Lecture V

In this lecture I will present the proof of meromorphic continuation of Eisenstein series.

1. Let G be a group over $\mathbb{Q}$, Γ – an arithmetic subgroup. We denote by G the group $G(\mathbb{R})$ of $\mathbb{R}$-points of G. Our main object of study is the automorphic space y.

Let P be a class of parabolic subgroups defined over $\mathbb{Q}$. For such P we constructed the space y_P in lecture 4.

We denote by $\mathfrak{a}_P$ the Lie algebra of the $\mathbb{Q}$-split component A_P of the center of L_P. A_P and hence $\mathfrak{a}$ act on the space y_P and we consider L^2-twisted actions of A_P and $\mathfrak{a}_P$ on $F(y_P)$.

2. Let f be an automorphic form on y_P. Then it is $\mathfrak{a}_P$-finite and hence can be decomposed $f = \Sigma f_x$ over some characters $x \in S_P := (\mathfrak{a}_P^*)_{\mathbb{C}}$. If this sum is over a finite subset $R \subset S_P$ we say that $\mathrm{supp}_{S_P}(f) \subset R$.

3. We would like to characterize every automorphic form f on y by support of all the cuspidal components of its constant terms.

More precisely, for any parabolic Q consider the form $f_Q = C_{QG}(f) \in F(y_Q)$, take its cuspidal component $f_{Q,\mathrm{cusp}}$ and decompose it as a sum over characters $x \in S_Q$.

The collection of all pairs $(Q, x \in S_Q)$ we call *cuspidal exponents* of f, and the corresponding functions $f_{Q,\mathrm{cusp},x} \in F_{\mathrm{cusp}}(y_Q)^x$ the *cuspidal coefficients* of f.

A similar definition works for $f \in \mathcal{A}(F_p)$, only here cuspidal exponents are taken only for $Q \subset P$.

Lemma. *If all cuspidal coefficients of an automorphic form f vanish then $f = 0$.*

Proof by induction.

3. If $P \subset P'$ then clearly cuspidal coefficients of the function $C_{P,P'}(f)$ are just a part of cuspidal coefficients of $f \in \mathcal{A}(y_{P'})$ – namely coefficients corresponding to parabolic subgroups $Q \subset P$.

Geometric lemma allows us also to restrict cuspidal coefficients of Eisenstein operators whenever they converge.

Suppose we are given an automorphic form $\varphi \in \mathcal{A}(y_P)$. Denote by B the set of cuspidal exponents of φ and φ_B, the collection of corresponding cuspidal coefficients.

Consider now the form $f = E_{GP}(\varphi)$ and assume that the series converges absolutely.

Proposition. *Cuspidal exponents of f are of the form $(Q, w(B_{P'}))$, where Q is a parabolic subgroup, $w : Q \to P' \subset P$ an association of parabolic subgroups and $B_{P'}$ are cuspidal are cuspidal exponents of the form φ corresponding to P'.*

Moreover cuspidal coefficients of f are sums of R_w (cusp. coefficients of φ).

This proposition follows from the geometric lemma, since in the formula $C_{QG}(f) = C_{QG} \circ E_{GP}(\varphi) = \sum_w E_{QQ'} \circ R_w \circ C_{P'P}$ the terms on the right with $Q \neq Q'$ will not give any contribution to the cuspidal part.

4. Fix $\varphi \in \mathcal{A}(y_{\mathcal{P}})$, consider the family of automorphic forms $u_s = \varphi \cdot a^s$ parametrized by points $s \in S_P$ and set $E(\varphi, s) := E(u_s)$.

As was shown in Borel's lecture this series is absolutely converging for $Re\ s \gg 0$ in the C-Weyl chamber in $\mathfrak{a}_P^*$.

Theorem. *$E(\varphi, s)$ has meromorphic continuation.*

Let us write the system of equations which are satisfied by $E(\varphi, s)$ in the domain of convergence.

(i) $\varphi \in \mathcal{A}_\sigma$ for some finite subset $\sigma \in Ivv/c$. Hence we have an equation

$$e_\sigma E(\varphi, s) = E(\varphi, s) \ .$$

(ii) There exists an ideal $I \subset z(L_P)$ of finite codimension such that $I\varphi = 0$. This implies that $I_s u_s = 0$, where I_s is the shift of the ideal I by $s \in s_{\mathcal{P}}$. Hence we have an equation

$$\tilde{I}_s E(\varphi, s) = 0$$

where $\tilde{I}_s = I_s \cap z(G)$.

Let us denote the system of equations (i) and (ii) by $\Xi'(s)$.

Theorem. *$\Xi'(s)$ is a system of equations in the space $F^{\mathrm{temp}}(y)$ of locally finite type.*

We will discuss the proof of this theorem in the appendix.

Now we would like to write the third equation which fixes our function f uniquely (in some domain of the variety of parameters S_P).

5. Let Q be a parabolic subgroup. Denote by $W_{Q\to P}$ the subset of $W_Q\backslash W/W_P$ which consists of elements w which give association of Q with a parabolic subgroup $P' \subset P$.

We know from proposition 3 above that Q cuspidal exponents of $E(\varphi, s)$ are contained in the union of $w(B+s)$, $w \in W_{Q\to P}$.

If $Q \subset P$ we have an element $w = e \in W_{Q\to P}$. The corresponding exponents $(B+s)$ of $E(\varphi, s)$ we will call *principle exponents*: all other exponents we call *additional exponents*. It is clear that for generic $s \in S$ these sets are disjoint.

Now for every $w \in W_{Q\to P}$ consider ideal $J = J_w \subset \mathcal{O}(S_{P'})$ which consist of functions (polynomial) on $S_{P'}$ which kill $c_{P'P}(\varphi)_{\text{cusp}}$. Clearly the ideal $J+s$ kills $c_{P'P}(u_s)_{\text{cusp}}$, and $w(J+s) \in \mathcal{O}(S_Q)$ kills $R_w c_{P'P}(u_s)_{\text{cusp}}$. Now let us write equations (iii):

Now let us write equations (iii):

(iii)$'$ $\prod_{w\in W_{Q\to P}} w(J+s)c_{QG}(f) = 0$

(iii)$''$ $\prod_{w\in W^{\text{add}}_{Q\to P}} w(J+s)c_{QG}(f) = \prod_{w\in W^{\text{add}}_{Q\to P}} (wJ+s)c_{QP}(u_s).$

Here $W^{\text{add}}_{Q\to P}$ is the set $W_{Q\to P}\backslash\{e\}$.

These are linear equations for a function f (which is a solution of the system $\Xi'(s)$ and hence an automorphic form).

For generic s these equations just mean that cuspidal support of f lies in the union of $w(B+s)$, ad that coefficients of principle exponents are the same as coefficients of u_s.

In order to prove the theorem about meromorphic continuation of $E(\varphi, s)$ we would use the continuations principle. It is based on the following

Proposition. *There exists an open domain $v \subset S_P$ such that the system of equations $\Xi(s) = \Xi'(s)\cup$ (iii) has a unique solution.*

This proposition is based on a key lemma which shows that under some conditions one cuspidal exponent dictates coefficients of another exponent.

Namely, let Q, Q' be two parabolic subgroups. We say that they are adjacent if there exists a parabolic subgroup $\mathcal{R}$ containing Q, Q' whose semisimple rank is $s.s.\text{rank}Q + 1 = s.s.\text{rank}Q' + 1$ and a nontrivial element $w \in W_Q$ s.t. $wQ \approx Q'$ (associated inside R). In other words Q, Q' are maximal parabolic subgroups in $\mathcal{R}$ which are now trivially associated inside $\mathcal{R}$.

6. Key lemma.

Let Q', Q'' be two classes of parabolic subgroups in G. We say that they are adjacent if we can realize Q' and Q'' as subgroups of G such that

(i) $Q' \cap Q''$ contains Levi subgroups of Q' and Q''.

(ii) Consider parabolic subgroup R generated by Q' and Q''. Then Q' and Q'' are maximal (non-trivial) parabolic subgroups in R.

Let us consider the space $L_{Q',Q''} = \{\xi \in \mathfrak{a}_{Q'} | (\xi, \mathfrak{a}^*_R) = 0\}$. This is a one-dimensional space, and it has a canonical nonzero functional $\rho_L := \rho_{Q'} - \rho_R$.

For any weight $\eta \in S_{Q'}$ we can define a number $\lambda(\eta) = (\eta | L_{Q',Q''})/\rho_L$.

Key lemma. *Let f be any automorphic form on $G, (Q', Q'')$ – adjacent parabolic subgroups. Let χ be a weight in $S_{Q'}$ Denote by $\tilde{\chi}$ the same weight considered as an element of $S_{Q''}$ when we identify $S_{Q'}$ and $S_{Q''}$ by restricting characters to the intersection $Q' \cap Q''$.*

Suppose that $Re\lambda(x) \gg 0$. Then the coefficient of cuspidal exponent $f_{Q',\mathrm{cusp},\chi}$ completely determines the coefficient of the cuspidal exponent $f_{Q'',\mathrm{cusp},\tilde{\chi}}$.

Based on the key lemma the proof proceeds as follows.

Choose a vector η inside the Weyl chamber $c_{\mathcal{P}} \subset \mathfrak{a}^*_{\mathcal{P}}$. Fix a very large number $\Lambda \in \mathbb{R}$ and consider an open domain $U \subset S_{\mathcal{P}}$ defined by $\Lambda\eta + B$, where B is the unit ball in $S_{\mathcal{P}}$ We claim that for s in domain U principle exponents of any form f determine all additional exponents.

Indeed, let Q be a parabolic subgroup and $w : Q \sim P' \subset P$ be its association with a subgroup $P' \subset P$.

We can and will assume that w has minimal length ℓ in its double coset $W_Q \backslash W / W_P$.

Then we can construct a sequence of adjacent parabolic groups $Q = Q_0 \sim Q_1 \sim Q_2 \sim \ldots \sim Q_\ell = P'$ in such a way that the element η will be positive with respect to any pair (Q_i, Q_{i+1}).

This implies that for any exponent $\chi \in \Lambda\eta + B + R$ its real part will be $\gg 0$ for any pair (Q_i, Q_{i+1}).

By the key lemma this implies that the χ component of $P' = Q_\ell$ the determines χ-component of $Q_{\ell-1}$, which determines χ-component of $Q_{\ell-2}$ and so on to χ-component of $Q = Q_0$. Thus the principle component χ determined corresponding component for Q. QED

8. Proof of key lemma.

Step 1. We can replace our form f by $f' = C_{RG}(f) \in \mathcal{A}(y_R)$.

Step 2. As usual we can compare K-finite forms on y_R with K-finite automorphic forms on its Levi group L_R. The groups Q', Q'' induce two adjacent parabolic subgroups in L_R.

Thus we can reduce the proof to the case $R = G$.

Step 3. We can replace G by its derived group and assume G is semisimple.

Step 4. Using Langlands decomposition we can assume that he form f has cuspidal components only on one class of associate parabolic subgroups, in our case just Q' and Q''.

Step 5. Decomposing f with respect to the action of $z(G)$ we can assume that the cuspidal component of f on Q' and Q'' appear only at exponents X at Q' and $\tilde{X}$ at Q''.

Step 6. L^2-argument. Suppose that $f_{Q',\mathrm{cusp},X} = 0$. Then the only nontrivial constant term of f is the cuspidal exponent $f_{Q'',\mathrm{cusp},\tilde{X}}$ (since Q', Q'' are maximal subgroups). Since exponent $\tilde{X}$ is negative with respect to Q'' and the cuspidal coefficient $f_{Q'',\mathrm{cusp},X}$ is rapidly decreasing on the derived group of L''_Q we can conclude that this constant term lies in $L^2(Z_T)$, where Z_T is a Siegel domain.

Since f is well approximated by its constant terms it lies in $L^2(y)$.

Now the same argument as for $SL(2)$ shows that if $Re\ x$ is very large then the eigenvalue of the Casimir operator on f should be real and positive, which contradicts the fact that it is a negative operator.