Second adjointness for representations of reductive p-adic groups

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§0. Introduction

0.1. In this paper, which was written in 1987, I continue the investigation of induced representations of reductive p-adic groups, started in [BZ]. The main tools of the investigation are induction functors i_{GM} and Jacquet functors r_{MG}. More precisely, let G be a reductive p-adic group and Alg G the category of algebraic (in other terminology, smooth) representations of G. For any parabolic subgroup $P < G$ with Levi component M we define the induction functor i_{GM} : Alg $M \to$ Alg G and Jacquet functor r_{MG} : Alg $G \to$ Alg M as in [BZ].

Frobenius reciprocity implies that functor r_{MG} is left adjoint to i_{GM}. Recently, I have discovered to my great surprise, that these functors are also adjoint in the opposite direction. More precisely, let $\overline{P}$ be the parabolic subgroup opposite to P with Levi component M. Then we can define functors $\overline{i}_{G,M}$: Alg $M \to$ Alg G and $\overline{r}_{MG}$: Alg $G \to$ Alg M in the same way as i and r, but using $\overline{P}$ instead of P.

Main theorem. *Functor i_{GM} is left adjoint to $\overline{r}_{MG}$, and $\overline{i}_{GM}$ is left adjoint to r_{MG}.*

This innocent-looking statement is in fact very powerful. For instance, it implicitly contains the strong admissibility theorem (indeed, it implies that functors r_{MG} commute with direct product and hence products of quasicuspidal representations are quasicuspidal. But this means that for a given open subgroup $K \subset G$ there exists a uniform bound on supports of all K-invariant matrix coefficients of all cuspidal representations of G, i.e. all these supports lie in some subset $S \subset G$, compact modulo center).

The aim of this paper is to prove the main theorem and to show how it implies many important results about induced representation: description of the center of category Alg G, matrix Paley-Wiener theorem, cohomological duality in Alg G.

More precise versions of the theorem are formulated in §. They allow to prove Zelevinsky's conjecture, that duality, which he defined on the Grothendieck group of representations of $GL(n)$, actually carries irreducible representations into irreducible ones (see [Z]). I should add, that this way of proving Zelevinsky's conjecture was suggested to me by V. Drinfeld many years ago. He explained to me that for the group $G = SL(2)$, Ext^1 (trivial representation) = Steinberg representation.

0.2. Contragredient properties of functor r_{HG}.

Another, essentially equivalent, form of the main theorem describes how to compute contragredient representations of $r_{MG}(\pi)$. For induction functor we have the Frobenius reciprocity $(i_{GM}(\rho))\tilde{} = i_{GM}(\tilde{\rho})$, where $\tilde{}$ denotes the contragredient representation (see [B2]).

Theorem. *There is a functorial isomorphism*

$$(r_{MG}(\pi))\tilde{} \approx \overline{r}_{MG}(\tilde{\pi}), \quad \pi \in \mathrm{Alg}\ G\ .$$

0.3. Matrix Paley-Wiener theorem.

Let ρ be an irreducible cuspidal representation of M. Consider the family of induced representations $\pi_\chi = i_{GM}(\chi \cdot \rho)$, parametrized by unramified characters χ of M, with underlying family of vector spaces E_χ.

Let $H = H(G)$ be the algebra of compactly supported locally constant measures on G. Any element $h \in H(G)$ induces the family of operators $h_\chi = \pi_\chi(h) : E_\chi \to E_\chi$.

This family has the following properties:

(PW1) h_χ is a regular function of parameter χ (unramified characters of M form a group isomorphic to $(\mathbb{C}^*)^\ell$ and function h_χ is algebraic on $(\mathbb{C}^*)^\ell$).

(PW2) There exists an open subgroup $K \subset G$ such that operators h_χ are left and right invariant with respect to $\pi(K)$.

(PW3) For any intertwaining operator $A : E_\chi \to E_{\chi'}$ one has $h_{\chi'} \circ A = A \circ h_\chi$.

Theorem. *Let $a_\chi : E_\chi \to E_\chi$ be a family of operators, satisfying* (PW1)-(PW3). *Then $a_\chi = h_\chi$ for some $h \in H(G)$.*

Remark. It is clear, that it is sufficient to check property (PW3) only on Zariski dense subsets of parameters χ and χ'.

This theorem follows easily from the following corollary of the main theorem: functor i_{GM} carries projective generators into projective generators.

0.4. Cohomological duality theorem.

Let us denote by γ_L and γ_R left and right actions of G on $H(G)$. For any $\pi \in \mathrm{Alg}\ G$, we can consider spaces $\mathrm{Ext}^i(\pi) = \mathrm{Ext}^i_{A(G)}(\pi, (\gamma_L, H(G)))$ as G-modules, using right action γ_R.

Theorem. *If π is irreducible then for exactly one index i $\mathrm{Ext}^i(\pi) \neq 0$. Moreover, representation $\mathrm{Ext}^i(\pi)$ is irreducible and $\pi \mapsto \mathrm{Ext}^i(\pi)$ defines a duality on the set of equivalence classes of irreducible algebraic representations of G.*

§1. Generalities from Algebra and Category Theory

1.1. Idempotented Algebras and Nondegenerate Modules.

We consider a class of rings slightly more general than rings with identity.

Definition. An associative ring $\mathcal{H}$ is called an *idempotented ring* if for each finite subset $\{x_i\} \in \mathcal{H}$ there exists an idempotent $e \in \mathcal{H}$ such that $ex_i = x_i = x_i e$ for all i.

Example. Each ring with identity is an idempotented ring. More generally, let $\mathcal{H}_\alpha$, $\alpha \in I$, be a direct system of rings and $\mathcal{H} = \varinjlim_{\alpha \in I} \mathcal{H}_\alpha$. Suppose that the ordered set I is filtered (i.e. for each $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha < \gamma, \beta < \gamma$) and all $\mathcal{H}_\alpha$ are rings with identities (but ring homomorphisms $\mathcal{H}_\alpha \longrightarrow \mathcal{H}_\beta$ for $\alpha < \beta$ are not supposed to map identities into identities). Then $\mathcal{H}$ is an idempotented ring.

In fact, any idempotented ring can be presented in such a way. Namely, consider the set $I = \mathrm{Idem}\mathcal{H}$ of idempotents in $\mathcal{H}$ with partial order $e \le f$ if $e\mathcal{H}e \subset f\mathcal{H}f$. Then $\mathcal{H} = \varinjlim_{e \in I} e\mathcal{H}e$, where $e\mathcal{H}e$ is the ring with identity e.

Usually we consider $\mathcal{H}$ to be an algebra over some field k and call $\mathcal{H}$ an idempotented algebra.

A (left) module M over an idempotented ring $\mathcal{H}$ is called *nondegenerate* if $\mathcal{H}M = M$ or equivalently $\varinjlim_{e \in \mathrm{Idem}\mathcal{H}} eM = M$. If $\mathcal{H}$ is a ring with identity, this is just the usual condition that 1 acts on M as identity.

The category of nondegenerate $\mathcal{H}$-modules we denote $\mathcal{M}(\mathcal{H})$. Each $\mathcal{H}$-module M contains the maximal nondegenerate submodule $\mathcal{H}M$, which we call the *nondegenerate part* of M. It is easy to see that $\mathcal{M}(\mathcal{H})$ is an abelian category with direct limits and filtered direct limits in $\mathcal{M}(\mathcal{H})$ are exact. Category $\mathcal{M}(\mathcal{H})$ also has arbitrary direct products (and, hence, inverse limits). Namely, for a family $\{M_\alpha \in \mathcal{M}(\mathcal{H})\}$ the product $\prod_\alpha M_\alpha$ in $\mathcal{M}(\mathcal{H})$ is equal to the nondegenerate part of the set theoretic direct product,

$$\prod_\alpha M_\alpha = \mathcal{H}\left(\prod_\alpha^{set} M_\alpha\right) = \varinjlim_{e \in \mathrm{Idem}\mathcal{H}} \left(\prod_\alpha (eM_\alpha)\right) .$$

1.2. Projective and Injective $\mathcal{H}$-Modules.

For each idempotent $e \in \mathcal{H}$ the functor $M \to eM$ is exact on $\mathcal{M}(\mathcal{H})$. Since $eM = \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}e, M)$, it shows, that $\mathcal{H}e$ is a finitely generated projective object in $\mathcal{M}(\mathcal{H})$. The family of modules $\mathcal{H}e$ for $e \in \mathrm{Idem}\mathcal{H}$ form a system of projective generators for category $\mathcal{M}(\mathcal{H})$. In particular, $\mathcal{M}(\mathcal{H})$ has enough projective objects, i.e. each module $M \in \mathcal{M}(\mathcal{H})$ is a quotient of a projective one.

Similarly, one can see that $\mathcal{M}(\mathcal{H})$ has enough injective objects. Namely for each $e \in \mathrm{Idem}\mathcal{H}$ and each injective $\mathbb{Z}$-module U denote by $I(e, U)$ the nondegenerate part of $\mathcal{H}$-module $\mathrm{Hom}_{\mathbb{Z}}(e\mathcal{H}, U)$. Then the functor $M \to \mathrm{Hom}_{\mathcal{H}}(M, I(e, U)) = \mathrm{Hom}_{\mathbb{Z}}(eM, U)$ is exact on $\mathcal{M}(\mathcal{H})$, i.e. $I(e, U)$ is an injective object, and $\{I(e, U)\}$ form a system of injective cogenerators in $\mathcal{M}(\mathcal{H})$.

We will denote by $\mathcal{M}^R(\mathcal{H})$ the category of nondegenerate right $\mathcal{H}$-modules, which we identify with category $\mathcal{M}(\mathcal{H}^\circ)$, where $\mathcal{H}^\circ$ is the opposite algebra. We define in a usual way the tensor product $M' \bigotimes_{\mathcal{H}} M$ of nondegenerate right and left $\mathcal{H}$-modules. It is easy to see that all the usual properties of $\bigotimes_{\mathcal{H}}$ hold in this case; we will use them freely. Note, that formula $M' \bigotimes_{\mathcal{H}} (\mathcal{H}e) = M'e$ shows, that $\mathcal{H}e$ is a flat $\mathcal{H}$-module, which implies that all projective $\mathcal{H}$-modules are flat.

Let $\mathcal{H}$ be an idempotented algebra over a field k. For each $\mathcal{H}$-module $M \in \mathcal{M}(\mathcal{H})$ we define the contragredient module $\widetilde{M} \in \mathcal{M}^R(\mathcal{H})$ as a nondegenerate part of the dual space $M^* = \mathrm{Hom}_k(M,k)$, i.e. $\widetilde{M} = \varinjlim_{e \in \mathrm{Idem}\,\mathcal{H}} (M^e)^*$. Similarly we define the functor $\sim$: $\mathcal{M}^R(\mathcal{H}) \to \mathcal{M}(\mathcal{H})$. It is easy to check that $\sim$ is an exact contravariant functor, with duality property $\mathrm{Hom}_{\mathcal{H}}(\widetilde{M}, N) = \mathrm{Hom}_{\mathcal{H}}(\widetilde{N}, M), M \in \mathcal{M}(\mathcal{H}), N \in \mathcal{M}^R(\mathcal{H})$. In particular, $\sim$ maps projective objects into injective ones.

1.3. Hecke Algebras.

Let G be an ℓ-group, i.e. a Hausdorf topological group, which has a basis of neighbourhoods of $e \in G$, consisting of open compact subgroups (see [BZ1]). Let $\mathcal{H} = \mathcal{H}(G)$ be the Hecke algebra of locally constant distributions (or complex valued measures) on G with compact support. Then $\mathcal{H}$ is an idempotented algebra (over $\mathbb{C}$) and category $\mathcal{M}(\mathcal{H}(G))$ is naturally identified with category $\mathcal{M}(G)$ of G-modules (see..., [BZ1]).

Let $K \subset G$ be an open compact subgroup, $e_K \subset \mathcal{H}(G)$ be the normalized Haar measure on K. Then $e_K \mathcal{H}(G) e_K$ is the subalgebra $\mathcal{H}_K(G)$ of K-bi-invariant measures. The system of idempotents $\{e_K\}$ is cofinal in Idem $\mathcal{H}(G)$, i.e. $\mathcal{H}(G) = \lim_{\overrightarrow{K}} \mathcal{H}_K(G)$.

The involution $\iota : g \mapsto g^{-1}$ on G defines the natural antiautomorphism $\iota : \mathcal{H}(G) \longrightarrow \mathcal{H}(G)$. Using this antiautomorphism we will usually identify $\mathcal{M}(\mathcal{H})$ with $\mathcal{M}^R(\mathcal{H})$, though sometimes it is move convenient to separate them.

1.4. Jordan-Hölder Content of a Module.

We want to describe some general properties of the category $\mathcal{M}(\mathcal{H})$. It is convenient to do it in a more general setting.

Let $\mathcal{M}$ be an abelian category with (arbitrary) direct sums (and, hence, direct limits). We will assume that $\mathcal{M}$ satisfies some axioms.

$(A1)$ Filtered direct limits in $\mathcal{M}$ are exact.

In [Gr] this axiom is called AB... It is equivalent (see [ ]) to

$(A1')$ Let $M_\alpha \subset M$ be a filtered system of submodules, $N \subset M$. Then

$$N \cap \left(\sum_\alpha M_\alpha \right) = \sum_\alpha (N \cap M_\alpha) \ .$$

An object $M \in \mathcal{M}$ is called *finitely generated* if for any filtered system of proper subobjects $M_\alpha \subset M$ the subobject $\sum_\alpha M_\alpha \subset M$ is proper. For a finitely generated object M the functor $\mathrm{Hom}(M, *) : \mathcal{M} \to Ab$ preserves direct sums.

An object $M \in \mathcal{M}$ is called *noetherian*, if every of its subobjects is finitely generated or, equivalently, if each ascending chain of subobjects $M_1 \subset M_2 \subset ...$ of M is stable.

Category $\mathcal{M}$ is called locally *noetherian* if each finitely generated object of $\mathcal{M}$ is noetherian.

$(A2)$ Every object $M \in \mathcal{M}$ is a union of finitely generated subobjects.

In order to avoid set-theoretical troubles we also add

$(A3)$ Isomorphism classes of finitely generated objects in $\mathcal{M}$ form a set.

We denote by $Irr\mathcal{M}$ the set of isomorphism classes of irreducible (i.e. simple) objects in $\mathcal{M}$. For every $E \in \mathcal{M}$ we denote by $JH(E) \subset Irr\mathcal{M}$ the subset of irreducible subquotients of E.

For each idempotented ring $\mathcal{H}$ the category $\mathcal{M} = \mathcal{M}(\mathcal{H})$ satisfies axioms $A1-A3$. We will denote $Irr(\mathcal{M}(\mathcal{H}))$ by $Irr\mathcal{H}$ and $Irr\mathcal{M}(G)$ by $IrrG$ (see 1.3).

Lemma. (i) *Let $E' \subset E$. Then $JH(E) = JH(E') \cup JH(E/E')$.*

(ii) *$JH(E) = \emptyset$ iff $E = 0$*

(iii) *If $E_\alpha \subset E$, then $JH\left(\sum_\alpha E_\alpha\right) = \bigcup_\alpha JH(E_\alpha)$.*

Proof:

(i) is clear.

(ii) Let $E \neq 0$. By $A2$ E has a nonzero finitely generated submodule E'. By Zorn's lemma E' has an irreducible quotient, i.e. $JH(E) \neq \emptyset$.

(iii) Let $I = \{\alpha\}$ be the indexing set of E_α. If I is finite, the statement follows from (i) by induction. Hence, replacing system $\{E_\alpha\}$ by a system, consisting of finite sums of E_α we can assume, that $\{E_\alpha\}$ is a filtered direct system. Let $Q = E'/E''$ be a simple subquotient of $\sum_\alpha E_\alpha$, i.e. $E'' \subsetneqq E' \subset \sum_\alpha E_\alpha$. Suppose that for all α $Q \notin JH(E_\alpha)$. Then for every α $E' \cap \left(E'' + E_\alpha\right) = E''$. By $A1'$ $E' \cap \sum_\alpha (E'' + E_\alpha) = \sum E' \cap (E'' + E_\alpha) = E''$, which contradicts the inclusion $E' \subset \sum_\alpha E_\alpha$, since $E'' \neq E'$.

1.5. Decomposition of Categories. Suppose that the category $\mathcal{M}$ is split into a product of two subcategories $\mathcal{M} = \mathcal{M}' \times \mathcal{M}''$. This splitting induces a disjoint union decomposition $Irr\mathcal{M} = Irr\mathcal{M}' \cup Irr\mathcal{M}''$. We want to show that this decomposition completely describes the splitting.

For each subset $S \subset Irr\mathcal{M}$ denote by $\mathcal{M}(S)$ the full subcategory of $\mathcal{M}$ defined by $\mathcal{M}(S) = \{E \in \mathcal{M} \mid JH(E) \subset S\}$. Lemma 1.4 shows that $\mathcal{M}(S)$ is an abelian subcategory, closed with respect to subquotients, extensions and direct limits. For every $E \in \mathcal{M}$ we denote by E_S the union of all submodules $E' \subset E$, which lie in $\mathcal{M}(S)$. Then E_S also lies in $\mathcal{M}(S)$. Let $S' \subset Irr\mathcal{M}$ be another subset, which does not intersect S. Then for each $E \in \mathcal{M}(S) \cap \mathcal{M}(S')$ we have $JH(E) = \emptyset$, i.e. $E = 0$. This implies that the categories $\mathcal{M}(S), \mathcal{M}(S')$ are orthogonal, i.e. $\mathrm{Hom}_{\mathcal{M}}(E, E') = 0$ for $E \in \mathcal{M}(S)$, $E' \in \mathcal{M}(S')$. Also for every $E \in \mathcal{M}$ $E_S \cap E_{S'} = 0$, i.e. $E \supset E_S \oplus E_{S'}$.

Definition. We say that a subset $S \in Irr\mathcal{M}$ *splits* an object $E \in \mathcal{M}$ if $E = E_S \oplus E_{\overline{S}}$, where $\overline{S} = Irr\mathcal{M}\backslash S$. We say that S *splits* $\mathcal{M}$ if it splits all objects in $\mathcal{M}$.

More generally, suppose we have a disjoint union decomposition $Irr\mathcal{M} = \bigcup_{\alpha \in A} S_\alpha$. We say that this decomposition $\{S_\alpha\}$ splits E if $E = \bigoplus\limits_{\alpha \in A} E_{S_\alpha}$. We say that the decomposition $\{S_\alpha\}$ splits $\mathcal{M}$ if it splits all objects in $\mathcal{M}$. In this case $\mathcal{M}$ is equivalent to the category $\prod\limits_{\alpha \in A} \mathcal{M}(S_\alpha)$.

Lemma. *Let $Irr\mathcal{M} = \bigcup_{\alpha \in A} S_\alpha$ be a disjoint union decomposition. Suppose it splits an object $E \in \mathcal{M}$. Then it splits all subquotients of E.*

Proof: Let $E = \bigoplus_{\alpha \in A} E_\alpha$, $E_\alpha \in \mathcal{M}(S_\alpha)$. It is sufficient to check that for every subobject $L \subset EL = \sum_\alpha (L \cap E_\alpha)$. Put $C = L/\sum_\alpha (L \cap E_\alpha)$. Then for α

$$JH(C) \subset JH\left(L/L \cap E_\alpha\right) \subset JH\left(E/E_\alpha\right) \subset \bigcup_{\beta \neq \alpha} JH\left(E_\beta\right) \subset \overline{S_\alpha}\ .$$

This implies, that $JH(C) \subset \bigcap_\alpha \left(\overline{S_\alpha}\right) = \emptyset$, i.e. $C = 0$.

Remark. Let $\mathcal{H}$ be an idempotented ring. Suppose category $\mathcal{M} = \mathcal{M}(\mathcal{H})$ has a decomposition $\mathcal{M} = \prod\limits_\alpha \mathcal{M}_\alpha$. Applying this decomposition to the $\mathcal{H}$-module $\mathcal{H}$ we see that $\mathcal{H} = \bigoplus\limits_\alpha \mathcal{H}_\alpha$. Since right multiplications in $\mathcal{H}$ are morphisms in $\mathcal{M}(\mathcal{H})$ all $\mathcal{H}_\alpha$ are two-sided ideals. It is easy to see that $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{H}_\alpha)$.

Conversely, each decomposition $\mathcal{H} = \bigoplus\limits_\alpha \mathcal{H}_\alpha$ of $\mathcal{H}$ into a direct sum of two-sided ideals leads to the decomposition $\mathcal{M}(\mathcal{H}) = \prod\limits_\alpha \mathcal{M}(\mathcal{H}_\alpha)$.

1.6. Realization of an Abelian Category as a Category of Modules. Let $\mathcal{M}$ be an abelian category, satisfying $A1 - A3$. Let $P \in \mathcal{M}$ be a finitely generated projective object, $\Lambda = \mathrm{End}_{\mathcal{M}}(P)^\circ$ ($^\circ$ denotes the opposite algebra).

We define the functor $r = r_P : \mathcal{M} \to \mathcal{M}(\Lambda)$ by $r(E) = \mathrm{Hom}_{\mathcal{M}}(P, E)$. It is exact and commutes with direct sums. Functor r has a left adjoint functor $i = i_P : \mathcal{M}(\Lambda) \to \mathcal{M}$. Indeed, every Λ-module M can be presented as a cokernel of a morphism ν_M of free Λ-modules $\nu_M : \bigoplus\limits_\alpha \Lambda \longrightarrow \bigoplus\limits_\beta \Lambda$, where ν_M is given by a matrix $\{\nu_{\alpha\beta} \in \Lambda\}$. We define $i(M)$ as a cokernel of a morphism $\nu' : \bigoplus\limits_\alpha P \longrightarrow \bigoplus\limits_\beta P$, where ν' is given by the same matrix $\{\nu_{\alpha\beta} \in \Lambda\}$. In case when $\mathcal{M} = \mathcal{M}(\mathcal{H})$ the functor i can be described as $i(M) = P \bigotimes\limits_\Lambda M$.

Lemma. *Suppose that P is a generator of the category $\mathcal{M}$, i.e. the functor r is faithful, or, equivalently,* $\mathrm{Hom}_{\mathcal{M}}(P, Q) \neq 0$ *for $Q \in Irr\mathcal{M}$. Then functor r and i are inverse and define an equivalence of categories*

$$\mathcal{M} \underset{r}{\overset{i}{\leftrightarrows}} \mathcal{M}(\Lambda).$$

Proof: See [...]

This lemma allows us to realize $\mathcal{M}$ as a category of modules over some algebra with identity. This realization is not unique, it depends on the choice of P. Let us describe the relation between two such realizations.

Let A be an algebra with identity, $P \in \mathcal{M}(A)$ a finitely generated projective generator, $\Lambda = (\mathrm{End}_A P)^\circ$. Then P is an $A-\Lambda$-bimodule. We define a dual $\Lambda - A$-bimodule P^* by $P^* = \mathrm{Hom}_A(P, A)$.

Proposition. *P^* is a finitely generated projective generator in $\mathcal{M}(\Lambda)$,* $\mathrm{End}_\Lambda(P^*) =$ *A° and the functors $i : \mathcal{M}(\Lambda) \to \mathcal{M}(A)$, $r : \mathcal{M}(A) \to \mathcal{M}(\Lambda)$ are canonically isomorphic to $r(E) = P^* \bigotimes\limits_A E, E \in \mathcal{M}(A)$ and $i(M) = \mathrm{Hom}_\Lambda(P^*, M)$, $M \in \mathcal{M}(\Lambda)$.*

Proof:

Step 1. For any $E \in \mathcal{M}(A)$ the natural morphism $P^* \bigotimes\limits_A E \to \mathrm{Hom}_A(P, E) = r(E)$ is an isomorphism.

Indeed, this is true for $P = A$, hence for $P = A^n$ and hence for P which is a direct summand of A^n.

Step 2. Since P is a generator of $\mathcal{M}(A)$, A is a direct summand of P^n for some natural n. Hence $r(A) = P^*$ is a direct summand of $r(P)^n = \Lambda^n$, i.e. P^* is a finitely generated projective Λ-module.

Step 3. Since functors r and i are mutually inverse, we have

$$\mathrm{Hom}_A(E, i(M)) = \mathrm{Hom}_\Lambda(r(E), M) =$$
$$= \mathrm{Hom}_\Lambda\left(P^* \bigotimes_A E, M\right) = \mathrm{Hom}_A(E, \mathrm{Hom}_\Lambda\,(P^*, M))$$

which implies that $i(M)$ is canonically isomorphic to $\mathrm{Hom}_\Lambda\,(P^*, M)$. Since the functor i is faithful, P^* is a generator of $\mathcal{M}(\Lambda)$.

Step 4. We have $r(P) = \mathrm{Hom}_A(P, P) = \Lambda \in \mathcal{M}(\Lambda)$, $r(A) = P^* \bigotimes\limits_A A = P^*$ and hence $i(\Lambda) = \mathrm{Hom}_\Lambda(P^*, \Lambda) = P$, $i(P^*) = \mathrm{Hom}_\Lambda(P^*, P^*) = A \in \mathcal{M}(A)$. This implies that as an algebra $\mathrm{End}_\Lambda(P^*) = A^\circ$.

Corollary. *P is a right projective Λ-module and* $\mathrm{End}_\Lambda(P) = A$.

Indeed, since $P = \mathrm{Hom}_\Lambda(P^*, \Lambda)$, it is a right projective Λ-module, dual to P^*. Hence $\mathrm{End}_\Lambda(P) = \mathrm{End}_\Lambda(P^*)^\circ = A$.

1.7. Realization of a Subcategory as a Category of Modules.

Let $P \in \mathcal{M}$ be a finitely generated projective object, which we do not suppose to be a generator. Consider subset $S = S_P \subset Irr M$ of irreducible quotients of P. We say that P *splits* the category $\mathcal{M}$ if the subset S splits $\mathcal{M}$, i.e. $\mathcal{M} = \mathcal{M}(S) \times \mathcal{M}(\overline{S})$. (see...).

Corollary. *Suppose P splits $\mathcal{M}$. Then functors r, i give equivalence of categories $\mathcal{M}(S) \underset{r}{\overset{i}{\leftrightarrows}} \mathcal{M}(\Lambda)$. Moreover, $\mathcal{M}(\overline{S}) = \{E \in \mathcal{M}(S) \mid \mathrm{Hom}(P, E) = 0\}$ $\mathcal{M}(S) = \{E \in \mathcal{M}(S) \mid E$ is a quotient of $\bigoplus\limits_\alpha P\}$.*

This easily follows from 1.6.

Example. Let $\mathcal{H}$ be an idempotented ring $\mathcal{M} = \mathcal{M}(\mathcal{H})$. Choose an idempotent $e \in \mathcal{H}$ and put $P = \mathcal{H}e$. Then P is a finitely generated projective object in $\mathcal{M}$, $\Lambda = (\mathrm{End}_{\mathcal{M}} P)^{\circ}$ coincides with the subalgebra $e\mathcal{H}e \subset \mathcal{H}$ and functors $r : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\Lambda)$, $i : \mathcal{M}(\Lambda) \to \mathcal{M}(\mathcal{H})$ are given by $r(E) = eE$, $i(M) = P \bigotimes_{\Lambda} M$.

We say that idempotent e splits $\mathcal{M}$ if the subset $S = S_e = \{\omega \in Irr\mathcal{M} \mid e\omega \neq 0\}$ splits $\mathcal{M}$. In this case functors r and i give equivalence of categories $\mathcal{M}(S) \overset{i}{\underset{r}{\leftrightarrows}} \mathcal{M}(\Lambda)$ and $\mathcal{M}(S) = \{E \in \mathcal{M}(\mathcal{H}) \mid E$ is generated by $eE\}$, $\mathcal{M}(\overline{S}) = \{E \in \mathcal{M}(\mathcal{H}) \mid eE = 0\}$.

1.8. The Central Algebra of $\mathcal{M}$.
Let $\mathcal{M}$ be an abelian category.

Definition. The central algebra $Z(\mathcal{M})$ is defined as $Z(\mathcal{M}) = \mathrm{End}(Id_{\mathcal{M}})$, where $Id_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is the identity functor. In other words, an element $z \in Z(\mathcal{M})$ is a collection of morphisms $z_M : M \to M$ for all $M \in Ob\mathcal{M}$, such that for each morphism

$$\alpha : M \to N \qquad z_N \circ \alpha = \alpha \circ z_M \ .$$

If $\mathcal{M} = \mathcal{M}(\mathcal{H})$ or $\mathcal{M}(G)$ we will also use notations $Z(\mathcal{H})$ or $Z(G)$ instead of $Z(\mathcal{M}(\mathcal{H}))$ or $Z(\mathcal{M}(G))$.

Lemma. *Let $\mathcal{H}$ be an idempotented ring. Then the morphism $z \mapsto z_{\mathcal{H}}$ identifies $Z(\mathcal{H})$ with the algebra $\mathrm{End}_{\mathcal{H}\times\mathcal{H}^{\circ}}(\mathcal{H})$ of endomorphisms of $\mathcal{H}$ which commute with right and left multiplications. In particular, if $\mathcal{H}$ has an identity, $Z(\mathcal{H})$ is isomorphic to the center of $\mathcal{H}$.*

Proof: is straightforward, see...

Corollary. *Let P be a finitely generated projective generator in $\mathcal{M}$, $\Lambda = (\mathrm{End}_{\Lambda} P)^{\circ}$. Then the natural morphism $z \mapsto z_P \in \Lambda$ gives an isomorphism of $Z(M)$ with the center of Λ.*

This follows from the lemma and 1.6.

§2. Decomposition theorem

2.0. Let G be a connected reductive p-adic group, $\Theta(G)$ the set of infinitesimal characters of G, $\Theta(G) = \cup\Theta$ its decomposition into the union of connected components. For each Θ consider the subset $S_\Theta = \inf \cdot \mathrm{ch}^{-1}(\Theta) \subset IrrG$ and denote by $\mathcal{M}(\Theta) = \mathcal{M}(G, S_\Theta)$ the corresponding subcategory in $\mathcal{M}(G)$, $\mathcal{M}(\Theta) = \{E \in \mathcal{M}(G) | JH(E) \subset S_\Theta\}$ (see 1). In this section we prove the following

Decomposition theorem. $\mathcal{M}(G) = \prod_{\Theta} \mathcal{M}(\Theta)$, *where Θ runs through all connected components of $\Theta(G)$.*

Our proof follows the proof in [ ] with slight modifications, which we will use later.

Generalization. Let B be a commutative algebra with identity. Put $\mathcal{M}(\Theta; B) = \{E \in \mathcal{M}(G; B) | E \in \mathcal{M}(\Theta) \quad \text{is } G\text{-module}\}$. Then decomposition theorem implies that $\mathcal{M}(G; B) = \prod_{\Theta} \mathcal{M}(\Theta; B)$.

2.1. Separation of compactly supported G-modules.

Let G be an arbitrary ℓ-group as in 1. A G-module E is called *compactly supported* if for each open compact subgroup $K \subset G$ and each $\xi \in E$ the function $g \mapsto (e_K g e_K)\xi$ has a compact support of G. This implies that E has compactly supported matrix coefficients. Using this fact and arguing exactly like in a case of compact groups, one can prove the following.

Proposition. (see [ ]). *Let V be a finitely generated compactly supported G-module. Then V is admissible and has finite length. The finite subset $S = JH(V) \subset IrrG$ splits the category $\mathcal{M}(G)$ and each module $E \in \mathcal{M}(G; S)$ is completely reducible.*

2.2. Separation of cuspidal components.

Let G be a reductive p-adic group. If the center $Z(G)$ of G is compact, cuspidal G-modules are compactly supported and we can use 2.1 to separate them. In general they are compactly supported modulo center $Z(G)$. To study this case we will use the following property of G.

(*) G has an open normal subgroup G^0 such that $Z(G) \cap G^0$ is compact, $Z(G) \cdot G^0$ has finite index in G and the group $\Lambda = G/G^0$ is a lattice, i.e. is isomorphic to $\mathbb{Z}^d$, $d \in \mathbb{Z}^+$.

It is easy to see that such a subgroup G^0 is unique. By definition the group $\Psi(G)$ of unramified characters of G coincides with

$$\text{Hom } (\Lambda, \mathbb{C}^*) = \{\psi : G \to \mathbb{C}^* : \psi|_{G^0} = 1\} .$$

Lemma. *Let (ρ, V) be a simple G-module. Then*

(i) *$\rho|_{G^0}$ is completely reducible of finite length. The subset $S_\rho = JH(\rho|_{G^0}) \subset IrrG^0$ is finite and is a G-orbit of the natural action of G on $IrrG^0$.*

(ii) *The correspondence $\rho \mapsto S_\rho$ gives a bijection of the set of $\Psi(G)$ - orbits in $IrrG$ and G-orbits in $IrrG^0$, i.e. $S_\rho = S_{\rho'}$ iff $\rho' \approx \psi\rho$ for some $\psi \in \Psi(G)$.*

(iii) *The stabilizer $St(\rho, \Psi)$ of ρ in $\Psi(G)$ is finite. If we choose for each $\psi \in \mathrm{St}\ (\rho, \Psi)$ a nonzero morphism $\alpha_\psi : (\rho, V) \to (\psi\rho, V)$, then $\{\alpha_\psi\}$ is a $\mathbb{C}$-basis of $\mathrm{End}_{G^0}(V)$.*

Proof: (i), (ii) are proven in [ ]. (iii) Put $A = \mathrm{End}_{G^0}(V)$ and define the action of G on A by $g(a) = \rho(g)a\rho(g)^{-1}$. This action is trivial on G^0. Because of Schur's lemma it is also trivial on $Z(G)$, so it is an action of the finite abelian group $G/G^0 \cdot Z(G)$. Using this we can decompose $A = \oplus A_\psi$, where A_ψ are eigenspaces of the action. But $A_\psi = \mathrm{Hom}_G(\rho, \psi\rho) = \mathbb{C} \cdot a_\psi$ by Schur's lemma, i.e. $A = \bigoplus_\psi \mathbb{C} \cdot a_\psi$ with $\psi \in St(\rho, \Psi) \subset \text{Hom } (G/G^0 \cdot Z(G), \mathbb{C}^*)$.

Harish-Chandra theorem. (see [ ]) *Let π be a quasicuspidal G-module, i.e. $r_{MG}(\pi) = 0$ for all subgroups $M \subsetneqq G$. Then it is compactly supported modulo center, i.e. $\pi|_{G^0}$ is compactly supported.*

Corollary. *Let (ρ, V) be a cuspidal irreducible G-module. Then the cuspidal component $D = \Psi(G) \cdot \rho \subset IrrG$ splits the category $\mathcal{M}(G)$.*

Proof: Put $S = S_\rho = JH(\rho|_{G^0}) \subset IrrG^0$. By 2.1 every G-module E has a decomposition $E = E_S \oplus E_{\bar{S}}$ with $E \in \mathcal{M}(G^0; S)$, $E_{\bar{S}} \in \mathcal{M}(G^0; \overline{S})$. Since this decomposition is canonical it is G-invariant, i.e. E_S and $E_{\bar{S}}$ are G-submodules. Lemma 2.2 implies that $E_S \in \mathcal{M}(G, D)$, $E_{\bar{S}}^{\perp} \in \mathcal{M}(G, \overline{D})$.

2.3. Functors i_{GM} and r_{MG}.

In order to deal with noncuspidal components we will use functors i_{GM} and r_{MG}. Let us recall some elementary properties of these functors. For simplicity we consider only the case when M is a standard Levi subgroup.

(i) Transitivity. Let $M < N < G$. Then $i_{GM} = i_{GN} \circ i_{NM}$, $r_{MG} = r_{MN} \circ r_{NG}$ (canonical isomorphisms).

(ii) Functor r_{MG} is left adjoint to i_{GM} (canonical adjointness). See [ ].

(iii) Functors i_{GM} and r_{MG} are exact and preserve direct sums. See [ ]

(iv) There exists a functorial isomorphism $i_{GM}(\tilde{\sigma}) = (i_{GM}(\sigma))\tilde{}$, $\sigma \in \mathcal{M}(M)$ (canonical isomorphism). See [ ]

(v) Functor r_{GM} maps finitely generated G-modules into finitely generated M-modules. See [ ].

(vi) Composition of functors r and i.

We need some notations. For each $w \in W_G$ we fix a representative $\overline{w} \in$ Norm (M_0, G). For each subgroup $H \subset G$ we put $w(H) = \bar{w} H \bar{w}^{-1}$ and denote by w the corresponding functor $w : \mathcal{M}(H) \to \mathcal{M}(w(H))$.

Let $M, N < G$. Each double coset $W_N \backslash W_G / W_M$ has a unique representative of minimal length; we denote the set of these representatives by W_G^{NM}. For each $w \in W_G^{NM}$ we put

$$M_w = M \cap w^{-1}(N) < M, \quad N_w = w(M_w) = w(M) \cap N < N \ .$$

Composition theorem. *Consider functors $F, F_w : \mathcal{M}(M) \to \mathcal{M}(N)$, for $w \in W_G^{NM}$, defined by $F = r_{NG} \circ i_{GM}$, $F_w = i_{NN_w} \circ w \circ r_{M_w M}$. More precisely, choose any ordering $\{w_1, \ldots, w_r\}$ of W_G^{NM} such that $w_i < w_j$ implies $i \geq j$ (here $<$ is the standard partial order on W, see [ ]). Then F has a canonical filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_V = F$ and F_i / F_{i-1} is canonically isomorphic to F_{w_i}.*

See the proof in [ ]. Canonicity of isomorphisms in .... and .... is discussed in appendix ...

(vii) Let $K \subset G$ be an open compact subgroup. We will use the following simple lemma, which describes K-invariant vectors in induced G-modules.

Lemma. *Let (P, M) be a standard parabolic pair. Fix a system $(g_1, \ldots, g_n)$ of representatives of double cosets $P\backslash G/K$ and consider open compact subgroups $\Gamma_1, \ldots, \Gamma_n \subset M$ defined by $\Gamma_i = pr_{P\to M}(P \cap gKg^{-1})$. Also fix a Haar measure on the unipotent radical $U \subset P$. Then for every $V \in \mathcal{M}(M)$ and $E = i_{GM}(V)$ there exists a canonical functorial isomorphism $E^K \approx \bigoplus_{i=1}^{n} V^{\Gamma_i}$.*

Proof: is straightforward.

2.4. Functors $i_{G,D}$ and $r_{D,G}$ and their properties.

Let (M, D) be a standard cuspidal block (notation $(M, D) < (G, \Theta(G))$). It means that $M < G$ and D is a cuspidal component of $\Theta(M)$. The subset $\Theta = i_{GM}(D) \subset \Theta(G)$ is a connected component. We say that the component Θ corresponds to the block (M, D) and use the notation $(M, D) < (G, \Theta)$. Another standard cuspidal block (N, D') corresponds to the same component Θ if and only if there exists $w \in W_G$ such that $w(M, D) = (N, D')$, i.e. $N = w(M)$, $D' = w(D)$. In this case we say that (N, D') is *associate* to (M, D) (notation $(N, D') \sim (M, D)$).

Standard cuspidal blocks will play a role similar to standard Levi subgroups. By 2.2. $\mathcal{M}(D)$ is a direct summand of $\mathcal{M}(M)$. We denote by $in_D : \mathcal{M}(D) \to \mathcal{M}(M)$ and $pr_D : \mathcal{M}(M) \to \mathcal{M}(D)$ the corresponding inclusion and projection functors.

Consider the functors

$$i_{GD} = i_{GM} \circ in_D : \mathcal{M}(D) \to \mathcal{M}(G)$$
$$r_{DG} = pr_D \circ r_{MG} : \mathcal{M}(G) \to \mathcal{M}(D) \ .$$

The following properties of these functors immediately follow from 2.3.

(i) r_{DG} is left adjoint to i_{GD}.

(ii) i_{GD} and r_{DG} are exact and preserve direct sums.

(iii)

Composition theorem. *Let (M, D), (N, D') be standard cuspidal blocks $F : r_{D'G} \circ i_{GD} : \mathcal{M}(D) \to \mathcal{M}(D')$. Then $F = 0$ unless $(M, D) \sim (N, D')$. If they are associate, F is glued from the functors $w : \mathcal{M}(D) \to \mathcal{M}(D')$, where $w \in \{w \in W_G^{NM} | w(M, D) = (N, D')\}$.*

Proof: By composition theorem F is glued from $pr_{D'} \circ i_{NN_w} \circ w \circ r_{M_w M} \circ in_D$. If $M_w \neq M$, we have $r_{M_w M} \circ in_D = 0$. If $N_w \neq N$, we have $pr_{D'} \circ i_{NN_w} = 0$ (as right adjoint to $r_{N_w N} \circ in_{D'} = 0$). This proves the theorem.

Proposition. *(i) The system of functors r_{DG} for all $(M, D,) < (G, \Theta(G))$ is faithful, i.e. $r_{DG}(E) = 0$ for all (M, D) implies that $E = 0$.*

(ii) *Fix a connected component $\Theta \subset \Theta(G)$. Then the system of functors r_{DG} with $(M,D) < (G,\Theta)$ is faithful on $\mathcal{M}(\Theta)$.*

(iii) *Let E be a G-module such that $r_{D'G}(E) = 0$ for all standard cuspidal blocks (N,D') which do not correspond to the component Θ. Then $E \in \mathcal{M}(\Theta)$.*

(iv) *Conversely, if $E \in \mathcal{M}(\Theta)$, then $r_{D'G}(E) = 0$ for $(N,D') \not< (G,\Theta)$.*

(v) *If $(M,D) < (G,\Theta)$, then $i_{GD}(\mathcal{M}(D)) \subset \mathcal{M}(\Theta)$.*

Lemma. *Let $w \in IrrG$, $\theta = \inf \cdot \mathrm{ch}\ w \in \Theta(G)$ and Θ be a connected component of θ. There exists a cuspidal block $(M,D) < (G,\Theta)$ such that $r_{DG}(w) \neq 0$. For each cuspidal block (N,D') which does not correspond to Θ $r_{D'G}(w) = 0$.*

Proof: We can find a cuspidal pair (M,ρ) such that $M < G$ and $w \in i_{GM}(\rho)$. Let $D \subset \Theta(M)$ be a connected component of ρ. Then $\mathrm{Hom}_G(w, i_{GD}(\rho)) = \mathrm{Hom}(r_{DG}(w),\rho) \neq 0$, i.e. $r_{DG}(w) \neq 0$. If $(N,D') \not\sim (M,D)$, then $r_{D'G}(w) \subset r_{D'G} \circ i_{GD}(\rho) = 0$ by composition theorem 2.4 (iii).

Proof of the proposition. Since functors r_{DG} are exact the lemma implies (i), (ii) and (iii). Since for $(N,D') \not\sim (M,D)$ $r_{D'G} \circ i_{GD} = 0$, (iii) implies (iv).

Let us prove (iv). Let (N,D') be a standard Levi block such that the corresponding component Θ' differs from Θ. Put $V = r_{D'G}(E) \in \mathcal{M}(D')$. By (v) $i_{GN}(v) \in \mathcal{M}(\Theta')$ and hence $\mathrm{Hom}_N(V,V) = \mathrm{Hom}_N(r_{D'G}(E),V) = \mathrm{Hom}_G(E, i_{GD'}(V)) = 0$, i.e. $V = 0$.

Corollary. *Let $N < G$, $\Theta \subset \Theta(G)$ be a connected component. Consider all components $\Theta_N \subset i_{GN}^{-1}(\Theta) \subset \Theta(N)$ and the corresponding product category $\mathcal{M}' = \prod_{\Theta_N} \mathcal{M}(\Theta_N)$. Then*

$$i_{GN}(\mathcal{M}') \subset \mathcal{M}(\Theta) \text{ and } r_{NG}(\mathcal{M}(\Theta)) \subset \mathcal{M}' .$$

Proof: it easily follows from the composition theorem in 2.3 and the proposition.

2.5. Proof of decomposition theorem.

Step 1. For each standard cuspidal block (M,D) define a functor $T_D = i_{GD} \circ r_{DG} : \mathcal{M}(G) \to \mathcal{M}(G)$. Since the functor r_{GD} is left adjoint to i_{GD}, for each G-module E we have a canonical functorial morphism $\alpha_D : E \to T_D(E)$. If $L \subset E$, then the restriction $\alpha_D|_L : L \to T_D(E)$ corresponds to the morphism $r_{DG}(L) \to r_{DG}(E)$. Since the functor r_{DG} is exact, this morphism is an inclusion. This proves that $\alpha_D(L) = 0$ if and only if $r_{DG}(L) = 0$.

Step 2. Consider the product morphism

$$\alpha = \prod_{(M,D)} \alpha_D : E \to \prod_{(M,D)} T_D(E)$$

where the product is over all standard cuspidal blocks (M,D). Then $r_{DG}(\mathrm{Ker}\ \alpha) = 0$ for all (M,D), and, since $\{r\}$ is a faithful system of functors, $\mathrm{Ker}\ \alpha = 0$.

Step 3. We want to show that the decomposition $IrrG = \bigcup_{\Theta} S_\Theta$ splits a G-module E. Since $E \subset \prod_{(M,D)} T_D(E)$ it is sufficient to check that $\{S_\Theta\}$ splits this product (see 1...). By proposition 2.4. (v) $\{S_\Theta\}$ splits $\bigoplus_{(M,D)} T_D(E)$, hence it would be sufficient to prove that $\bigoplus_{(M,D)} T_D(E) \approx \prod_{(M,D)} T_D(E)$. This follows from the following general statement.

(*) Let $V_\Theta \in \mathcal{M}(\Theta) : \Theta \subset \Theta(G)$. Then $\bigoplus_{\Theta} V_\Theta \approx \prod_{\Theta} V_\Theta$.

Step 4. As we saw in ... $\prod_{\Theta} V_\Theta = \lim_{\overrightarrow{K}} (\prod_{\Theta} V_\Theta^K)$. Hence (*) follows from

(**) Let $K \subset G$ be an open compact subgroup. Then $V_\Theta^K = 0$ for all but a finite number of components Θ, so $\bigoplus_{\Theta} V_\Theta^K = \prod_{\Theta} V_\Theta^K$.

Put $S_K = \{L \in IrrG | L^K \neq 0\}$, $\Theta_K(G) = \inf \cdot \mathrm{ch} \cdot S_K$. If $V_\Theta^K \neq 0$, then V_Θ has an irreducible subquotient in S_K. Hence (**) follows from

(***) $\Theta_K(G)$ is a union of a finite number of components.

Let $\Theta \subset \Theta(G)$ be a connected component $(M, D) < (G, \Theta)$, $(\rho, V) \in D$. For every ψ put $E_\psi = i_{GM}(\psi\rho) \in \mathcal{M}(G)$. The lemma 2.3 ( ) shows that the space E_ψ^K does not depend on ψ and is equal to $\bigoplus_i V^{\Gamma_i}$. For a given infinitesimal character $\theta = (M, \psi\rho) \in \Theta$ the fiber $\inf \cdot \mathrm{ch}^{-1}(\Theta) \subset IrrG$ coincides with $JH(E_\psi)$. This implies, that $\theta \in \Theta_K(G)$ iff $\bigoplus_i V^{\Gamma_i} \neq 0$. Hence Θ either lies in $\Theta_K(G)$ or does not intersect it, i.e. $\Theta_K(G)$ is a union of components. Moreover, $\Theta \subset \Theta_K(G)$ iff $D \subset \Theta_{\Gamma_i}(M)$ for some i. So, using induction in dim M, we should estimate only the number of cuspidal components. In other words (***) follows from

(***′) $\Theta_K(G)$ contains a finite number of cuspidal connected components.

Step 5. Using 2.2 we see that (***′) is equivalent to

(****) $Irr_K G^0$ has a finite number of compactly supported G^0-modules.

This statement is deduced in [ ] from the following

Uniform admissibiliy theorem. *Let $K \subset G$ be an open compact subgroup. There exists an effective constant $C = C(G, K)$ such that for each simple G-module L $\dim L^K \leq C(G, K)$.*

Remark. The proof in [ ] does not give an effective estimate for the number and type of cuspidal components in $\Theta(G)$. In .... we will give an effective estimate.

2.6. The faithfulness of the functor r_{DG}.

Fix a connected component $\Theta \subset \Theta(G)$. As we saw in 2.4. the system of functors $\{r_{DG} | (M,D) < (G,\Theta)\}$ is faithful on $\mathcal{M}(\Theta)$. In fact, each of these functors is faithful. This fact allows us to simplify notations in many proofs.

Proposition. *Let $(M,D) < (G,\Theta)$. Then the functor r_{DG} is faithful on $\mathcal{M}(\Theta)$. In particular, for every G-module $E \in \mathcal{M}(\Theta)$ the morphism $\alpha_D : E \to T_D E$, described in 2.5 is an inclusion.*

The proof is based on the following lemma, due to Casselman

Lemma. *Let $M < G$ be a maximal Levi subgroup, $D \subset \Theta(M)$ a cuspidal component, $\rho \in D$. Suppose that for some $w \in W_G$, $wM < G$ and $w(M,D) \neq (M,D)$. Then the G-module $\pi = i_{GM}(\rho)$ is irreducible.*

Proof:

Step 1. Let $R(G)$ be the Grothendieck group of G-modules of finite length. By Langlands theory $R(G)$ is generated by $i_{GN}(\psi\sigma)$, where $N < G$, $\psi \in \Psi(N)$, $\sigma \in IrrN$ is a tempered N-module.

Consider the infinitesimal character θ, corresponding to (M,ρ) and a subgroup $R(\theta) \subset R(G)$, generated by G-modules with infinitesimal character θ. Let $i_{GN}(\psi\sigma) \in R(\theta)$. If $N \neq G$ then, since M is maximal, $(N,\psi\sigma)$ is conjugate to (M,ρ) and hence $i_{GN}(\psi\sigma) \approx \pi$. Hence if exclude the possibility $N = G$, then $R(\theta) = \mathbb{Z}\cdot\pi$, i.e. π is irreducible.

Suppose there exists a tempered G-module $\sigma \in IrrG$, and $\psi \in \Psi(G)$ such that $\psi\sigma \in R(\theta)$. Replacing ρ by $\psi^{-1}\rho$ we can assume that $\psi = 1$, i.e. inf. ch. $\sigma = \theta$. Replacing the cuspidal pair (M,ρ) by a conjugate one we can assume, that $\sigma \subsetneq \pi$.

Step 2. Since M is a maximal Levi subgroup, there exist modulo W_M, only one nontrivial element $w \in W_G$ such that $wM < G$ (see [ ]).

Put $N = wM$, $D' = wD$, $\pi' = i_{GN}(w\rho)$. We have $r_{DG}(\pi) = \rho$, $r_{D'G}(\pi) = w\rho$. Since the system of functors r_{DG}, $r_{D'G}$ is faithful on $\mathcal{M}(\theta)$ and $r_{DG}(\sigma) \neq 0$, this implies that $r_{DG}(\sigma) = \rho$, $r_{D'G}(\pi/\sigma) = w\rho$ and hence $r_{D'G}(\sigma) = 0$. This shows that π has length 2. Similarly, π' has length 2. Since $\sigma \in JH(\pi') = JH(\pi)$ and $\sigma \not\subset \pi'$, there exists a nontrivial morphism $\pi' \to \sigma$.

Step 3. For every G-module τ denote by τ^+ the Hermitian contragredient G-module. Then $\sigma^+ \approx \sigma$, since σ is tempered and hence unitary. Also ρ^+ lies on the same component D as ρ, since D contains some unitary M-modules. This implies that $\tau = (\pi')^+$ has a form $\tau = i_{GN}(\rho')$ with $\rho' \in D'$.

Nontrivial morphism $\pi' \to \sigma$ gives a nontrivial morphism $\sigma = \sigma^+ \to \tau$. But $\mathrm{Hom}_G(\sigma,\tau) = \mathrm{Hom}_G(\sigma, i_{GN}(\rho')) = \mathrm{Hom}_N(r_{D'G}(\sigma),\rho')$ i.e. $r_{D'G}(\sigma) \neq 0$, which contradicts Step 2. This contradiction proves the lemma.

Proof of the proposition.

Let $E \in \mathcal{M}(\Theta)$, $E \neq 0$. We have to prove that $r_{DG}(E) \neq 0$. By ... we can find a standard cuspidal block (N,D'), associate to (M,D) such that $r_{D'G}(E) \neq$

0. Let $(N, D') = w(M, D)$, $w \in W_G$. We call the map $w : M \to N$ elementary if there exists a Levi subgroup $L < G$ such that $M < L$, $N < L$, $w \in W_L$ and M is a maximal Levi subgroup in L. It is shown in [ ] that any map $w : M \to N$ can be obtained as a composition of elementary maps. Hence we can assume that $w : M \to N$ is elementary.

Let $\Theta' = i_{LM}(D) = i_{LN}(D') \subset \Theta(L)$, $V = r_{LG}(E) \in \mathcal{M}(L)$. Since $r_{D'L}(V) = r_{D'G}(E) \neq 0$, V has a nontrivial D'-component. Hence replacing G by L and E by the Θ'-component of V we can assume that $M < G$ is a maximal Levi subgroup. We can also assume that $(M, D) \neq (N, D')$, otherwise $r_{DG}(E) = r_{D'G}(E) \neq 0$. Choose an irreducible subquotient $w \in E$. Then $w \in JH(i_{GM}(\rho))$ for some $\rho \in D$. By the lemma, $i_{GM}(\rho)$ is irreducible, i.e. $w = i_{GM}(\rho)$. This implies that $r_{DG}(w) \neq 0$ and hence $r_{DG}(E) \neq 0$.

Thus we have proved that r_{DG} is faithful on $\mathcal{M}(\Theta)$. The same arguments as in 2.5 show that $\alpha_D : E \to T_D E$ is an inclusion.

§3. Decomposition of category $\mathcal{M}(G)$ with respect to a compact subgroup

3.1. Let $K \subset G$ be an open compact subgroup $H_K = H_K(G)$. Put $S_K = \{L \subset IrrG | L^K \neq 0\}$. We say that the subgroup K *splits* $\mathcal{M}(G)$ if the subset S_K splits $\mathcal{M}(G)$, i.e. $\mathcal{M}(G) = \mathcal{M}(S_K) \times \mathcal{M}(\overline{S}_K)$. As shown in ... in this case we have

$$\mathcal{M}(S_K) = \{E \in \mathcal{M}(G) | E \text{is generated by } E^K\},$$
$$\mathcal{M}(\overline{S}_K) = \{E \in \mathcal{M}(G) | E^K = 0\}$$

and the functors

$$r : \mathcal{M}(S_K) \to \mathcal{M}(H_K),$$
$$i : \mathcal{M}(H_K) \to \mathcal{M}(S_K)$$

given by $r(E) = E^K$, $i(M) = H \bigotimes_{H_K} M$ are mutually inverse equivalences of categories.

We want to show that there are a lot of subgroups K which split $\mathcal{M}(G)$. In order to do this we describe some geometrical sufficient conditions on K.

First of all, let us notice, that if S_K is a union of subsets S_Θ for some components Θ, then K splits $\mathcal{M}(G)$. In fact, one can prove that any splitting subset $S \subset IrrG$ is a union of S_Θ (it follows, for instance, from the description of $Z(\mathcal{M}(G))$ below). So we want to find conditions which imply that S_K is a union of S_Θ.

3.2. Let $P \subset G$ be a parabolic subgroup $M = P/U$. For a compact open subgroup $K \subset G$ put $K_P = K \cap P$, $K_M = pr_{P \to M}(K_P)$. Let $K \subset G$, $\Gamma \subset M$ be open compact subgroups. Consider the following conditions on K and Γ.

(I) For each $g \in G$ the subgroup $({}^g K)_M \subset M$ contains a subgroup, conjugate to Γ.

(II) For any open subgroup $N \subset G$ the subset $(pr_{P\to M})^{-1}(\Gamma)\cdot N$ contains a subgroup conjugate to K.

Note that these conditions are invariant with respect to conjugation of P, K or Γ.

Lemma. (see.....).

(i) *Suppose K, Γ satisfy I. Then for each M-module V, $V^\Gamma = 0 \Rightarrow i_{GM}(V)^K = 0$.*

(ii) *Suppose K, Γ satisfy II. Then for each G-module E, $E^K = 0 \Rightarrow r^P_{MG}(E)^\Gamma = 0$.*

Proof:

(i) Follows from Lemma ....

(ii) V is isomorphic to E_v as Γ-module (see ...). Denote by $A : E \to E_V$ the natural projection. Suppose that $E^r_V \neq 0$ and choose $\xi \in E$ such that $v = A\xi \in E^r_V\backslash 0$. Let N be the stabilizer of ξ in g. Then for each $g \in pr_{P\to M}(r)^{-1}\cdot N$ we have

$$A(g\xi) = A(\gamma n)\xi = \gamma A(n\xi) = \gamma A\xi = \gamma v = v \ .$$

Choose a subgroup $K' \subset pr^{-1}(\Gamma)\cdot U$, conjugate to K. Then $A(e_{K'}\xi) = v \neq 0$, i.e. $E^{K'} \neq 0$ and $E^K \neq 0$.

Corollary. *Let $K \subset G$ be an open compact subgroup such that for each parabolic subgroup P the pair $K, \Gamma = K_M \subset M$ satisfy both conditions I and II. Then S_K is a union of S_Θ and hence K splits $\mathcal{M}(G)$.*

Proof: Let $\Theta \subset \Theta(G)$ be a connected component $(M, D) < (G, \Theta)$ a corresponding standard cuspidal block. Let $L \in S_\Theta$. Then by ... $r_{DG}(L) \neq 0$, so for some $\psi \in \Psi(M)$ there exists an epimorphism $r_{DG}(L) \to \psi\rho$ and an inclusion $L \to i_{GM}(\psi\rho)$. Hence

$$L^K = 0 \Rightarrow r_{DG}(L)^\Gamma = 0 \Rightarrow V^\Gamma = 0 \text{and}$$
$$V^\Gamma = 0 \Rightarrow i_{GM}(\psi\rho) = 0 \Rightarrow L^K = 0 \ .$$

Thus the condition $L^K = 0$ does not depend on $L \in S_\Theta$, i.e. either $S_\Theta \subset S_K$ or $S_\Theta \subset \overline{S}_K$.

Remarks.

(i) It is sufficient to check condition (I) for (finite number of) representatives $\{g_i\}$ of double cosets $P\backslash G/\text{Norm } K$. In particular, if K is a congruence subgroup, which is normalized by the maximal compact subgroup K_0, then Iwasava decomposition $G = PK_0$ implies that I holds for $\Gamma = K_M$.

(ii) Let $(P, \overline{P})$ be a parabolic pair. Suppose that $K \subset U\ \Gamma\ \overline{U}$, where $\Gamma \subset M = P \cap \overline{P}$. Then condition II holds. Indeed, put $C = pr_{U\Gamma\overline{U}\to\vec{V}}(K)$. Then we can find $a \in Z(M)$ for which ${}^aC$ is arbitrarily small, and hence lie in N, which implies ${}^aK \subset U\ \Gamma\ N$.

Examples.

(1) A congruence subgroup K of a nonzero level is normalized by K_0 and satisfies $KU \cup K_M\overline{U}$ for each standard parabolic pair $(P, \overline{P})$. Hence it splits $\mathcal{M}(G)$.

(2) Let I be an Iwahori subgroup (see ...). Then it is easy to see that $I \subset UI_M\overline{U}$. Choosing representatives $w \in W = K_0/I$ in $P\backslash G/I$ it is easy to check that K, K_M satisfy condition I for each standard parabolic subgroup p. Thus I splits $\mathcal{M}(G)$. Another proof of the fact see in [ ]. In this case S_I consists of one component S_Θ.

(3) The maximal compact subgroup K_0 does not split $\mathcal{M}(G)$ since trivial and Steinberg G-modules $\mathbb{C}$ and St lie over the same component $\Theta \subset \Theta(G)$, but $\mathbb{C}^{K_0} \neq 0$ while $St^{K_0} = 0$.

§4. Noetherian properties of $\mathcal{M}(G)$

4.1. Structure of category $\mathcal{M}(D)$ for a cuspidal component D.
Let $D \subset \Theta(G)$ be a cuspidal component. Fix $(\rho, V) \in D$. Denote by F the algebra of regular functions on algebraic variety $\Psi(G)$. It coincides with the group algebra of the lattice $L = G/G^0$ and hence has a natural structure of $G - F$-module. This module describes a universal $\psi(G)$-family of unramified characters of G since its specialization at a point $\psi \in \Psi(G)$ is $\mathbb{C}_\psi$.

We denote by $\Pi(\rho)$ the $G - F$-modules $\Pi(\rho) = F \bigotimes\limits_{\mathbb{C}} V$. As G-module $\Pi(\rho)$ does not depend on the choice of a point $\rho \in D$ (up to a noncanonical isomorphism). So we denote this G-module as $\Pi(D)$.

For every $\psi \in$ Stab $(\rho, \Psi(G))$ we choose an isomorphism $\alpha_{_F} : (\rho, V) \to (\psi\rho, V)$ and extend it to the automorphism of $\Pi(D)$ by $\alpha_\psi(v, f) = \alpha_\psi(v) \otimes \psi(f)$, where $\psi(f)$ is defined as $\psi(f)(\psi_1) = f(\psi^{-1}\psi_1)$.

Proposition. *Let $D \subset \Theta(G)$ be a cuspidal component, $(\rho, V) \in D$.*

(i) $\Pi(D)$ *is a finitely generated projective generator in the category* $\mathcal{M}(D)$.

(ii) $\operatorname{End}_G \Pi(D) = \bigoplus\limits_{\psi} F \cdot a_\psi$ *where* $\psi \in \operatorname{Stab}(\rho, \Psi(G))$.

Proof:

(i) Since $F = \operatorname{ind}_{G^0}^G(\mathbb{C})$, where $\mathbb{C}$ is the trivial G^0-module, $\Pi(D) = \operatorname{ind}_{G^0}^G(\rho|_{_{G^0}})$. Hence for every G-module E we have $\operatorname{Hom}_G(\Pi(D), E) = \operatorname{Hom}_{G^0}(V, E)$. If $E \in \mathcal{M}(D)$ its restriction to G^0 is completely reducible (see 2.1), i.e. the functor $E \mapsto \operatorname{Hom}_G(\Pi(D), E) = \operatorname{Hom}_{G^0}(V, E)$ is exact and faithful. Hence $\Pi(D)$ is a projective generator of $\mathcal{M}(D)$. Since G^0 is open in G, $\Pi(D)$ is finitely generated.

(ii) $\operatorname{Hom}_G(\Pi(D), \Pi(D)) = \operatorname{Hom}_{G^0}(V, F \otimes V) = = F \bigotimes\limits_{\mathbb{C}} \operatorname{Hom}_{G^0}(V, V)$, so the statement follows from 2.2.

Using ... we see that the category $\mathcal{M}(D)$ has a fairly simple description. Namely, put $\Lambda = \mathrm{End}_G(\Pi(D))^0$. Then $\mathcal{M}(D)$ is equivalent to the category $\mathcal{M}(\Lambda)$. The algebra Λ is a free module over the subalgebra F with generators a_ψ, i.e. $\Lambda = \bigoplus_\psi F \cdot a_\psi$ with $\psi \in \mathrm{Stab}(\rho, \Psi(G))$, and following relations

(a) $a_\psi f a_\psi^{-1} = \psi(f)$, $f \in F$.

(b) $a_\psi a_\chi = c(\psi, \chi) a_{\psi\chi}$, where $c(\psi, \chi) \in \mathbb{C}$ are some constants, defining a projective representation of $\mathrm{Stab}(\rho, \Psi(G))$ in V.

Corollary. *(i) The center $Z(\mathcal{M}(D))$ of the category $\mathcal{M}(D)$ is isomorphic to the algebra $Z(D) \subset F \subset \mathrm{End}(\Pi(D))$ of regular functions on D.*

(ii) Category $\mathcal{M}(D)$ is locally noetherian.

(iii) Every finitely generated G-module $E \in \mathcal{M}(D)$ is $Z(D)$ admissible.

Proof:

(i) Relations (a) - (b) show that $Z(D)$ coincides with the center of Λ. Using ... we see that it coincides with $Z(\mathcal{M}(D))$.

(ii) Since $D \approx \Psi(G)/\mathrm{Stab}(\rho, \Psi(G))$, F, and hence Λ, is a finitely generated $Z(D)$-module. Since $Z(D)$ is a noetherian algebra, the category $\mathcal{M}(\Lambda) \approx \mathcal{M}(D)$ is locally noetherian.

(iii) Since $\rho\big|_{G^0}$ is admissible (see 2.1), $\Pi(D)$ is F-admissible and hence $Z(D)$-admissible. Since any finitely generated G-module $E \in \mathcal{M}(D)$ is a quotient of $\Pi(D)^n$, $n \in \mathbb{Z}^+$, it is also $Z(D)$ admissible.

4.2. Noetherian properties of $\mathcal{M}(G)$.

Theorem. *Category $\mathcal{M}(G)$ is locally noetherian. Functors r and i map finitely generated modules into finitely generated ones.*

Proof:

Step 1. Functor r maps finitely generated modules into finitely generated ones. This easily follows from Iwasava decomposition (see [ ]).

Step 2. Let (M, D) be a standard cuspidal block, $V \in \mathcal{M}(D)$ be a finitely generated M-module. Then G-module $E = i_{GD}(V)$ is noetherian.

Let $\Theta = i_{GM}(D) \subset \Theta(G)$. Then $E \in \mathcal{M}(\Theta)$ (see ...). Since the functor r_{DG} is faithful and exact on $\mathcal{M}(\Theta)$ it is sufficient to check, that $r_{DG}(E)$ is noetherian. But by 2.4 $r_{DG}(E) = r_{DG} \circ i_{GD}(V)$ is glued from M-modules wV, $w \in W(D)$, each of which is noetherian by Proposition 4.1.

Step 3. Let E be a finitely generated G-module. Then it is noetherian. Indeed, by 2... E imbeds into $\bigoplus_{(M,D)} T_D E$. Since it is finitely generated, its image lies in a finite sum. Using Steps 1,2 we see that each G-module $T_D E = r_{GD} \circ r_{DG}(E)$ is noetherian, and hence E is noetherian.

Step 4. Let $N < G$, $V \in \mathcal{M}(N)$ be a noetherian M-module. Then $i_{GN}(V)$ is noetherian G-module.

Repeating arguments in Step 3 we see that V is contained in a finite sum $\bigoplus_{(M,D)} i_{ND} \circ r_{DN}(V)$. Hence $i_{GN}(V)$ is contained in a finite sum $\oplus i_{GD} \circ r_{DN}(V)$, which is noetherian by Steps 1,2.

Generalization. Let B be a commutative noetherian $\mathbb{C}$-algebra with identity. Then category $\mathcal{M}(G;B)$ is locally noetherian, and functors i, r map noetherian $G-B$-modules into noetherian ones.

Generalization. Let B be a commutative algebra with identity. Then $\mathcal{M}(D;B) \approx \mathcal{M}(\Lambda \bigotimes_{\mathbb{C}} B)$, $Z(\mathcal{M}(D,B)) = Z(D) \bigotimes_{\mathbb{C}} B$. If B is noetherian, then $Z(D) \bigotimes_{\mathbb{C}} B$ is noetherian, since $Z(D)$ is a finitely generated $\mathbb{C}$-algebra. This implies that $\mathcal{M}(D,B)$ is noetherian.

§5. Stabilization Theorem

5.1. Let $K \subset G$ be an open compact subgroup. For each $g \in G$ we put $h(g) = e_k g e_k \in \mathcal{H}_K$, where g stands for δ-distribution at g. In other words, $h(g)$ is the unique normalized bi-K-invariant measure, supported on KgK.

In some cases we have equalities $h(a^i) = h(a)^i$ for $i \geq 0$ or $h(ab) = h(a)h(b)$. (geometrically it means that $K^i gK = (KgK)^i$ and $KabK = KaKbk$ respectively). We want to describe some sufficient conditions for these equalities. Essentially these conditions mean that a, b are dominant with respect to some parabolic pair.

Definition. Let $(P, \overline{P})$ be a parabolic pair. We say that subgroup K is *in a good position* with respect to $(P, \overline{P})$ if

$$(*) \qquad K = K_- \Gamma K_+ \text{ ,where } K_- = K \cap \overline{U},\ \Gamma = K \cap M,\ K_+ = K \cap U \ .$$

Suppose $(P, \overline{P})$ and K are in a good position. We call element $a \in M$ dominant with respect to $(P, \overline{P}, K)$ if

$$(**) \qquad a^{-1} K_- a \subset K_- \ , \quad a \Gamma a^{-1} = \Gamma \ , \quad a K_+ a^{-1} \subset K_+ \ .$$

For each compact subgroup $C \subset G$ we denote by e_c the distribution on G, which is the image of the normalized Haar measure on c. If K is in a good position with respect to $(P, \overline{P})$, we have

$$e_K = e_{K_-} e_\Gamma e_{K_+} = e_{K_+} e_\Gamma e_{K_-} \ .$$

If a, b are dominant with respect to $(P, \overline{P}, K)$ we have $h(ab) = h(a)h(b)$. Indeed,

$$KaKbK = KaK_+ \Gamma K_- bK = K(aK_+ a^{-1})(a \Gamma a^{-1}) ab (b^{-1} K_- b) K = KabK \ .$$

Example. Let $A \subset Z(M_0)$ be the maximal split torus, $\Lambda = \mathrm{Hom}_{alg.gr.}(A, F^*)$ its character lattice, $\Sigma \subset \Lambda$ the root system of G and $\Sigma^+ \subset \Sigma$ the system of positive roots, corresponding to P_0. Put $A^+ = \{a \in A \mid |\alpha(a)| \leq 1$ for all $\alpha \in \Sigma^+\}$. Then there exist arbitrary small open compact subgroups $K \subset G$ (congruence subgroups) such that $(P_0, \overline{P}_0)$ and K are in a good position, and all elements $a \in A^+$ are dominant with respect to $(P_0, \overline{P}, K)$. In particular, $\mathcal{H}_K$ contains a very big commutative subalgebra $\mathcal{A} = \mathrm{span}\{h(a) | a \in A^+\}$.

In fact these congruence subgroups are in a good position with respect to each standard parabolic pair $(P, \overline{P})$ and all elements in $A^+ \cap Z(M)$ are dominant with respect to $(P, \overline{P}, K)$ (see [ ]).

5.2. To each element $g \in G$ naturally corresponds a parabolic pair. Namely, put $P_g = \{x \in G |$ the sequence $g^i x g^{-i}$, $i = 1, 2, \ldots,$ is bounded in $G\}$.

Statement. P_g is a parabolic subgroup of G, $\left(P_g, P_{g^{-1}}\right)$ is a parabolic pair.

For regular semisimple g the statement is proved in [c]. It is enough for our purposes.

Definition. Let $(P, \overline{P})$ be a parabolic pair. We say that an element $a \in M$ is *strictly dominant* with respect to $(P, \overline{P})$ if $(P, \overline{P}) = (P_a, P_{a^{-1}})$. Geometrically it means that operators $Ad\, a|_U$ and $Ad\, a^{-1}|_{\overline{U}}$ are strictly contractable and the family of operators $\{Ad\, a^i \mid i \in \mathbb{Z}\}$ is uniformly bounded on M.

Let $(P, \overline{P})$ and K be in a good position. We say that an element $a \in M$ is *strictly dominant with respect to* $(P, \overline{P}, K)$ if it is dominant and strictly dominant with respect to $(P, \overline{P})$.

Lemma. *(i) Let $g \in G$, $(P, \overline{P}) = (P_g, P_{g^{-1}})$. There exist arbitrary small open subgroups $K \subset G$ in a good position with respect to $(P, \overline{P})$ such that g is strictly dominant with respect to $(P, \overline{P}, K)$.*

(ii) Let K be in a good position with respect to $(P, \overline{P})$. There exist an element $a \in Z(M)$ strictly dominant with respect to $(P, \overline{P}, K)$.

Proof: Statement (i) is proved in [ ], (ii) is straightforward.

Fix an element strictly dominant with respect to $(P, \overline{P}, K)$ and consider increasing sequences of subgroups

$$U_n = a^{-n} K_+ a^n \subset U \quad , \quad \overline{U}_n = a^n K_- a^{-n} \subset \overline{U} \, .$$

When $n \to \infty$ these subgroups become arbitrary large, when $n \to -\infty$ they become arbitrary small.

Put $h = h(a)$. Using formulae in 5.1, we get for $n \geq 0$

$$h^n = e_K a^n e_K$$
$$e_K a^n = a^n e_{U_n} e_\Gamma e_{\overline{U}_{-n}}$$
$$h^n = e_K a^n e_K = a^n e_{U_n} e_K$$
and similarly
$$h^n = e_K e_{\overline{U}_n} a^n .$$

Proposition. *Let E be a G-module, E_U the space of U-coinvariants of E (see...) and $A : E \to E_U$ the natural M-equivariant projection. Denote by A_k the corresponding morphism $A_K : E^K \to E^U_\Gamma = (E_U)^\Gamma$. Then*

(i) $A_K h^n = a^n A_K$.

(ii) *For $\xi \in e^K$ $h^n\xi = 0$ iff $e_{U_n}\xi = 0$*

In particular

$$\operatorname{Ker} A_K = \bigcup_n \operatorname{Ker} e_{U_n} \mid_{E^K} = \left\{\xi \in E^K \mid h^n\xi = 0 \quad \textit{for larger}\right\} .$$

(iii) *If $\xi \in E$ is $\overline{U}_{-n}$-invariant, then $a^n e_\Gamma A\xi = Ae_K a^n\xi$. In particular, for each $\eta \in E^\Gamma_U$ $a^n\eta \in \operatorname{Im} A_K$ for large n, i.e. $\bigcup\limits_n a^{-n} \operatorname{Im} A_K = E^\Gamma_U$.*

Proof: Formula $h^n = a^n e_{U_n} e_K$ implies (i). Since the operator a on E^Γ_U is invertible, it also implies (ii). Using formula $a^n e_{U_n} e_\Gamma e_{\overline{U}_{-n}} = e_K a^n$ we see that $a^n e_\Gamma A\left(e_{\overline{U}_{-n}}\xi\right) = a^n A e_{U_n} e_\Gamma e_{\overline{U}_{-n}}\xi = Aa^n e_{U_n} e_\Gamma e_{\overline{U}_{-n}}\xi = Ae_K a^n\xi$ which proves (iii).

This proposition means, that space E^Γ_U together with operator g is naturally isomorphic to the localization of E^K with respect to operator h.

5.3. Stabilization Theorem. Let $(P, \overline{P})$ be a parabolic pair, $K \subset G$ an open compact subgroup, in a good position with respect to $(P, \overline{P})$. Denote by $C = C_K$ a constant in uniform admissibility theorem (see.....), i.e. a bound for $\dim L^K$ for $L \in IrrG$.

Let $a \in M$ be an element strictly dominant with respect to $(P, \overline{P}, K)$. Put $h = h(a) \in \mathcal{H}_K$. For each G-module E consider h as an endomorphism of E^K.

Stabilization theorem. (i) *For each G-module E there exists a unique decomposition $E^K = E^K_0 \oplus E^K_*$ into h-invariant subspaces such that $h^c E^K_0 = 0$ and h is invertible on E^K_*. Namely, $E^K_0 = \operatorname{Ker} h^n$, $E^K_* = \operatorname{Im} h^n$ for any $n \geq C$.*

(ii) *Let $C \subset U$, $\overline{C} \subset \overline{U}$ be sufficiently large open compact subgroups. Then for each G-module E*

$$E^K_0 = E^K \cap \operatorname{Ker} e_C \ , \quad E^K_* = e_K e_{\overline{C}} E \ .$$

In particular, E^K_0, E^K_* do not depend on the choice of a.

(iii) *Consider the natural morphism $A_K : E^K \to E^\Gamma_U$. Then $E^K_0 = \operatorname{Ker} A_K$, $A_K : E^K_* \to E^\Gamma_U$ is an isomorphism.*

Proof: Using formulas $h^n = a^n e_{U_n} e_K = e_K e_{\overline{U}_n} a^n$, we see that (i) implies (ii) for subgroups $C \supset U_n = a^{-n} K_+ a^n$, $\overline{C} \supset \overline{U}_n = a^n K_- a^{-n}$. Using proposition 5.2 we see that (i) implies (iii). Hence it is enough to prove (i).

Step 1. Let L be a $\mathbb{C}[x]$-module, i.e. a vector space with an endomorphism x. We say that L is *x-stable* if L has an x-invariant decomposition $L = L_0 \oplus L_*$ such that $xL_0 = 0$ and x is invertible on L_*. Clearly, L is x-stable $\iff L = \operatorname{Ker} x \oplus \operatorname{Im} x \iff \operatorname{Ker} x^2 = \operatorname{Ker} x$, $\operatorname{Im} x^2 = \operatorname{Im} x \iff x$ is invertible on $L/\operatorname{Ker} x \approx \operatorname{Im} x$.

It is easy to check that the direct sum of x-stable modules is x-stable and for each morphism $\alpha : L \to L'$ of x-stable $\mathbb{C}[x]$-modules $\operatorname{Ker} \alpha$ and $\operatorname{Coker} \alpha$ are x-stable $\mathbb{C}[x]$-modules.

Step 2. Denote by $\mathcal{M}' \subset \mathcal{M}(G)$ the subcategory of G-modules E such that E^k is h^C-stable. We have to show that $\mathcal{M}' = \mathcal{M}(G)$.

As follows from Step 1 direct sums of modules in $\mathcal{M}'$ and kernels and cokernels of morphisms of modules in $\mathcal{M}'$ lie in $\mathcal{M}'$.

Also, $\mathcal{M}'$ contains all irreducible G-modules. Indeed, for each irreducible G-module L $\dim L^K \leq C$, and hence the sequence of subspaces $\operatorname{Im} h^i$ is constant for $i \geq C$, i.e. h is invertible on $\operatorname{Im} h^C$.

Step 3. Let B be a commutative noetherian $\mathbb{C}$-algebra, E B-admissible $\sigma - B$-module. Suppose that $r_{MG}(E)$ is B-admissible $M - B$-module. Then for some $n > 0$ E^K is h^n-stable.

Indeed, since E^K is noetherian B-module, the sequence of submodules $\operatorname{Ker} h^n$ is stable. By proposition....$\operatorname{Ker} A^K = \bigcup_n \operatorname{Ker} h^n$, and hence $\operatorname{Ker} A_K = \operatorname{Ker} h^n$ for some $n > 0$.

By proposition..... E_U^Γ is a union of B-submodules $a^{-n} \operatorname{Im} A_k$. Since E_U^Γ is finitely generated B-module it is equal to $a^{-\Gamma} \operatorname{Im} A_K$ for some $\Gamma > 0$. Since a is invertible on E_U^r we see that $E_U^\Gamma = \operatorname{Im} A_k = E^k/\operatorname{Ker} A_K$.

Thus the operator h is invertible on $E^K/\operatorname{Ker} A_k = E^K/\operatorname{Ker} h^n$, which implies that E^K is h^n stable.

Step 4. Let (N, D) be a standard cuspidal block, $(\rho, V) \in D$, $\Pi(D) = F \otimes V$ be $G-F$-module described in... Put $(\Pi, E) = i_{G_M}(\Pi(D))$. Then for some $n > 0$ E^K is h^n-stable.

It is sufficient to check that E and $r_{MG}(E)$ are F-admissible modules. By composition theorem $r_{MG}(E)$ is glued from M-modules $i_{MM_w} \circ w(\Pi(D))$. Hence $\mathcal{F}$-admissibility of E and $r_{MG}(E)$ follows from the following.

Lemma. *The functor $i_{GM} : \mathcal{M}(A, B) \to \mathcal{M}(G, B)$ maps B-admissible modules into B-admissible ones.*

This lemma is an immediate consequence of lemma...

Step 5. Module (Π, E) is step 4 which lies in $\mathcal{M}'$, i.e. E^K is h^C-stable. Indeed, it is sufficient to check that $\operatorname{Ker} h^n \subset \operatorname{Ker} h^C$. Let $\xi \in \ker h^n$, $\xi' = h^C \xi$. For each $\psi \in \Psi(M)$ consider specialization morphism $\Pi(D) \to \psi\rho$ and the corresponding morphism $\alpha_\psi : E \to E_\psi = i_{GM}(\psi\rho)$.

Lemma. (see [ ]) *For generic ψ G-module E_ψ is irreducible.*

This lemma implies that for generic ψ $E_\psi \in \mathcal{M}'$. Since $h^n \alpha_\psi(\xi) = 0$, this implies that $\alpha_\psi(\xi') = h^C \alpha_\psi(\xi) = 0$ and hence $\xi' = 0$.

Step 6. Let (N, D) be a standard cuspidal block. Then $i_{GN}(\mathcal{M}(D)) \subset \mathcal{M}'$.

Let $\sigma \in \mathcal{M}(D)$. Since $\Pi(D)$ is a projective generator in $\mathcal{M}(D)$ we can represent σ as a cokernel of some morphism $\gamma : \oplus_\alpha \Pi(D) \to \oplus_\beta \Pi(D)$. Then $i_{GN}(\sigma) = \operatorname{Coker}(\oplus_\alpha \Pi \to \oplus_\beta \Pi)$ (since functor i_{G_N} is exact and preserves direct sums). Since $\Pi \in \mathcal{M}'$ Step 2 implies that $\sigma \in \mathcal{M}'$.

Step 7. Each G-module E lies in $\mathcal{M}'$. Indeed, we can embed E into module $E' = \oplus_{(N,D)} i_{GD} \circ r_{DG}(E)$ as in..... By Step 6 $E' \in \mathcal{M}'$. Similarly we embed E'/E into $E'' \in \mathcal{M}'$. Then $E = \ker(E' \to E'')$ lies in $\mathcal{M}'$ by step 2.

5.4. Corollaries and Remarks to the Stabilization Theorem.

Generalized Jacquet Lemma. *Let K be in a good position with respect to $(P, \overline{P})$. Then for each G-module E the morphism $A_K : E^K \to E_U^\Gamma$ is an epimorphism. Moreover, it has a right inverse morphism B, functorial in E, i.e. E_U^Γ can be realized in a natural way as a direct summand of E^K.*

Corollary. *Functor r_{MG}^P maps B-admissible $G-B$-modules into B-admissible $M-B$-modules.*

We will prove more a general result.

Let B be a commutative $\mathbb{C}$-algebra with identity. Fix a class of objects $C \subset \mathcal{M}(B)$ closed with respect to isomorphisms, finite direct sums and taking of direct summands (i.e. for $x \oplus y \approx Z,\ Z \in C$ iff $X, Y \in C$). Examples: C is the class of finitely generated B-modules, or the class of projective B-modules, or the class of flat B-modules and so on. We say that $G-B$-module E is of C-type if for each open compact subgroup $K \subset G$ B-module E^K lies in C.

Proposition. *Fix a class $C \subset \mathcal{M}(B)$ as above. Then functors $i_{GM}^P : \mathcal{M}(M, B) \to \mathcal{M}(G, B)$ $r_{MG}^P : \mathcal{M}(G, B) \to \mathcal{M}(M, B)$ map C-type modules into C-type modules.*

Proof: For functor i_{GM} this follows from lemma..... Let E be a $G-B$-module of type C and $\Gamma_0 \subset M$ an open compact subgroup. Choose an open compact subgroup $K \subset G$ in a good position with respect to $(P, \overline{P})$ such that $\Gamma = K \cap M \subset \Gamma_0$. Then $E_U^{\Gamma_0}$ is a direct summand of E_U^Γ, which is a direct summand of E^K. Hence B-module $E^{\Gamma_0} U$ lies in C, which proves the proposition for functor r_{MG}.

Remark. 1. Consider the decreasing sequence of right ideals $J_n = h^n \mathcal{H}_K \subset \mathcal{H}_K$. Applying stabilization theorem to G-module $\mathcal{H}(G)e_K$ we see that it is stable, namely

$$(*) \qquad J_n = J_C \quad \text{for} \quad n \geq C\ .$$

In fact this statement is equivalent to the theorem. Indeed, it implies that $\operatorname{Im} h^n = \operatorname{Im} h^C$ for each G-module E. Using the natural anti-involution of $\mathcal{H}(G)$, given by the antiautomorphism $g \mapsto g^{-1}$ on G, we can deduce from (*) that $\mathcal{H}_K h^n = \mathcal{H}_K h^C$ for $n \geq C$, which implies that $\operatorname{Ker} h^n = \operatorname{Ker} h^C$.

Note, that (*) is purely geometrical statement, which has nothing to do with the representation theory. It would be very interesting to find a direct geometrical proof of (∗). Such proof would probably give a reasonably precise estimate for constant C in $(*)$. I was able to find such proof for congruence subgroups in $\mathrm{GL}(Z)$, but not for higher rank. Another form of the statement $(*)$, which does not involve the choice of a, is $(**)$ For sufficiently large open compact subgroups $C \subset U$ the ideal $J_C = e_K e_C \mathcal{H}(G) e_K$ does not depend on C. Namely, this is true for $C \supset a^{-C} K_+ a^C$.

5.5. An Effective Bound of the Number of Cuspidal Components With a Given Conductor.

Fix an open compact subgroup $K \subset G$. We want to give an effective bound of the number of cuspidal components $D \subset \Theta_K(G)$.

Let E-be a G-module, $\xi \in E^K, \widetilde{\xi} \in \widetilde{E}^K$. We denote by $\varphi_{\widetilde{\xi},\xi}$ the matrix coefficient $\varphi_{\widetilde{\xi},\xi}(g) = \langle \widetilde{\xi}, g\widetilde{\xi} \rangle$.

Proposition. *There exists a compact subset $S \subset G^\circ$, which can be effectively described in terms of G and K, such that for each quasicuspidal G-module E, $\xi \in E^K$, $\widetilde{\xi} \in \widetilde{E}^K$ the matrix coefficient $\varphi_{\overline{\xi},\xi}$ vanishes on $G^\circ \backslash S$.*

This proposition gives a desired bound. Indeed, let $D_1, \ldots, D_r$ be different cuspidal components in $\Theta_K(G)$, $V_i \in D_i$, $0 \neq \xi_i \in V_i^K$, $0 \neq \widetilde{\xi}_i \in V_i^K$, $\varphi_i = \varphi_{\overline{\xi}_i,\xi_i}$ for $i = 1, \ldots, r$. By 2.. matrix coefficients φ_i are linearly independent on G°. Since they vanish on $G^\circ \backslash S$ and are K-biinvariant, their number r is less or equal to $\#(K \backslash S / K)$.

Proof of Proposition. Let $A \subset Z(M_0)$ be the maximal split torus, L the lattice of coweights of A, which we will identify with the quotient $L = A/A^\circ$ of A by its maximal compact subgroup. Let $L^\circ = L \cap G^\circ$ be the semisimple part of L, $L^{\circ +} = L^\circ \cap A^+$, where A^+ is defined in example 5.1. In other words, $L^{\circ +} = \{a \in L \mid (\alpha, a) \leq 0 \text{ for all } \alpha \in \Sigma^+\}$ is the Weyl chamber, corresponding to P_0.

Let us fix a homomorphism $L \to A$, inverse to the projection $A \to L$, and using it identifies L with a subgroup of A. By Cartan decomposition there exists a compact subset $\Omega \subset G^\circ$ such that $G^\circ = \Omega^{-1} L^{\circ +} \Omega$.

Choose a congruence subgroup K', which lies in the open subset $\bigcap_{x \in \Omega} xKx^{-1}$ and denote by $C = C_{K'}$ the constant in uniform admissibility theorem for K'. Put $S^\circ = L^{\circ +} \backslash \left[L^{\circ +} + c(L^{\circ +} \backslash 0) \right]$, $S = \Omega^{-1} S^\circ \Omega$. We claim that S is a desired subset. First of all, since $L^{\circ +}$ is a strictly convex cone, set S° is finite, i.e.., S is compact. Let E be a quasicuspidal G-module, $\overline{\xi} \in E^K$, $\xi \in E^K$, $g \in G^\circ \backslash S$. We want to show that $\varphi_{\overline{\xi},xi}(g) = 0$. By definition $g = x^{-1} a' y$, where $x, y \in \Omega$, $a' \in L^{\circ +}$ is of the form $a' = b + ca$, $b \in L^{\circ,+}$, $a \in L^{\circ +} \backslash 0$. Put $h(a) = e_{K'} a e_{K'}$ and similarly for a', b. Since $a \in L^{\circ +} \backslash 0$ the corresponding parabolic subgroup P_a differs from G, i.e. $r_{MG}^P(E) = 0$. Hence for each vector $\eta \in E$ $h(a)^n \eta = 0$ for large n and by the stabilization theorem, $h(a)^C \eta = 0$. Hence

$$\varphi_{\widetilde{\xi}\xi}(g) = \varphi_{x\widetilde{\xi},y\xi}(a') = \left(x\widetilde{\xi}, a' y\xi \right) = \left(x\widetilde{\xi}, h(a') y\xi \right) = \left(x\widetilde{\xi}, h(b) h(a)^C y\xi \right) = 0$$

Here we used that vectors $x\widetilde{\xi}$ and $y\xi$ are K'-invariant. Formula $h(a') = h(b)h(a)^C$ follows from 5.1. Note, that addition in L becomes multiplication, when L is considered as a subgroup of G.

Remark. All bounds we described are effective, but quite excessive. The most excessive is the estimate for the constant $C = C_K$ in the proof of uniform admissibility theorem. It would be interesting to find more precise bounds.

§6. Main Theorems About Functors Randi

6.1. Pairing Between $\widetilde{E}_{\overline{U}}$ and E_U.

Let $(P, \overline{P})$ be a parabolic pair. For each G-module E denote by $\widetilde{E}$ the contragredient G-module and consider M-modules $\widetilde{E}_{\overline{U}} = (\widetilde{E})_{\overline{U}}$ and E_U.

Theorem. *There exists a unique pairing $\{\quad\} : \widetilde{E}_{\overline{U}} \times E_U \to \mathbb{C}$ satisfying the following condition on the asymptotic of matrix coefficients.*

(ASS) *Let $K \subset G$ be an open compact subgroup, $a \in M$ be an element strictly dominant with respect to $(P, \overline{P})$. Then there exists n_0, depending only on a and K, such that for each $\widetilde{\xi} \in \widetilde{E}$, $\xi \in E$, $i > n_0$ $(\overline{\xi}, a^i\xi) = \{\overline{A}\widetilde{\xi}, A^i\xi\}$ (here $\overline{A} : \widetilde{E} \to \widetilde{E}_{\overline{U}}$, $A : E \to E_U$ are natural projections).*

The pairing $\{\quad\}$ is M-equivariant, functorial in E and it gives an isomorphism of M-modules $\widetilde{E}_{\overline{U}} \xrightarrow{\sim} (E_U)^\sim$.

Corollary. *There exists a canonical functorial isomorphism $r^{\overline{P}}_{MG}(\widetilde{E}) \approx \left(r^P_{MG}(E)\right)^\sim$. In particular, for a standard Levi subgroup $M < G$ $\overline{r}_{MG}(\widetilde{E}) = r_{MG}(E)^\sim$.*

Proof: Indeed, by definition $r^{\overline{P}}_{MG}(\widetilde{E}) = \widetilde{E}_{\overline{U}} \otimes \Delta^{1/2}_{\overline{U}}, r^P_{MG}(E) = E_U \otimes \Delta^{1/2}_U$. Since $\Delta_{\overline{U}}$ and Δ_U are canonically dual (see appendix.....), the theorem implies the corollary.

6.2. Proof of Theorem 6.1.

Step 1. Let $K \subset G$ be an open compact subgroup in a good position with respect to $(P, \overline{P})$, $\Gamma = K \cap M$. First let us define the pairing $\{\quad\} : \widetilde{E}^r_{\overline{U}} \times E^r_U \to \mathbb{C}$. By the stabilization theorem $A_K : E^K_* \to E^\Gamma_U$ is an isomorphism, so we can identify E^Γ_U with a subspace $E^K_* \subset E^K$. Applying the stabilization theorem to the parabolic pair $(\overline{P}, P)$, subgroup K and G-module $\widetilde{E}$ we can identify $\widetilde{E}^\Gamma_{\overline{U}}$ with the subspace $\widetilde{E}^K_* \subset \widetilde{E}^K$. Then the restriction of the pairing $(\quad,\quad) : \widetilde{E}^K \times E^K \to \mathbb{C}$ defines a pairing $\{\quad\} : \widetilde{E}^\Gamma_{\overline{U}} \times E^\Gamma_U \to \mathbb{C}$.

Step 2. Choose an element $a \in M$ strictly dominant with respect to $P, \overline{P}, K$ (see 5.2) and put $h = h(a)$, $h^* = h(a^{-1})$. For each $\widetilde{\xi} \in \widetilde{E}^K$, $\xi \in E^K$ we have

$$(\widetilde{\xi}, a^n\xi) = (\widetilde{\xi}, h^n\xi) = (\widetilde{\xi}, h^n\xi) = \left((h^*)^n\widetilde{\xi}, \xi\right) .$$

Using stability theorem, we see that for $n > C_K(\widetilde{\xi}, a^a\xi)$ depends only on projections of $\widetilde{\xi}$ on $\widetilde{E}^K_*$ and of ξ and E^K_*. This shows that the pairing $\{\quad\}$ satisfies condition (ASS) for a and K. Since h is invertible on E^K_*, $\{\quad\}$ is uniquely determined by condition (ASS).

Step 3. Let $K' \subset K$ be a smaller subgroup, such that a is strictly dominant with respect to $P, \overline{P}, K'$. Consider the corresponding pairing $\{\ \ \}:' \widetilde{E}_{\overline{U}}^{\Gamma'} \times E_U^{\Gamma'} \to \mathbb{C}$. It satisfies (ASS) and by uniqueness property of $\{\ \ \}$ the restriction of $\{\ \ \}'$ to $\widetilde{E}_{\overline{U}}^{\Gamma} \times E_U^{\Gamma}$ coincides with $\{\ \ \}$. Hence, choosing smaller and smaller subgroups K, we can define a pairing $\{\ \ \} : \widetilde{E}_{\overline{U}} \times E_{\overline{U}} \to \mathbb{C}$ satisfying (ASS), and this pairing is unique. By construction the pairing $\{\ \ \}$ does not depend on a. This implies that it is M-equivariant.

Step 4. For each subgroup K the space $\widetilde{E}^K$ is dual to E^K and the operator h^* in $\widetilde{E}^K$ is dual to the operator h in E^K. Hence $\widetilde{E}_*^K$ is dual to E_*^K. By definition of $\{\ \ \}$ $\widetilde{E}_{\overline{U}}^{\Gamma} \approx \widetilde{E}_*^K$ is dual to $E_U^{\Gamma} \approx E_*^K$, which implies that $\{\ \ \}$ gives an isomorphism of $\widetilde{E}_{\overline{U}}$ with module $(E_U)^{\sim}$ contragredient to E_U.

6.3. Completion of σ-Modules. We want to describe the pairing $\{\ \ \}$ in a more direct and visual way, using the notion of completion of G-modules.

Definition. Let E be a G-module. We define its completion $E^\wedge$ in any of three equivalent ways

(i) $E^\wedge = \mathrm{Hom}_G(\mathcal{H}(G), E)$.

(ii) $E^\wedge = \varprojlim_{K} E^K$, where the inverse limit is over all open compact subgroups $K \subset G$ and for $K' \subset K$ the connecting morphism $E^{K;} \to E^K$ is given by $\xi \mapsto e_K \xi$.

(iii) $E^\wedge$ is the completion of E in the topology, generated by open subset $\mathrm{Ker}\, e_K$ for open compact subgroups $K \subset G$.

The algebra $D_C(G)$ of compactly supported distributions on G acts on the completion $E^\wedge$ by $d\xi^\wedge(h) = \xi^\wedge(h * d)$. This action is continuous in the topology, described in (iii) and its restriction to $E \subset E^\wedge$ coincides with the natural action of $D_C(G)$ on E. In particular, G acts on $E^\wedge$, but this representation usually is not smooth. The smooth part of $E^\wedge$ coincides with $E = \mathcal{H}(G)E^\wedge$.

It is easy to check that the functor $E \mapsto E^\wedge$ is exact and faithful. Moreover, if $E' \subset E$, then $(E')^\wedge = \mathrm{Closure}\, E'$ in $E^\wedge = \{\xi^\wedge \in E^\wedge \mid \mathcal{H}(G)\xi^\wedge \subset E' \subset E\}$.

It is easy to check that $(\widetilde{L})^\wedge \approx L^*$ (the dual space). This gives the following realization of $E^\wedge$, convenient for computations:

Let us realize E as a submodule of $\widetilde{L}$ for some G-module L and then $E^\wedge$ can be described as

$$E^\wedge = \{\xi^* \in L^* \mid \mathcal{H}(G)\xi^* \subset E \subset \widetilde{L}\} .$$

6.4.

Theorem. *Let $(P, \overline{P})$ be a parabolic pair, E a G-module. Then there exists a canonical isomorphism*

$$\overline{A} : (E^\wedge)^U \tilde{\to} (E_{\overline{U}})^\wedge$$

where $(E^\wedge)^U$ is the space of U-invariants in $E^\wedge$. For each $\xi^\wedge \subset (E^\wedge)^U$ the vector $\eta^\wedge = \overline{A}\xi^\wedge$ is uniquely characterized by the following property.

(*) *For each subgroup $K \subset G$ in a good position with respect to $(P, \overline{P})$ $\overline{A}e_K\xi^\wedge = e_r\eta^\wedge$.*

This theorem allows us to give another description of the pairing $\{\quad\}$ in theorem 6.1. Namely, applying it to G-module $\widetilde{E}$ we see that $((\widetilde{E})^\wedge)^U = (E^*)^U = (E_U)^*$ is canonically isomorphic to $(\widetilde{E}_{\overline{U}})^\wedge$. Hence $\widetilde{E}_{\overline{U}} =$ smooth part of $(\widetilde{E}_{\overline{U}})^\wedge =$ smooth part of $(E_U)^* = (E_U)^\sim$, which is the statement of theorem 6.1.

Proof of the Theorem.

Step 1. Let $K' \subset K \subset G$ be open compact subgroups in a good position with respect to $(P, \overline{P})$. Then for each $\xi' \in E_*^{K'}$ $e_K \xi' \in E_*^K$ and $A e_K \xi' = e_r A \xi'$. Indeed, let $C \subset U$ be a very large open compact subgroup, $L = e_C E$. By stabilization theorem (applied to $\overline{P}, P, K$) $E_*^K = e_K L$ and $E_*^{K'} = e_{K'} L$, which implies that $E_*^K = e_K E_*^{K'}$. Moreover, for each $\eta \in L$ $A(e_K \eta) = A(e_{K_+} e_r e_{K_-} \eta) = e_r A(e_{K_-} \eta) = e_r A(\eta)$ and similarly for K'. Hence if $\xi' = e_{K'} \eta$, we have $A(e_K \xi') = e_\Gamma A(\eta) = e_\Gamma (e_{\Gamma'} A(\eta)) = e_\Gamma A(\xi')$.

Step 2. Consider the inverse system $\{e_K\}$ where K runs through all good subgroups (i.e. open compact subgroups in a good position with respect to $(P, \overline{P})$). Step 1 shows that $\{E_*^K\}$ form a subsystem in $\{E_K\}$ and $\overline{A} : E_*^K \simeq E_{\overline{U}}^\Gamma$ gives an isomorphism of this subsystem with the system $\{E_{\overline{U}}^\Gamma\}$. This allows us to identify $(E_{\overline{U}})^\wedge = \varprojlim_K E^\Gamma)\overline{U}$, with the subspace $E_*^\wedge = \varprojlim_K (E_*^K) \subset \varprojlim_K (E^K) = E^\wedge$. Clearly $E_*^\wedge = \{\xi^\wedge \in E^\wedge \mid e_K \xi^\wedge \subset E_*^K$ for all good $K\}$.

Step 3. Let us prove that $E_*^\wedge = (E^\wedge)^U$. Indeed $P\xi^\wedge \in E_*^\wedge \iff$ for all good K $e_K \xi^\wedge \in E_*^K P \iff$ for all good K and all open compact subgroups $C \subset U$ $e_K \xi^\wedge \in e_K e_c E\mathbb{P} \iff$ for all $C \subset V$ and all good K $e_K \xi^\wedge \in e_K e_C E P \iff$ for all $C \subset U$, $\xi^\wedge$ lies in the closure of $e_c E \iff$ for all $C \subset U$, $e_C \xi^\wedge = \xi^\wedge$.

This last condition implies that $\xi^\wedge$ is U-univariant. Conversely, suppose that $\xi^\wedge$ is U-invariant and prove that for each $C \subset U$ $e_C \xi^\wedge = \xi^\wedge$. Choose a small subgroup $K \subset G$ normalized by C. Then the vector $\xi = e_K \xi^\wedge$ is C-invariant which implies that $e_C \xi = \xi$. Hence $e_K e_C \xi^\wedge = e_C e_K \xi^\wedge = e_C \xi = \xi = e_K \xi^\wedge$. Since this is true for arbitrary small K, $e_C \xi^\wedge = \xi^\wedge$.

6.5. Second Adjointness of Functors i and r.

Theorem. *Let $(P, \overline{P})$ be a parabolic pair, $M = P \cap \overline{P}$. Then the functor $i_{GM}^P : \mathcal{M}(M) \to \mathcal{M}(G)$ is canonically left adjoint to the functor $r_{MG}^{\overline{P}} : \mathcal{M}(G) \to \mathcal{M}(M)$. In particular, for a standard Levi subgroup $M < G$ the functor i_{GM} is left adjoint to $\overline{r}_{MG}$.*

This theorem follows from Theorem 6.4 and the following form of Frobenius reciprocity.

Proposition. *Let G be an ℓ-group (see...), $H \subset G$ a closed subgroup. Define the induction functor* $\mathrm{ind} : \mathcal{M}(H) \to \mathcal{M}(G)$ *as in ([ ]), i.e., for $V \in \mathcal{M}(H)$ we define G-module $E = \mathrm{ind}(G, H, V)$ as*

$E = \{f : G \to V \mid f(hg) = hf(g)$ for $h \in H$, support of f is compact modulo H and f is locally constant$\}$.

Define the twisted induction functor $\mathrm{ind}^\Delta(V)) = \mathrm{ind}\left(V \otimes \Delta_G \Delta_H^{-1}\right)$. *Then for each $V \in \mathcal{M}(H)$, $E \in \mathcal{M}(G)$ there is a canonical functorial isomorphism*

$$\mathrm{Hom}_G\left(\mathrm{ind}^\Delta(V), E\right) = \mathrm{Hom}_G(V, E^\wedge) .$$

In other words, the functor ind^{Δ} *is left adjoint to the functor* S*, given by* $S(E) =$ H*-smooth part of* $E^{\wedge}$.

Proof of Proposition. Let $S(G)$ be the space of locally constant compactly supported functions on G with left action of G. We have a canonical isomorphism $\mathcal{H}(G) = S(G) \otimes \Delta_G (f \otimes \mathcal{M} \to f \cdot \mathcal{M})$. We will identify $S(G)$ with $\mathrm{ind}(G, 1, \mathbb{C})$ (since G acts on $\mathrm{ind}(\mathbb{C})$ from the right, this identification involves change $g \mapsto g^{-1}$). By transitivity of induction we have $\mathrm{ind}(G, H, S(H)) = S(G)$.

This implies, that

$$\mathrm{ind}^{\Delta}(\mathcal{H}(H)) = \mathrm{ind}(S(H) \cdot \Delta_G \cdot \Delta_H^{-1} \cdot \Delta_H) = \Delta_G \cdot \mathrm{ind}(S(H)) = \Delta_G \cdot S(G) = \mathcal{H}(G) \, .$$

Since ind^{Δ} is an exact functor, preserving direct sums and $\mathcal{H}(H)$ is a projective generator of $\mathcal{M}(H)$, $\mathrm{ind}^{\Delta}(V) = \mathcal{H}(G) \bigotimes_{\mathcal{H}(H)} V$. This implies, that

$$\begin{aligned} \mathrm{Hom}_G(\mathrm{ind}^{\Delta}(V), E) = \mathrm{Hom}_G \left(\mathcal{H}(G) \bigotimes_{\mathcal{H}(H)} V, E \right) = \\ = \mathrm{Hom}_H \left(V, \mathrm{Hom}_G(\mathcal{H}(G), E) \right) = \mathrm{Hom}_H(V, E^{\wedge}) = \\ = \mathrm{Hom}_H(V, S(E)) \, . \end{aligned}$$

All isomorphisms above are canonical.

Remark. Let us describe explicitly morphism $\alpha : V \to \mathrm{ind}^{\Delta}(V)^{\wedge}$, corresponding to identity morphism of $\mathrm{ind}^{\Delta}(V)$. For $v \in V$ we define $\alpha(v) \in \mathrm{ind}^{\Delta}(V)^{\wedge}$ by condition, that for each open compact subgroup $K \subset G$ the function $f_K = e_K \alpha(v) \in \mathrm{ind}^{\Delta}(V)$ has the following form and vanishes outside of HK and

$$f(hK) = h e_{H \cap K} v \otimes \mathcal{M}_G \otimes \mathcal{M}_H^{-1} \left(\mathcal{M}_G(K)^{-1} \mathcal{M}_H(H \cap K) \right) \, .$$

where $\mathcal{M}_G \in \Delta_G, \mathcal{M}_H \in \Delta_H$.

Proof of the Theorem. Let $V \subset \mathcal{M}(M), E \in \mathcal{M}(G)$. Using canonical isomorphisms $\Delta_G \Delta_P^{-1} = \Delta_U^{-1}$ and $\Delta_U^{-1} = \Delta_{\overline{U}}$ we have

$$\begin{aligned} \mathrm{Hom}_G \left(i_{GM}^{P}(V), E \right) = \mathrm{Hom}_G \left(\mathrm{ind}^{\Delta}(G, P, V \otimes \Delta_U^{1/2}) E \right) = \\ = \mathrm{Hom}_P \left(V \otimes \Delta_U^{1/2}, E^{\vee} \right) = \mathrm{Hom}_M \left(V \otimes \Delta_U^{1/2}, (E^{\vee})^U \right) = \\ = \mathrm{Hom}_M \left(V, E_{\overline{U}} \otimes \Delta_{\overline{U}}^{1/2} \right) = \mathrm{Hom}_M \left(V, r_{MG}^{\overline{P}} E \right) \, . \end{aligned}$$

Remark. Let us write explicitly morphism $\alpha : V \to \overline{r}_{MG} i_{GM} V$. Let $v \in V$. Choose a subgroup K, in a good position with respect to $(P, \overline{P})$, such that $e_r v = v$. Then $\alpha(v)$ is represented by $\mathcal{M}_{jf}^{1/2}$, where $f : G \to V \otimes_{\Delta} U^{-1/2}$ is supported on PK and for $k \in K$ $f(K) = v \mathcal{M}_U^{-1/2} \ \mathcal{M}_G^{-1}(K) \mathcal{M}_P(K \cap P)$.

Here $\mathcal{M}_U \in D_U, \mathcal{M}_{\overline{U}} \in \Delta_{\overline{U}}$ are dual and $\mathcal{M}_G = \mathcal{M}_{\overline{U}} \cdot \mathcal{M}_P$. In particular, $\mathcal{M}_G^{-1}(K) \ \mathcal{M}_P(K \cap P) = \mathcal{M}_{\overline{U}}^{-1}(K_-)$. Identifying $\mathcal{M}_U^{-1/2}$ with $\mathcal{M}_{\overline{U}}^{1/2}$ we can write

$$\int\limits_{\overline{U}} \alpha(v) = \left(\int_{K_- v \cdot \mathcal{M}_{\overline{U}}} \right) \mathcal{M}_{\overline{U}}^{-1}(K_-) = v \ .$$

This shows that α coincides with the morphism in the composition theorem, corresponding to the big cell $P\overline{P}$ and the point $w = 1 \in P\overline{P}$ (see.....).