1 Lecture IV

1. Let G be a reductive group over $\mathbb{R}$. We denote by $\mathfrak{a}(G)$ the connected component of the split component of the center $C(G)$ of the group G.

The group $\mathfrak{a}(G)$ is a linear space over $\mathbb{R}$ (we identify it with its Lie algebra).

We can define this group in a different, more geometric way. Namely consider the group $\mathfrak{a}^*(G) = Hom(G, \mathbb{R}^{+*})$. This again is a commutative group with the structure of a linear space over $\mathbb{R}$, and $\mathfrak{a}(G)$ is just the dual space.

2. Now we would like to construct from group G, a linear $\mathbb{R}$ space $\mathfrak{a} = \mathfrak{a}_G$ and a vast system $\phi(G) \subset \mathfrak{a}$ which allows to analyze the structure of parabolic subgroups in G.

The idea is that we choose a minimal parabolic subgroup $P_0 \subset G$ and set $\mathfrak{a} = \mathfrak{a}(P_0)$. The important thing is that this construction is absolutely canonical. This follows from the following general observation.

Let P be a conjugacy class of parabolic subgroups in G. We denote by X_P the set of these subgroups: we can consider it as a topological space or as an algebraic variety over $\mathbb{R}$.

The group G-acts on the space X_P.

Proposition. (i) *The action of G on X_P is transitive*

(ii) *For any point $P \in X_P$ the stabilizer $Stap(P; G)$ of this point in G equals P. This is a standard result about parabolic subgroups. It implies that we can identify X_P with G/P. In fact this is the way how usually the flag variety X_P is defined. However, it reflects a very wrong point of view since the notation G/P creates an impression that the construction depends on the choice of the parabolic subgroup P, which we have seen that X_P is described purely in terms of the group G.*

Similarly, suppose we have some functorial construction $P \mapsto F(P)$ (e.g. $F(P) = U_p$ or $F(P) = L_p = P/U_p$ or $F(P) = \mathfrak{a}(L_p)$). Then it gives us a

G-equivariant fibration F over the flag variety X_P. Especially interesting is the case when this fibration is constant; this happens if the adjoint action of P on P defines trivial action of P on $F(P)$. In this case $F(P)$ is canonically the same for all P in X_P and this gives us a canonical object $F(P)$.

Examples. The functor $F(P) = L_p$ is not invariant with respect to conjugation (if L_p is not commutative), so we can not define the object $F(P)$.

However, if we take functor $F(P) = \mathfrak{a}(L_p)$ then P acts trivially on $F(P)$ and hence we have canonically defined vector space $\mathfrak{a}(P) = \mathfrak{a}(L_p)$.

Similarly, we can define a canonical algebra $z(P); = z(L_P)$.

3. Definition. Starting with the group G we define (canonically) the Cartan space $\mathfrak{a}$ of G to be an $\mathbb{R}$-vector space $\mathfrak{a}(P_0)$ where P_0 is the conjugacy class of minimal parabolic subgroup.

Let us list structures which we have on the Cartan space $\mathfrak{a}$.

(i) We have a root system ϕ which comes from identification $\mathfrak{a} = \mathfrak{a}(L_{p_0})$ for any minimal parabolic subgroup $P_0 \subset G$.

(ii) We have a distinguished a system of positive roots $\phi^+ \subset \phi$ which correspond to roots in U_{P_0}. In particular, we have the system of simple roots $\Delta \subset \phi$.

(iii) To any conjugacy class of parabolic subgroups P corresponds a canonical subspace $\mathfrak{a}(P) \subset \mathfrak{a}$ (for example $\mathfrak{a}(P_0) = \mathfrak{a}$).

The correspondence $P \mapsto \Delta \cap \mathfrak{a}(P)$ is a bijection between classes P and subsets of Δ.

4. Fix a class of parabolic subgroups P. Let us define the canonical G-fibrations over X_P. $\mathcal{G}_P = X_P \times G$ with diagonal action of G,

$$g(x,h) = (gx, gh)$$

Aff_p is a quotient of $\mathcal{G}_P$ with respect to a bundle of subgroup U_p over X_P. In other words fibers of Aff_p over a point $P \in X_P$ is $U_p \backslash G$; action of G from the left is the same as for $\mathcal{G}_P$. Note that both these fibrations have an action of G on the right which preserves fibers.

5. Let us go back to automorphic forms.

First consider the case $G = SL(2,\mathbb{R})$ and Γ a lattice in G (for example a congruence subgroup of $SL(2,\mathbb{Z})$).

As before we would like to study the automorphic representation $(\Pi, G, F(y))$, where $y = \Gamma\backslash G$. We assume that y is not compact.

As before we want to introduce spaces y_P and $Z_{\mathcal{P}}$ which describe the behaviour of the space y at ∞ (here $\mathcal{P}$ is the class of minimal parabolic subgroups in $G = SL(2,\mathbb{R})$).

Definition. A parabolic subgroup $P \in X_P$ is called *cuspidal* (with respect to Γ) if $\Gamma_{U_p}\backslash U_p$ is compact. (Here as before: $\Gamma_{U_p} = \Gamma \cap U_p$). We let X_P, cusp denote the subset of cuspidal points in X_P.

Proposition. The group Γ acts on the set $X_{P,cusp}$ with finite number of orbits.

This is a part of reduction theory. Now define the space $y'_{\mathcal{P}}$ is the preimage of $X_{P,cusp}$ in Aff_p; in other words $y'_P = \amalg_{P-cusp} U_p\backslash G$. The group Γ acts on y'_P from the left and we set $y_P = \Gamma\backslash y_{P'}$.

We can write a more explicit formula for y_P. Let $P_1, \ldots, P_r$ be representatives of Γ-orbits in $X_{P,cusp}$. Then it is clear that y_P is a disjoint union $\amalg_{i=1,\ldots,r} \Gamma_{P_i} \cdot U_{P_i}\backslash G$.

In particular, if there is only one orbit, then $y_{\mathcal{P}} = \Gamma_p U_p\backslash G$. Similarly we define the space Z'_P as preimage of $X_{P,cusp}$ in the fibration $\mathfrak{g}_{\mathcal{P}}$ and set $Z = \Gamma\backslash Z'_{\mathcal{P}}$. Clearly $Z = \amalg \Gamma_{P_i}\backslash G$.

As before we have canonical action of the group G on $F(y_P)$ and $F(Z)$, we have canonical projections $Z \to y_P$, $Z \to y$ and we have a canonical (twisted) action of $z(L_P) = S(\mathfrak{a})$ on $F(y_{\mathcal{P}})$ such that $z(G) \subset z(L_{\mathcal{P}})$.

6. In order to define Siegel domains $y_{P,T} \subset y_P$ and $Z_T C Z$ we have to introduce the function $a : y_P \to \mathfrak{a}$. We want this function to satisfy

(i) $a(yk) = a(y)$, $k \in k$ – maximal compact.

(ii) $a(\xi y) = \xi + a(y)$, $\xi \in \mathfrak{a}$. Here we consider the geometric left action of the group $\mathfrak{a} = \mathfrak{a}(L_p)$ on the space $y_{\mathcal{P}}$.

These two conditions define the map a on every component of the space $y_{\mathcal{P}}$ uniquely up to translation.

Since we do not know how to compare maps a on different components we simply choose some map $a : y_P \to \mathfrak{a}$ which satisfies (i), (ii). Now for every $T \in \mathfrak{a}$ we define

$$y_{P,T} = a^{-1}(T + C_{\mathfrak{a}})$$
$$Z_T = a^{-1}(T + C_{\mathfrak{a}}$$

where $C_{\mathfrak{a}}$ is the Weyl chamber in $\mathfrak{a}$ (in case of $SL(2)$ $\mathfrak{a} \approx \mathbb{R}$, $C_{\mathfrak{a}} = \mathbb{R}^+$).

The reduction theory tells us that $pr : Z_T \to y$ is onto when $T \ll 0$ and has fibers with bounded number of elements. After this the proof works exactly in the same way as for $\Gamma = SL(2, \mathbb{R})$. The only difference is that now y_P and Z_T have several (finite number) components, which do not affect the proof.

7. Now consider a reductive group $\underline{\mathfrak{g}}$ over $\mathbb{R}$ and set $G = \underline{\mathfrak{g}}(\mathbb{R})$, $\Gamma \subset \underline{\mathfrak{g}}(Q)$-arithmetic subgroup. In case when $\underline{\mathfrak{g}}$ is not split over $\mathbb{Q}$ the geometry of the space $\mathfrak{a}$ and root system ϕ should be different from what we described. The easy translation to the new case is given as follows:

(i) We consider only algebraic groups over $\mathbb{Q}$. For a reductive group G we define $\mathfrak{a}_\Gamma(G)$ to be a connected component of the $\mathbb{Q}$-split torus in the center $C(G)$. As before we can describe the space $\mathfrak{a}_\Gamma(G)^*$ directly: it consists of characters $X : G \to \mathbb{R}^{+*}$ which are trivial on the subgroup Γ.

In this case cuspidal parabolic subgroups are just $\mathbb{Q}$-rational parabolics. We can define the space $\mathfrak{a} = \mathfrak{a}_\Gamma(P_0)$ where P_0 is a minimal $\mathbb{Q}$-rational parabolic subgroup.

As before, we construct a root system ϕ in or (this is just a root system of $\underline{\mathfrak{g}}$), subsystems ϕ^+ and Δ in $\phi, \ldots, G(\mathbb{Q})$ conjugacy class $\mathcal{P}$ of rational parabolic subgroups correspond to a subset of Δ.

For every $\mathcal{P}$ we construct G-spaces $y_{\mathcal{P}}$ and $Z_{\mathcal{P}}$ as before. Of spaces $Z_{\mathcal{P}}$ we are mostly interested in the space $Z = Z_{P_0}$.

For every space y_P we have canonical representation $(\Pi_P, G, F(y_{\mathcal{P}}))$. Note that in this case, after we fixed an L^2-structure on the space $F(y)$, all spaces $F(y_P)$ inherit canonical L^2-structures; indeed every unipotent subgroup U_p has canonical measure μ normalized by conditions $\mu(\Gamma_{U_{\mathcal{P}}} \backslash U_p) = 1$.

We have also a canonical action of the algebra $z(L_p)$ on the space $F(y_P)$ (as before this is a twisted action). We have $z(L_{\mathcal{P}}) > z(G)$ and $z(L_{\mathcal{P}})$ is a finitely generated $z(\in)$-module.

8. Spaces $F(y_P)$ for different classes P are connected by operators

$$E_{QP} : F(y_P) \longrightarrow F(y_Q) \ \text{ for } P < Q$$
$$C_{PQ} : F(y_Q) \longrightarrow F(y_P) \ \text{ for } P < Q \ .$$

These operators are G-equivariant, E_{QP} is formally adjoint to C_{RQ}.

Let us describe the operator E_{QP}. In order to do this consider a correspondence $X_{P<Q}$ between X_P and X_Q which consists of all pairs $P < Q$ (we consider only cuspidal parabolic subgroups). This defines a correspondence $y'_{P<Q}$ between y'_P and $y'_Q : y'_{\mathcal{P}} \xleftarrow{\alpha'} y'_{P<Q} \xrightarrow{\beta'} y'_Q$ ($y'_{P\subset G} = \amalg_{(P,Q)\in X_{P\subset Q}} U_Q \backslash G$). Since all constructions are Γ-invariant we get a correspondence $y_P \xleftarrow{\alpha} y_{P\subset Q} \xrightarrow{\beta} y_Q$ in which fibers $\beta^{-1}(Q?)$ are $\{P \subset Q\}$ and fibers of α are $\Gamma_{U_p} \cdot U_Q \backslash U_{\mathcal{P}}$. Now we define $E_{QP}(\varphi) = \alpha_* \beta^*(\varphi)$. This gives us a summation over all $Q \supset P$ which in case of one cusp is the summation over Γ_Q / Γ_P which we had before.

Operator C_{PQ} is defined by integration over $\Gamma_{U_p} \cdot U_Q \backslash U_p$, $C_{PQ}(f) = \beta_* \alpha^*(f)$. We call E_{QP} *Eisenstein operator* and C_{PQ}-constant term operator.

It is easy to check the following

Properties. (i) Let $R > Q > P$. Then

$$E_{RP} = E_{RQ} \circ E_{QP}$$

$$C_{PR} = C_{RQ} \cdot C_{QP}$$

(ii) Operators E_{QP} and C_{PQ} are G-equivariant. They also commute with the actions of the algebra $z(\mathbb{Q})$.

9. Important roll in the proof plays the following geometric lemma which describes the composition $C_{QG} \circ E_{GP}$.

The proof and the setup of this lemma are exactly the same as in the paper [BZ] where a similar fact is proven for functions, not operators. Let us briefly formulate the structure of the lemma. In more detail it is explained in the appendix.

First of all denote by W the Weyl group of the root system ϕ. For us important is that elements $w \in W$ parametrize $\underline{\mathfrak{g}}(\mathbb{Q})$ orbits in the space $X_{P_0,cusp} \times X_{P_0,cusp}$.

Every cuspidal parabolic class P defines a subgroup $W_P \in W$.

Geometric lemma. Let P, Q be cuspidal parabolic classes in G.

(i) Then operator $C_{QG} \circ E_{GP} = \sum_w T_w$, where the sum is over $w \in W_Q \backslash W / W_P$ and every operator T_w has the following form.

(ii) For a given w there exist parabolic cuspidal classes $Q_w \subset Q$ and $P_w \subset P$ an association $Q_w \overset{w}{\rightsquigarrow} P_w$ of parabolic subgroups such that the corresponding intertwining operator $R_w : F(y_{Prw}) \to F(y_{Q_w})$ satisfies

$$T_w = E_{QQ_w} \circ R_w \circ C_{P_w,P} \ .$$

(iii) We will also use the fact that the intertwining operator R_w we can decompose as a product of elementary intertwining operators

$$R_w = R_{\sigma_1} \dots R_{\sigma_e} \ .$$

(iv) For every $z \in z(L_{Q_w})$ we have $z \circ R = w = R_w \circ w(2)$, where $w(Z)$ is the corresponding element of $z(L_{P_w})$.