1 Lecture III

$$G = SL(2,\mathbb{R}) \ , \quad \Gamma = SL(2,\mathbb{Z}) \ , \quad y = \Gamma\backslash G \ .$$

1. Our goal is to construct some automorphic forms on y. Let us start with an automorphic form φ on y_p and consider a family of automorphic forms on y_p $u_s = \varphi \cdot y^s$. It is easy to see that for $Res \gg 0$ the Eisenstein operator $E(u_s)$ is given by an absolutely convergent series. Hence it defines an automorphic form on y, which we denote $E(\varphi; s)$. Clearly that this defines a function $s \to E(\varphi, s)$ from $\mathbb{C}$ to $F^{temp}(y)$ in the area $Res \gg 1$ and this function is holomorphic (see a more detailed discussion of this notion in the appendix).

Main theorem. The function $E(\varphi, s)$ has a meromorphic continuation to the whole plane $\mathbb{C}$ of parameter s. For every s this gives us an automorphic form $E(\varphi, s)$. These forms are called Eisenstein series.

2. First we prove this theorem in the simplest case $\varphi = y^{1/2}$ (this is the unique up to scalar k-invariant function such that $\delta\varphi = 0$; here δ is a generator of $z(L)$ and $\delta(y^{1/2+s}) = 2sy^{1/2+s}$. What we can tell about the Eisenstein series $E(s) := E(\varphi, s)$ when $Res \gg 0$.

(1) $E(s)$ is a well defined element of $F^{temp}(y)$.

(2)) (i) $E(s)$ is k-invariant.

(ii) $\delta^2 E(s) = 4s^2 E(s)$, where δ^2 we consider as an element of $z(G)$.

Let us note that if we have any function f on y which satisfies equations 2(i),(ii), then the constant term cf satisfies the same equation and hence

$$cf = y^{1/2}(\alpha(s)y^s + \beta(s) \cdot y^{-s})$$

and this is true for any s (for $s = 0$ we have to modify this a little bit). We will see that the function $E(s)$ for $Res \gg 0$ satisfies an additional condition

$$(3) \qquad \alpha(s) = 1 \ , \ i.e. \ cE(s) = y^{1/2}(y^s + \beta(s)y^{-s})$$

we will soon show the following

Proposition. *Consider the following system of equations for a function* $f \in F^{tem}(y)\Xi(s)$*:*

(i) $\delta^2 f = 4s^2 f$

(ii) f *is* $K-$ *invariant*

(iii) $cf = (y^s + ?y^{-s})y^{1/2}$.

Then in the domain $U = \{s \in \mathbb{C} \mid Res > 2\}$ this system has a unique solution.

3. Now I would like to invoke the following very general

Continuation principle. Let V be a topological vector space, S – a connected complex manifold.

Suppose that for every $s \in S$ we are given a system of linear (non-homogeneous) equations $\Xi(s)$ for a vector $v \in V$. We assume that this system of equations is holomorphic in s (see precise definition in the appendix).

Suppose we know that for s in some open non-empty domain $U \subset S$ the system $\Xi(s)$ has a *unique* solution $v(s) \in V$. Then this system has a unique solution $v(s)$ for almost all s and the function $s \mapsto v(s)$ extends to a meromorphic function on S with values in V.

This principle seems to be very general. I do not claim that it always holds, but it is applicable in a very broad range of situations.

Clearly that together with the proposition above this principle implies the derived meromorphic continuation of the Eisenstein series $E(s)$.

4. Let us discuss how to make a theorem from the continuation principle.

Definition. Let $\Xi(s)$ be a system of linear equations for a vector in a vector space V which holomorphically depends on point s of a complex manifold S. For every $s \in S$ we denote by $Sol(\Xi(s))$ the subset of vectors $v \in V$ which satisfy the system of equations $\Xi(s)$.

Definition. (i) We say that the system $\Xi(s)$ is *of finite type* if there exists a *finite dimensional* vector space L and a holomorphic family of morphisms $\lambda(s) : L \to V$ such that for every $s \in S$ $\lambda(s)(L) \supset Sol(\Xi(s))$.

(ii) We say that the system $\Xi(s)$ is of *locally finite type* if every point $s \in S$ has a neighborhood $W \exists s$ such that $\Xi(s)$ is of finite type over this neighborhood.

Theorem. *Suppose the system $(\Xi(s), V)$ is of locally finite type. Then the continuation principle holds for $\Xi(s)$.*

Proof. Step 1. Reduction to systems of finite type. Suppose we have already shown that the continuation principle holds for systems of finite type in a strong form where "for almost all s" means "for s in an open dense subset".

Let $S_0 \subset S$ be the set of points t in S such that the system $\Xi(s)$ has a unique solution in some neighborhood of t. Then for any point $s \in S$ there is a neighborhood W such that $\Xi(s)$ is of finite type on W, and hence by assumption above $S_0 \cap W$ is either empty or dense in W.

But this immediately implies that the closure $\overline{S}_0$ is open in S. Since S is connected and S_0 is not empty this implies that $\overline{S}_0 = S$, i.e. S_0 is dense in S.

Remark. In fact we will show that $S \backslash S_0$ is a proper analytic subset of S.

Step 2. Suppose that $\Xi(S)$ is a system of finite type. Fix a finite dimensional space L and a holomorphic family of morphisms $\lambda(s) : L \to U$ such that for all s $\lambda(s)L \supset Sol\Xi(s)$. Let us show that it is enough to establish continuation principle for the case when $\lambda(S) : L \to V$ is monomorphic for all s. Indeed, let us above a point $s_0 \in s$ such $k(s) = \dim(\lambda(s)(L))$ is maximal at point s_0. Choose a subspace $L_0 \subset L$ of dimension $k = k(s_0)$ such that $\lambda(S)L_0 = \lambda(S)L$. Then it is easy to see that for some open dense subset $S_0 \subset S$ $\lambda(S)L_0 = \lambda(S)L$ and $\lambda(S)$ is a monomorphism on L_0. Replacing L by L_0 and S by S_0 we can assume that $\lambda(S) : L \to V$ is mono for all s.

Step 3. Since for all $S \in S$ $\lambda(s) : L \to U$ is mono and $Sol\Xi(s) \subset L$ we can replace our space V by L and the system of equations Ξ by an induced system $\Xi'(s)$ on L. Hence we can assume that $L = V$ is finite dimensional.

Choose coordinates $x_1, \ldots, x_k$ on L. Then our system of equations can be written as some infinite system

$$\sum_i a_{i\alpha}(s)x_i = b_\alpha(s) \ ,$$

Pick a point $s \in U$. Since at this point our system has a unique solution, we can find indices $\alpha_1, \ldots, \alpha_k$ such that the determinant of $k \times k$ matrix $(a_i, \alpha_i(s))$ is not 0. But then this determinant is not 0 for some open dense subset $S_0 \subset S$.

Using Cramer's rule we can solve our $k \times k$ subsystem of equations $\widetilde{\Xi}(s)$ and get a meromorphic vector function $s \mapsto v(s)$ which is holomorphic for $s \in S_0$.

Note that for $s \in S_0 \cap U$ our function is the only solution of the system $\widetilde{\Xi}$ and hence it coincides with the only solution of the larger system of equations Ξ. Thus our function $v(s)$ is a meromorphic continuation of the solution of the system Ξ on U. This implies that it is a solution of Ξ everywhere on S_0.

This shows that on the dense subset $S_0 \subset S$ our function $v(s)$ represents *unique* solution for $\Xi(s)$. Moreover, we know that $v(s)$ extends to a meromorphic function on the whole set S. Q.E.D.

5. How to check that some system of equations $\Xi(s)$ in a vector space V is of locally finite type. For example, how to see that the equations (i), (ii), (iii) in Proposition 2 above for Eisenstein functions $E(s)$ are of locally finite type.

First we can forget equation (iii). We will show that the system (i), (ii) of linear homogeneous equations is already of locally finite type.

We will present the detailed proof later. But here I would like to make the following metamathematical remark.

Suppose we have a system of linear homogeneous equations $\Xi(s)$ in a

space V. If it is of locally finite type then we have
$(*)$ for every $s \in S$ the subspace $Sol\Xi(s) \subset V$ is finite dimensional.

So this is a necessary condition. Now suppose we have a *proof* that for a given holomorphic system of linear equations $\Xi(s)$ the condition $(*)$ holds. Then I claim that always we can tighten up this proof so that it will show that the system $\Xi(s)$ is of locally finite type. It will require somewhat more refined analysis, but it will be essentially the same proof.

In the case of the Eisenstein series we have already shown that for every $s \in S$ the solutions of the system of equations (i), (ii) in Proposition (2) from a finite dimensional subspace of automorphic forms $\mathcal{A}_{\sigma_0}^{I(s)}$. So I would like to emphasize that the proof of the fact that the system $\Xi(s)$ is of locally finite type is a not very interesting technical extension of things we have already done.

6. As we have seen the proof of meromorphic continuation of the function $E(s)$ (and the same will be true in a general case) consists of 2 rather independent parts:

(I) Show that the system $\Xi(s)$ is of locally finite type.

(II) Show that in some non-empty domain $U \subset S$ the system $\Xi(s)$ has a unique solution.

We will later describe how to modify the proof of finite dimensionality of spaces of automorphic forms to prove (I). Now I prefer to discuss more interesting uniqueness Part (II).

First of all notice that the equation (iii) in Proposition 2 is not really a linear equation. But we can safely replace it by a linear equation

$$(iii)' \qquad (\delta + 2s)cv = (\delta + 2s)y^{s+1/2}$$

since clearly for $s \neq 0$ this equation has exactly the same solutions (in the class of k-invariant functions).

Now how to prove that for $Res > 5$ our system has a unique solution. One solution is given by the Eisenstein series which converges in this region. Suppose we have two solutions f_1, f_2 and denote by $\widetilde{f}$ their difference, $\widetilde{f} = f_1 - f_2$. Then the function $\widetilde{f}$ is an automorphic form which satisfies

$$\begin{aligned} Df &= 4s^2 f \text{ for } D = \delta^2 \\ cf &= \beta \cdot y^{-s} \end{aligned}$$

We have shown before that any automorphic form is closely approximated by its constant term. This implies that the function $\widetilde{f}$ lies in $L^2(X, \mu)$.

Now this function is an eigenfunction of the operator $D = \delta^2$ which is Casimir operator $+1$. We know that the Casimir operator on $L^2(x, \mu)$ is selfadjoint and negative. Hence any L^2-eigenfunction of the operator D showed have eigenvalue which is real and ≤ 1. This shows that in the domain $Res > 1/2$ the system of equations $\Xi(s)$ has unique solution.

Remark. Technically it is a little difficult to analyse the unbounded operator Casimir and show that it is selfadjoint and negative. But as usual we can replace this consideration with the consideration of some bounded condition operators $\Pi(\alpha)$ and get the same result. We will do this in an appendix.

7. In order to finish the proof of meromorphic continuation of the function $E(s)$ we need to prove that in the area of convergence functions $E(s)$ satisfy equation (3) from Section 2. Namely, that

$$CE(s) = y^{1/2}(y^s + \beta(s)y^{-s}) \text{ for some } \beta(s)\ .$$

This follows from the following important proposition which gives some information about the composition of the operators C and E.

Proposition. *In the domain of convergence we can write $C \circ E = Id + I$, where I is some operator $I : F(y_s) \to F(y_s)$ which satisfies*

$$(*) \qquad I\delta = -\delta I \text{ and } I \text{ is } G - \text{invariant}$$

Clearly this proposition implies the formula above.

Proof of Proposition. $CE(\varphi) = \int_{u\in U/\Gamma_U} u(\sum_\gamma \ \gamma(\varphi)du$, where the sum is take over Γ/Γ_p.

Let us recall that we have a Bruhat decomposition for the group G, namely $G = G_e \cup G_\omega$, where $G_e = P$, $G_\omega = U_p\omega p$ with $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ – nontrivial element of the Weyl group. Let us break our sum in the expression for CE into two terms $I_e + I_w$ corresponding to decomposition $\Gamma = \Gamma_e \cup \Gamma_\omega$ of elements $\gamma \in \Gamma$. It is clear that $I_e(\varphi) \equiv \varphi$.

In order to investigate the operator I_ω let us choose a system of representatives R of left cosets $\Gamma_{U_p}\backslash\Gamma_\omega/\Gamma_p$. Then clearly we can write $I_\omega = \sum_{v\in R} I_v$, where $I_v(\varphi) = \int_{U\in U_p} Ur(\varphi)du$.

Since the function φ is left U_p-invariant and $r \in U_\omega P$ it is easy to see that $I_v(\delta\varphi) \equiv -\delta I_v(\varphi)$. Namely, we claim that it is true for arbitrary $v \in U_\omega P$, not necessarily the element of Γ. In order to prove this, we can multiply the element v by elements of U on the left and on the right. Then we can reduce the claim to the situation when $v \in \omega L_p$. In this case the formula follows from the fact that $I_r\Pi(\ell) = \Pi(\ell^{-1})I_r$ for any element ℓ of the center $C(L_p)$, where Π is the action of L_p on the space Aff_p. This formula is a simple computation.