of G on c is trivial). So we can twist χ and get

$$\chi_{\mathcal{D}}: \underset{\mathfrak{g}\otimes\mathcal{D}}{\overset{\shortparallel}{\mathfrak{g}_{\mathcal{D}}}} \to c_{\mathcal{D}} \quad \text{and} \quad \chi_{\mathcal{D},E}: \mathfrak{g}_E \otimes \mathcal{D} \to c_{\mathcal{D},E} = c_{\mathcal{D}}$$

($c_{\mathcal{D},E} = c_{\mathcal{D}}$ because the action of G on c is trivial).

So we get

$$\begin{array}{ccc} \Gamma(X, \mathfrak{g}_E \otimes \mathcal{D}) & \to & \Gamma(X, c_{\mathcal{D}}) \\ \psi & & \psi \\ \phi & & \chi_{\mathcal{D},E}(\phi) \end{array}$$

Now define $f: \mathcal{M} \to \mathcal{A}$ by $f(E,\phi) := \chi_{\mathcal{D},E}(\phi)$.

The open subsets $\mathcal{A}^\diamondsuit \subset \mathcal{A}^\heartsuit \subset \mathcal{A}$.

Definition. $\mathcal{A}^\heartsuit$ (heart) $:= \{\text{sections of } c_{\mathcal{D}} \text{ which are generically in } c_{\mathcal{D}}^{reg,ss}\}$

Why only generically? We have the discriminant map $\mathrm{Discr}: c \to \mathbb{A}^1$, which is a homogenous polynomial of degree $N := \{\text{number of rooots}\}$. It induces a map $\mathrm{Discr}_{\mathcal{D}}: c_{\mathcal{D}} \to \mathcal{D}^{\otimes N}$. So if $a \in \mathcal{A}$ we have $\mathrm{Discr}_{\mathcal{D}}(a) \in \Gamma(X, \mathcal{D}^{\otimes N})$. By definition, $a \in \mathcal{A}^\heartsuit$ if and only if $\mathrm{Discr}_{\mathcal{D}}(a)$ is not identically zero. Let Δ_a be the divisor of $\mathrm{Discr}_{\mathcal{D}}(a) \in \Gamma(X, \mathcal{D}^{\otimes N})$. This is an effective divisor.

It cannot happen that $\Delta_a = 0$ because $\deg \Delta_a = N \cdot \deg \mathcal{D}$.

diamond

<u>Definition.</u> $\mathcal{A}^\diamond := \{$sections of $C_{\mathcal{D}}$ such that Δ_a is multiplicity-free$\}$

Clearly $\mathcal{A}^\diamond \subset \mathcal{A}^\heartsuit \subset \mathcal{A}$ are non-empty open subsets. One can show that $\mathcal{A}^\diamond \neq \emptyset$.

<u>Definition.</u> $\mathcal{M}^\heartsuit := f^{-1}(\mathcal{A}^\heartsuit)$, $\mathcal{M}^\diamond := f^{-1}(\mathcal{A}^\diamond)$.

<u>Plan.</u>

① If $G = GL(n)$ then to $a \in \mathcal{A}$ one associates an n-sheeted covering $Y_a \to X$. Y_a is called the <u>spectral curve</u>. It may be singular. It turns out that

$a \in \mathcal{A}^\heartsuit \Leftrightarrow Y_a$ is reduced

$a \in \mathcal{A}^\diamond \Leftrightarrow$ Y_a is smooth, the critical points of the map $Y_a \to X$ are Morse (i.e., the ramification indices equal 2), and the critical values are distinct.

Let $\mathcal{M}_a := f^{-1}(a)$. Then for $a \in \mathcal{A}^\heartsuit$ one has

$\mathcal{M}_a = Coh(Y_a) := \{$torsion-free coherent sheaves on Y_a of rank 1$\}$

(By definition, "rank 1" means "rank 1 on the smooth locus").

A torsion-free coherent sheaf on a smooth curve is a vector bundle, so if $a \in \mathcal{A}^\diamond$ then $\mathcal{M}_a = \underline{\operatorname{Pic}} Y_a$, where $\underline{\operatorname{Pic}} Y_a$ denotes the <u>stack</u> of line bundles on Y_a.

In general, $\underline{\operatorname{Pic}} Y_a \subset \mathcal{M}_a$ and the Picard stack $\underline{\operatorname{Pic}} Y_a$ acts on $\mathcal{M}_a$ (one can tensor a coherent sheaf by a line bundle).

"Picard stack" is Grothendieck's name for "commutative group stack". In other words, "Picard stack" is to "stack" like "commutative algebraic group" is to "scheme".

② For <u>any</u> reductive group G one defines a Picard stack $\mathcal{P}_a$ acting on $\mathcal{M}_a$, $a \in \mathcal{A}$ (if $G = GL(n)$ then $\mathcal{P}_a = \underline{\mathrm{Pic}}\, Y_a$). As $a \in \mathcal{A}$ varies one gets a Picard stack $\mathcal{P}$ over $\mathcal{A}$ acting on $\mathcal{M}$ (over $\mathcal{A}$).

One also has $\mathcal{P} \xrightarrow{\sim} \mathcal{M}^{reg} \subset \mathcal{M}$, where $\mathcal{M}^{reg}$ is the open substack of $\mathcal{M}$ defined by

$$\mathcal{M}^{reg} := \{(E, \phi) \in \mathcal{M} \mid \phi \in \Gamma(X, \mathfrak{g}^{reg})\}$$

One shows that if $a \in \mathcal{A}^{\diamond}$ then $\mathcal{P}_a = \mathcal{M}_a$ and if $a \in \mathcal{A}^{\heartsuit}$ then $\mathcal{P}_a$ is dense in $\mathcal{M}_a$.

<u>Remark.</u> The density statement is not obvious: for $G = GL(n)$ it amounts to the Altman-Iarrobino-Kleiman theorem, which says that if Y is a curve with plane singularities then $\underline{\mathrm{Pic}}\, Y$ is dense in the stack of rank 1 torsion-free coherent sheaves. "Plane singularities" means that Y is locally embeddable into $\mathbb{A}^2$ (or equivalently, that $\dim T_y Y \leq 2$ for all $y \in Y$). We will see that spectral curves have plane singularities. It is also true that every curve with plane singularities is locally isomorphic to a spectral curve. Altman, Iarrobino, and Kleiman constructed a curve with more complicated singularities for which the density statement does not hold.

<u>On Ngo's strategy of the proof of the fundamental lemma.</u>

He proves a "product formula", which says that

$$\mathcal{M}_a / \mathcal{P}_a = \prod_{x \in \Delta_a} \mathcal{W}_{x,a}$$

where each stack $\mathcal{W}_{x,a}$ has "local nature" (it is described in terms of the formal neighborhood of x in X). It turns out that the fundamental lemma is (more or less) about counting $\mathbb{F}_q$-points on $\mathcal{W}_{x,a}$. Ngo's strategy is to use $\mathcal{M}_a/\mathcal{P}_a$ to understand $\mathcal{W}_{x,a}$.

Spectral curves.

Let $G = GL(n)$. Then

$$c = \left\{ \begin{matrix} \text{monic polynomials} \\ p \in k[t],\ \deg p = n \end{matrix} \right\} = \left\{ \begin{matrix} \text{finite subschemes} \\ \text{of } \mathbb{A}^1 \text{ of order } n \end{matrix} \right\}$$

$\mathbb{G}_m$ acts on c. So if ℓ is a 1-dimensional vector space over k (i.e., a $\mathbb{G}_m$-torsor over k) one can twist c by ℓ and get

$$c_\ell = \{ t^n + \alpha_1 t^{n-1} + \ldots + \alpha_0 \mid \alpha_i \in \ell^{\otimes i} \}.$$

<u>Exercise.</u> $c_\ell = \{$finite subschemes of ℓ of order $n\}$.

We have a line bundle $\mathcal{D} \to X$, and $\mathcal{A} := \Gamma(X, c_{\mathcal{D}})$.

<u>Definition.</u> A <u>spectral curve</u> is a closed subscheme $Y \subset \mathcal{D}$ such that the morphism $Y \to X$ is flat, finite, and of degree n.

Applying the exercise to $\ell = \mathcal{D}_x$ for each $x \in X$, we see that $\mathcal{A} := \Gamma(X, c_{\mathcal{D}})$ identifies with the space of spectral curves. The spectral curve corresponding to $a \in \mathcal{A}$ will be denoted by Y_a.

Clearly a point $a \in \mathcal{A}$ is in $\mathcal{A}^{\heartsuit}$ if and only if Y is reduced (here we are using that char $k = 0$ or char $k > n$).

<u>Exercise.</u> $a \in \mathcal{A}^{\diamond}$ if and only if Y is smooth, the critical points of the map $Y \to X$ are Morse, and its critical values are distinct.

Let us describe the Hitchin fiber $\mathcal{M}_a$, $a \in \mathcal{A}^\heartsuit$, in terms of the spectral curve Y_a.

Claim. $\mathcal{M}_a$ identifies with the stack of torsion-free coherent sheaves on Y_a of rank 1.

Construction. Given a Higgs bundle $(E, \phi: E \to \mathcal{D} \otimes E)$ we will construct a coherent sheaf on Y_a. We can think of ϕ as $\phi: \mathcal{D}^{-1} \to \mathrm{End}\, E$ or as an algebra morphism $\phi: \mathrm{Sym}(\mathcal{D}^{-1}) \to \mathrm{End}\, E$, where $\mathrm{Sym}(\mathcal{D}^{-1})$ is the symmetric algebra of $\mathcal{D}^{-1}$. So ϕ defines on E a structure of a sheaf of modules over $\mathrm{Sym}(\mathcal{D}^{-1})$. Let $\tilde{E}$ be the corresponding coherent sheaf on $\mathrm{Spec}\,\mathrm{Sym}(\mathcal{D}^{-1}) = \{\text{total space of } \mathcal{D}\}$. By the Hamilton-Cayley theorem, $\tilde{E}$ is supported on Y_a. Let $\pi: Y_a \to X$ be the projection, then $\pi_* \tilde{E} = E$. So $\tilde{E}$ is torsion-free.

Let us show that if $x \in X$ is a generic point and $y \in \pi^{-1}(x)$ then the fiber $\tilde{E}_y$ has dimension 1. Fix an isomorphism $\mathcal{D}_x \xrightarrow{\sim} k$, then ϕ_x becomes an operator $E_x \to E_x$. Our y corresponds to an eigenvalue λ_y of ϕ_x, and $\tilde{E}_y = \mathrm{Coker}(\phi_x - \lambda_y)$. If x is generic then λ_y has multiplicity 1, so $\dim \tilde{E}_y = 1$.

Exercise. (i) Describe the inverse construction.

(ii) Let $a \in \mathcal{A}$ be the polynomial $t^n + \alpha_1 t^{n-1} + \ldots + \alpha_n$, $\alpha_i \in \Gamma(X, \mathcal{D}^{\otimes i})$. Show that the trivial line bundle on Y_a corresponds to the following Higgs bundle $(E, \phi: E \to E \otimes \mathcal{D})$ on X:

$$E = \mathcal{O}_X \oplus \mathcal{D}^{-1} \oplus \ldots \oplus \mathcal{D}^{1-n},$$

$$\phi = \begin{pmatrix} 0 & & 0 & -\alpha_n \\ 1 & 0 & & \cdots \\ & 1 & & \\ 0 & & 1 \quad 0 & -\alpha_2 \\ & & 1 & -\alpha_1 \end{pmatrix}$$

Remark. In Exercise (ii) $\det E$ is nontrivial, which is unpleasant. To get rid of this drawback, one can modify the above correspondence between Higgs bundles on E and rank 1 torsion-free coherent sheaves on Y_a as follows: to a Higgs bundle (E, ϕ) associate $\tilde{E} \otimes \pi^* \mathcal{D}^{\frac{1-n}{2}}$ rather than $\tilde{E}$. (If n is even then one needs here a square root of $\mathcal{D}$).

Definition of P_a and the action $P_a \times M_a \to M_a$.

Now let G be any reductive group. Let $a \in \mathcal{A} = \Gamma(X, C_{\mathcal{D}})$. We have the group scheme of regular centralizers $J \to C$. Since $\mathbb{G}_m$ acts on $J \to C$ we can twist by the $\mathbb{G}_m$-torsor $\mathcal{D}$ and get

$$\begin{array}{ccc} & & X \\ & & \downarrow a \\ J_{\mathcal{D}} & \longrightarrow & C_{\mathcal{D}} \end{array}$$

Define $J_a := a^* J_{\mathcal{D}}$. Let $\mathcal{P}_a$ denote the stack of J_a-torsors. Since the group scheme J_a is commutative $\mathcal{P}_a$ is a Picard stack.

<u>Example.</u> Let $G = GL(n)$. Then we have the spectral curve Y_a and the projection $\pi: Y_a \to X$. The group $(J_a)_x$ identifies with the group of invertible functions on $\pi^{-1}(x)$. So a section of J_a over an open subset $U \subset X$ is the same as an invertible function on $Y_a \times_U X$. So a J_a-torsor on X is the same as a $\mathbb{G}_m$-torsor on Y_a, i.e., $\mathcal{P}_a = \underline{\mathrm{Pic}}\, Y_a$.

Now let us define the action $\mathcal{P}_a \times \mathcal{M}_a \to \mathcal{M}_a$. It will come from a certain action of J_a on (E, ϕ), (which will be defined) for any $(E,\phi) \in \mathcal{M}_a$; namely, given such action we can twist (E, ϕ) by any J_a-torsor.

It remains to define an action of J_a on (E,ϕ), i.e., a morphism $J_a \to \underline{\mathrm{Aut}}(E,\phi) \subset \underline{\mathrm{Aut}}\, E = G_E$. The question is local on X, so we can assume that E and $\mathcal{D}$ are trivial. Fix a trivialization of E and that of $\mathcal{D}$. Then ϕ becomes a map $\phi: X \to \mathfrak{g}$, $\underline{\mathrm{Aut}}(E,\phi)$ becomes a group subscheme of $G \times X$, and the fiber of $\underline{\mathrm{Aut}}(E,\phi)$ over $x \in X$ equals the centralizer $I_{\phi(x)}$. So what we need is a morphism $J_{a(x)} \to I_{\phi(x)}$, $a(x) = \chi(\phi(x))$. We defined it last time.

On the isomorphism $P_a \xrightarrow{\sim} \mathcal{M}_a^{reg}$.

The essential thing is to define a morphism $P_a \to \mathcal{M}_a^{reg}$ (proving that it is an isomorphism is easy). Since P_a acts on $\mathcal{M}_a$ this amounts to specifying the image of $0 \in P_a$. It is here that one needs a square root of $\mathcal{D}$.

<u>Example</u> (cf. Exercise (ii) on p. 8). Let us define a canonical section $\mathcal{A} \to \mathcal{M}$ for $G = SL(n)$. Let $a \in \mathcal{A}$ be the polynomial $t^n + \alpha_1 t^{n-1} + \ldots + \alpha_n$, $\alpha_i \in \Gamma(X, \mathcal{D}^{\otimes i})$. How to write the corresponding Higgs bundle (E, ϕ)? We take ϕ to be the "companion matrix"

$$\phi = \begin{pmatrix} & 0 & & -\alpha_n \\ 1 & & & \vdots \\ & \ddots & & \\ 0 & \ldots & 1, & -\alpha_1 \end{pmatrix}$$

But we need a rank n vector bundle E <u>with trivial determinant</u> such that ϕ makes sense as an operator $E \to E \otimes \mathcal{D}$. Answer: $E = \bigoplus_{i \in S} \mathcal{D}^{\otimes i}$, where $S = \left\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right\}$. So if n is even we need $\mathcal{D}' := \mathcal{D}^{1/2}$ to define E. A more scientific explanation of why one needs a square root of $\mathcal{D}$ can be found in §2.2.3 of Ngo's article.

<u>Generalization</u>. Let G be any reductive group. To define the canonical section $\mathcal{A} \to \mathcal{M}$ take E to be the pushforward of the $\mathbb{G}_m$-torsor $\mathcal{D}' = \mathcal{D}^{1/2}$ via $2\check{\rho}: \mathbb{G}_m \to G$. Here $2\check{\rho}$ is the sum of the positive coroots.