CONSTRUCTING DE RHAM-WITT COMPLEXES ON A BUDGET (DRAFT VERSION)

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Contents

1. Introduction

1.1. **The de Rham-Witt complex.** Let X be a smooth algebraic variety defined over a field k. The *algebraic de Rham cohomology* $\mathrm{H}^*_{\mathrm{dR}}(X)$ is defined as the hypercohomology of the de Rham complex

$$\Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots.$$

When $k = \mathbf{C}$ is the field of complex numbers, Grothendieck [11] showed that the algebraic de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X)$ is isomorphic to the usual de Rham cohomology of the underlying complex manifold $X(\mathbf{C})$ (and therefore also to the singular cohomology of the topological space $X(\mathbf{C})$, with complex coefficients). However, over fields of characteristic $p > 0$, algebraic de Rham cohomology is a less satisfactory invariant. This is due in part to the fact that it takes values in the category of vector spaces over k, and therefore consists entirely of p-torsion. To address this point, Berthelot and Grothendieck introduced the theory of *crystalline cohomology*. To every smooth algebraic variety over a perfect field k of characteristic p, this theory associates cohomology groups $\mathrm{H}^*_{\mathrm{crys}}(X)$ which are modules over the ring of Witt vectors $W(k)$, and can be regarded as integral versions of the de Rham cohomology groups $\mathrm{H}^*_{\mathrm{dR}}(X)$.

The crystalline cohomology groups of a variety X were originally defined as the cohomology groups of the structure sheaf of a certain ringed topos, called the *crystalline topos* of X. However, Bloch [6] (in the case of small dimension) and Deligne-Illusie [15] later gave an alternative description of crystalline cohomology, which is closer in spirit to the definition of algebraic de Rham cohomology. More precisely, they showed that the crystalline cohomology of a smooth variety X can be described as the hypercohomology of a complex of sheaves

$$\mathcal{W}\Omega^0_X \xrightarrow{d} \mathcal{W}\Omega^1_X \xrightarrow{d} \mathcal{W}\Omega^2_X \xrightarrow{d} \cdots$$

called the *de Rham-Witt complex of* X. The complex $\mathcal{W}\Omega^*_X$ is related to the usual de Rham complex by the following pair of results:

Theorem 1.1.1. *Let X be a smooth algebraic variety defined over a perfect field k. Then there is a canonical map of cochain complexes of sheaves $\mathcal{W}\Omega^*_X \to \Omega^*_X$, which induces a quasi-isomorphism $\mathcal{W}\Omega^*_X / p\mathcal{W}\Omega^*_X \to \Omega^*_X$.*

Theorem 1.1.2. *Let $\mathfrak{X}$ be a smooth formal scheme over $\operatorname{Spf} W(k)$ with central fiber $X = \operatorname{Spec} k \times_{\operatorname{Spf} W(k)} \mathfrak{X}$. Moreover, suppose that the Frobenius map $\varphi_X : X \to X$ extends to a map of formal schemes $\varphi_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}$. Then there is a natural quasi-isomorphism $\Omega^*_{\mathfrak{X}} \to \mathcal{W}\Omega^*_X$ of chain complexes of abelian sheaves on the underlying topological space of X (which is the same as the underlying topological space of $\mathfrak{X}$). Here $\Omega^*_{\mathfrak{X}}$ denotes the de Rham complex of the formal scheme $\mathfrak{X}$ relative to $W(k)$.*

Warning 1.1.3. In the situation of Theorem 1.1.2, the map of chain complexes $\Omega^*_{\mathfrak{X}} \to \mathcal{W}\Omega^*_X$ depends on the choice of map $\varphi_{\mathfrak{X}}$, though the induced map on cohomology groups is independent of that choice.

The proofs of Theorems 1.1.1 and 1.1.2 which appear in [15] depend on some rather laborious calculations. Our aim in this note is to present an alternate construction of the complex $\mathcal{W}\Omega^*_X$ (which agrees with the construction of Deligne and Illusie for smooth varieties; see Theorem 4.4.12) which can be used to prove Theorems 1.1.1 and 1.1.2 in an essentially calculation-free way: the only input we will need is the Cartier isomorphism (which we recall as Theorem 3.3.5) and some elementary homological algebra.

1.2. **Overview.** What is the essence of the construction of $\mathcal{W}\Omega^*_X$, and what does our paper offer? Suppose for simplicity $X = \operatorname{Spec} R$ is an affine scheme, smooth over the perfect field k. In this case, one first builds the de Rham complex Ω^*_R. This is a relatively concrete invariant of R and has a simple universal property: namely, Ω^*_R is a commutative differential graded algebra with a map $R \to (\Omega_R)^0$, and Ω^*_R is universal for this structure. The theory of crystalline cohomology (see, e.g, [1]) shows that this cochain complex of k-vector spaces, up to quasi-isomorphism, has a canonical lift to $W(k)$. The theory of the de Rham-Witt complex provides a commutative, differential graded algebra $\mathcal{W}\Omega^*_R$ over $W(k)$ equipped with a quasi-isomorphism $\mathcal{W}\Omega^*_R/p \to \Omega^*_R$, which provides a representative of crystalline cohomology. One can view $\mathcal{W}\Omega^*_R$ as a type of "rigidification:" crystalline cohomology naturally lives in the derived category, while de Rham-Witt theory provides an explicit cochain representative.

To construct the de Rham-Witt complex $\mathcal{W}\Omega^*_R$, we proceed as follows: choose a p-complete, p-torsion-free algebra $\widetilde{R}$ over $W(k)$ such that $\widetilde{R}/p \simeq R$; since R is smooth, there is no obstruction to doing this. The first approximation to $\mathcal{W}\Omega^*_R$ is the p-completed de Rham complex $\hat{\Omega}^*_{\widetilde{R}}$. On the one hand, $\hat{\Omega}^*_{\widetilde{R}}$ is a commutative, differential graded algebra over $W(k)$, and we have an isomorphism $\hat{\Omega}^*_{\widetilde{R}}/p \simeq \Omega^*_R$. On the other hand, the construction $\hat{\Omega}^*_{\widetilde{R}}$ depends on the choice of lift $\widetilde{R}$ of R, and is not a functor of R.

In order to make something functorial in R, we have to make a further choice: a lift $\varphi_{\widetilde{R}} : \widetilde{R} \to \widetilde{R}$ of the Frobenius on R. The key observation is that the choice of $\varphi_{\widetilde{R}}$ induces an operator

$$F : \hat{\Omega}^*_{\widetilde{R}} \to \hat{\Omega}^*_{\widetilde{R}}$$

of graded algebras such that

$$dF = pFd.$$

Namely, F is a type of divided Frobenius: it is defined on i-forms as $F = \varphi^*_{\widetilde{R}}/p^i$, since $\varphi^*_{\widetilde{R}}$ makes i-forms divisible by p^i. In the paper, we call such structures

Dieudonné complexes (Definition 2.1.1). We study the homological algebra of Dieudonné complexes in sections 2 and 3 of this paper.

A crucial point about the Dieudonné complex $(\hat{\Omega}^*_{\widetilde{R}}, d, F)$ (equipped with the de Rham differential and the operator F) is that it enjoys a special homological property: F implements the classical *Cartier isomorphism*

$$H^*(\Omega^*_R) \simeq \Omega^*_{R^{(1)}}$$

between the de Rham *cohomology* of R and differential forms on R (with a Frobenius twist). Given such a Dieudonné complex, we show that a general procedure called *strictification* will produce a new (generally much larger) Dieudonné complex quasi-isomorphic to the original one, equipped in addition with a *Verschiebung* operator V and suitably complete with respect to V. We construct $\mathcal{W}\Omega^*_R$ as the strictification of $\hat{\Omega}^*_{\widetilde{R}}$. Remarkably, strictification also has a rigidification function, as a short argument will show that the strictification does not depend on the choice of lifts $(\widetilde{R}, \varphi_{\widetilde{R}})$. In fact, just as one can define the de Rham complex as universal in the category of commutative, differential graded algebras, one can define the de Rham-Witt complex via an analogous universal property in an appropriate category of strict Dieudonné algebras (Definition 4.1.1).

In [15], a tower $\{\mathcal{W}_n\Omega^*_X\}$ of cochain complexes is constructed as an initial object in the category of *V-pro-complexes:* specifically, one looks at towers of commutative, differential graded algebras related by Verschiebung operators. One difference in our approach is that we work directly with the inverse limit $\mathcal{W}\Omega^*_X$ rather than the individual terms $\mathcal{W}_n\Omega^*_X$; the inverse limit is often easier to work with (e.g., as it is p-torsion-free). In addition, in our setup, the operator F on the de Rham-Witt complex plays the fundamental role, rather than V, which simplifies the relevant identities.

Finally, we mention a more highbrow motivation of $\mathcal{W}\Omega^*_X$, which we explore further in Section 8. As mentioned above, the general theory of crystalline cohomology shows that the de Rham complex, when considered as an object of the derived category $D(k)$, lifts to $W(k)$. In general, cohomology theories (e.g., produced via site-theoretic machines) produce elements of the derived category, defined up to quasi-isomorphism, rather than cochain complexes. In addition, from a homotopy-theoretic point of view, the representation of a cohomology theory in characteristic p by a commutative, differential graded algebra is a strong condition: for instance, mod p singular cohomology of spaces cannot generally be represented in this fashion, because of the existence of Steenrod operations.

The derived category $D(\mathbf{Z})$ of the integers is equipped with an operator $L\eta_p : D(\mathbf{Z}) \to D(\mathbf{Z})$, an additive (though not exact) functor which has the effect of reducing the p-torsion in cohomology by a small amount. The operator $L\eta_p$ was discovered in connection with crystalline cohomology, and has been used extensively in the recent work by Bhatt-Morrow-Scholze [4] on integral p-adic

Hodge theory. Namely, in [1], it was observed that the crystalline cohomology of an affine variety is equipped with a quasi-isomorphism with its own $L\eta_p$. Concretely, if $\widetilde{R}$ is a p-adically complete algebra such that R/p is smooth over a perfect field k, then the operator F discussed earlier yields a quasi-isomorphism $\hat{\Omega}^*_{\widetilde{R}} \simeq L\eta_p \hat{\Omega}^*_{\widetilde{R}}$. Furthermore, $L\eta_p$ implements a type of strictification. We show (Theorem 8.3.11) that the fixed points $\widehat{D(\mathbf{Z})}_p^{L\eta_p}$ of $L\eta_p$ on the p-complete derived category $\widehat{D(\mathbf{Z})}_p \subset D(\mathbf{Z})$ is equivalent to the category of strict Dieudonné complexes. In particular, the structure that crystalline cohomology is a fixed point of $L\eta_p$ (which is ultimately a form of the Cartier isomorphism) explains the strictification of crystalline cohomology given by the de Rham-Witt complex.

The theory of the de Rham-Witt complex has taken many forms since its original introduction in [15], and our approach is restricted to the classical case of algebras over $\mathbf{F}_p$. In particular, we do not address the relative de Rham-Witt complexes of Langer-Zink [20], or the absolute de Rham-Witt complexes considered by Hesselholt-Madsen [12] in relation to topological Hochschild homology.

1.3. **Notations.** Throughout this paper, we let p denote a fixed prime number. We say that an abelian group M is *p-torsion-free* if the map $p : M \to M$ is a monomorphism.

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2. Dieudonné Complexes

In this section, we introduce and study the category **KD** of Dieudonné complexes. In particular, we treat the theory of saturated and strict Dieudonné complexes and the operations of saturation and strictification. A key class of Dieudonné complexes is given by those of *Cartier type* (Definition 2.4.1) which shows that in certain cases, these operations are especially mild homologically.

2.1. The Category KD.

Definition 2.1.1. A *Dieudonné complex* is a triple (M^*, d, F), where (M^*, d) is a cochain complex of abelian groups and $F : M^* \to M^*$ is a map of graded abelian groups which satisfies the identity $dF(x) = pF(dx)$.

We let **KD** denote the category whose objects are Dieudonné complexes, where a morphism from (M^*, d, F) to (M'^*, d', F') in **KD** is a map of graded abelian groups $f : M^* \to M'^*$ satisfying $d'f(x) = f(dx)$ and $F'f(x) = f(Fx)$ for each $x \in M^*$.

Remark 2.1.2. Let (M^*, d, F) be a Dieudonné complex. We will generally abuse terminology by simply referring to the underlying graded abelian group M^* as a *Dieudonné complex*; in this case we implicitly assume that the data of d and F have also been specified. We will refer to F as the *Frobenius map* on the complex M^*.

In the situation of Definition 2.1.1, it will be convenient to interpret F as a map of cochain complexes.

Construction 2.1.3. Let (M^*, d) be a p-torsion-free cochain complex of abelian groups, which we identify with a subcomplex of the localization $M^*[p^{-1}]$. We define another subcomplex $(\eta_p M)^* \subseteq M^*[p^{-1}]$ as follows: for each integer n, we set $(\eta_p M)^n = \{x \in p^n M^n : dx \in p^{n+1} M^{n+1}\}$.

Remark 2.1.4. Let (M^*, d, F) be a Dieudonné complex, and suppose that M^* is p-torsion-free. Then F determines a map of cochain complexes $\alpha_F : M^* \to (\eta_p M)^*$, given by the formula $\alpha_F(x) = p^n F(x)$ for $x \in M^n$. Conversely, if (M^*, d) is a cochain complex which is p-torsion-free and $\alpha : M^* \to (\eta_p M)^*$ is a map of cochain complexes, then we can form a Dieudonné complex (M^*, d, F) by setting $F(x) = p^{-n}\alpha(x)$ for $x \in M^n$. These constructions are inverse to one another.

2.2. Saturated Dieudonné Complexes.

Definition 2.2.1. Let (M^*, d, F) be a Dieudonné complex. We will say that (M^*, d, F) is *saturated* if the following pair of conditions is satisfied:

(i) The graded abelian group M^* is p-torsion-free.

(ii) For each integer n, the map F induces an isomorphism of abelian groups

$$M^n \to \{x \in M^n : dx \in pM^{n+1}\}.$$

We let $\mathbf{KD}_{\mathrm{sat}}$ denote the full subcategory of $\mathbf{KD}$ spanned by the saturated Dieudonné complexes.

Remark 2.2.2. Let (M^*, d, F) be a Dieudonné complex, and suppose that M^* is p-torsion-free. Then (M^*, d, F) is a saturated Dieudonné complex if and only if the map $\alpha_F : M^* \to (\eta_p M)^*$ is an isomorphism (see Remark 2.1.4).

Remark 2.2.3. Let (M^*, d, F) be a saturated Dieudonné complex. Then the map $F : M^* \to M^*$ is injective. Moreover, the image of F contains the subcomplex pM^*. It follows that for each element $x \in M^n$, there exists a unique element $Vx \in M^n$ satisfying $F(Vx) = px$. We will refer to the map $V : M^* \to M^*$ as the *Verschiebung*.

Proposition 2.2.4. *Let (M^*, d, F) be a saturated Dieudonné complex. Then the Verschiebung $V : M^* \to M^*$ is an injective map which satisfies the identities*

(1) $$F \circ V = V \circ F = p(\mathrm{id})$$

(2) $$F \circ d \circ V = d \qquad p(d \circ V)$$

(3) $$V \circ d.$$

Proof. The relation $F \circ V = p(\mathrm{id})$ follows from the definition, and immediately implies that V is injective (since M^* is p-torsion-free). Precomposing with F, we obtain $F \circ V \circ F = pF$. Since F is injective, it follows that $V \circ F = p(\mathrm{id})$. We have $pd = (d \circ F \circ V) = p(F \circ d \circ V)$, so that $d = F \circ d \circ V$. Postcomposition with V then yields the identity $V \circ d = V \circ F \circ d \circ V = p(d \circ V)$. □

Proposition 2.2.5. *Let (M^*, d, F) be a saturated Dieudonné complex and let $r \geq 0$ be an integer. Then the map F^r induces an isomorphism $M^* \to \{x \in M^* : dx \in p^r M^{*+1}\}$.*

Proof. It is clear that F^r is injective (since F is injective), and the inclusion $F^r M^* \subseteq \{x \in M^* : dx \in p^r M^{*+1}\}$ follows from the calculation $d(F^r y) = p^r F^r(dy)$. We will prove the reverse inclusion by induction on r. Assume that $r > 0$ and that $x \in M^*$ satisfies $dx = p^r y$ for some $y \in M^{*+1}$. Then $d(p^r y) = 0$, so our assumption that M^* is p-torsion-free guarantees that $dy = 0$. Applying our assumption that M^* is a saturated Dieudonné complex to x and y, we can write $x = F(x')$ and $y = F(y')$ for some $x' \in M^*$ and $y' \in M^{*+1}$. We then compute

$$pF(dx') = d(F(x')) = dx = p^r y = p^r F(y').$$

Canceling F and p, we obtain $dx' = p^{r-1} y'$. It follows from our inductive hypothesis that $x' \in \mathrm{im}(F^{r-1})$, so that $x = F(x') \in \mathrm{im}(F^r)$. □

Remark 2.2.6. Let (M^*, d, F) be a saturated Dieudonné complex. It follows from Proposition 2.2.5 that every cycle $z \in M^*$ is infinitely divisible by F: that is, it belongs to $F^r M^*$ for each $r \geq 0$.

2.3. **Saturation of Dieudonné Complexes.** Let $f : M^* \to N^*$ be a morphism of Dieudonné complexes. We will say that f *exhibits* N^* *as a saturation of* M^* if N^* is saturated and, for every saturated Dieudonné complex K^*, composition with f induces a bijection

$$\mathrm{Hom}_{\mathbf{KD}}(N^*, K^*) \to \mathrm{Hom}_{\mathbf{KD}}(M^*, K^*). \tag{4}$$

In this case, the Dieudonné complex N^* (and the morphism f) are determined by M^* up to unique isomorphism; we will refer to N^* as the *saturation* of M^* and denote it by $\mathrm{Sat}(M^*)$.

Proposition 2.3.1. *Let M^* be a Dieudonné complex. Then M^* admits a saturation* $\mathrm{Sat}(M^*)$.

Proof. Let $T^* \subseteq M^*$ be the graded subgroup consisting of elements $x \in M^*$ satisfying $p^n x = 0$ for $n \gg 0$. Replacing M^* by the quotient M^*/T^*, we can reduce to the case where M^* is p-torsion-free. In this case, Remark 2.2.2 implies that the direct limit of the sequence

$$M^* \xrightarrow{\alpha_F} (\eta_p M)^* \xrightarrow{\eta_p(\alpha_F)} (\eta_p \eta_p M)^* \xrightarrow{\eta_p(\eta_p(\alpha_F))} (\eta_p \eta_p \eta_p M)^* \to \cdots$$

is a saturation of M^*. □

Corollary 2.3.2. *The inclusion functor $\mathbf{KD}_{\mathrm{sat}} \hookrightarrow \mathbf{KD}$ admits a left adjoint, given by the saturation construction $M^* \mapsto \mathrm{Sat}(M^*)$.*

Remark 2.3.3. Let (M^*, d, F) be a Dieudonné complex, and assume that M^* is p-torsion-free. Then we can identify M^* with a subgroup of the graded abelian group $M^*[F^{-1}]$, defined as the colimit $M^* \xrightarrow{F} M^* \xrightarrow{F} \ldots$, or equivalently defined as equivalence classes of elements $F^{-n}x$ for $x \in M^*$ in a natural way. Unwinding the definitions, we see that the saturation $\mathrm{Sat}(M^*)$ can be described more concretely as follows:

- As a graded abelian group, $\mathrm{Sat}(M^*)$ can be identified with the subgroup of $M^*[F^{-1}]$ consisting of those elements x satisfying $d(F^n(x)) \in p^n M^*$ for $n \gg 0$ (note that if this condition is satisfied for some nonnegative integer n, then it is also satisfied for every larger integer).
- The differential on $\mathrm{Sat}(M^*)$ is given by the construction $x \mapsto F^{-n} p^{-n} d(F^n x)$ for $n \gg 0$.
- The Frobenius map on $\mathrm{Sat}(M^*)$ is given by the restriction of the map $F : M^*[F^{-1}] \to M^*[F^{-1}]$.

Remark 2.3.4. For future reference, we record a slight variant of Remark 2.3.3. Let (M^*, d, F) be a p-torsion-free Dieudonné complex on which F is injective, and form the construction $M^*[F^{-1}]$ as above. The operator $d : M^* \to M^{*+1}$ extends to a map

$$d : M^*[F^{-1}] \to M^*[F^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[1/p]$$

given by the formula

$$d(F^{-n}x) = p^{-n}F^{-n}dx.$$

Then $\mathrm{Sat}(M^*) \subset M^*[F^{-1}]$ can be identified with the graded subgroup consisting of those elements $y = F^{-n}x \in M^*[F^{-1}]$ such that $dy \in M^*[F^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[1/p]$ actually belongs to $M^*[F^{-1}]$.

Namely, suppose that we can write $d(F^{-n}x) = F^{-m}z$ in $M^*[F^{-1}]$ for $x, z \in M^*$. Then $F^{-n}dx = p^n F^{-m}z$ and $F^{r-n}dx = p^n F^{r-m}dz$ for $r \gg 0$. It follows that $d(F^{r-n}x) = d(F^r F^{-n}x)$ is divisible by p^r for $r \gg 0$. It follows that $F^{-n}x$ belongs to $\mathrm{Sat}(M^*)$ as in the formula of Remark 2.3.3. Conversely, if $F^{-n}x$ belongs to $\mathrm{Sat}(M^*)$ as in the formula of Remark 2.3.3, then it is easy to see that $d(F^{-n}x) \in M^*[F^{-1}]$ as desired.

2.4. **The Cartier Criterion.** Let M^* be a Dieudonné complex. For each $x \in M^*$, the equation $dFx = pF(dx)$ shows that the image of Fx in the quotient complex M^*/pM^* is a cycle, and therefore represents an element in the cohomology $\mathrm{H}^*(M/pM)$. This construction determines a map of graded abelian groups $M^* \to \mathrm{H}^*(M/pM)$, which factors uniquely through the quotient M^*/pM^*.

Definition 2.4.1. We say that the Dieudonné complex M^* is of *Cartier type* if

- (i) The complex M^* is p-torsion-free.
- (ii) The Frobenius map F induces an isomorphism of graded abelian groups $M^*/pM^* \to \mathrm{H}^*(M/pM)$.

Theorem 2.4.2 (Cartier Criterion). *Let M^* be a Dieudonné complex which is of Cartier type. Then the canonical map $M^* \to \mathrm{Sat}(M^*)$ induces a quasi-isomorphism of cochain complexes $M^*/pM^* \to \mathrm{Sat}(M^*)/p\,\mathrm{Sat}(M^*)$.*

To prove Theorem 2.4.2, we will need to review some properties of the construction $M^* \mapsto (\eta_p M)^*$.

Construction 2.4.3. Let M^* be a cochain complex which is p-torsion-free. For each element $x \in (\eta_p M)^k$, we let $\overline{\gamma}(x) \in \mathrm{H}^k(M/pM)$ denote the cohomology class represented by $p^{-k}x$ (which is a cycle in M^*/pM^*, by virtue of our assumption that $x \in (\eta_p M)^k$). This construction determines a map of graded abelian groups $\overline{\gamma} : (\eta_p M)^* \to \mathrm{H}^*(M/pM)$. Since the codomain of $\overline{\gamma}$ is a p-torsion group, the map $\overline{\gamma}$ factors (uniquely) as a composition

$$(\eta_p M)^* \to (\eta_p M)^*/p(\eta_p M)^* \xrightarrow{\gamma} \mathrm{H}^*(M/pM).$$

Proposition 2.4.4 (Compare [4, Prop. 6.12]). *Let M^* be a cochain complex of abelian groups which is p-torsion-free. Then Construction 2.4.3 determines a quasi-isomorphism of cochain complexes*

$$\gamma : (\eta_p M)^*/p(\eta_p M)^* \to \mathrm{H}^*(M/pM).$$

Here we regard $\mathrm{H}^*(M/pM)$ as a cochain complex with respect to the Bockstein map $\beta : \mathrm{H}^*(M/pM) \to \mathrm{H}^{*+1}(M/pM)$ determined the short exact sequence of cochain complexes $0 \to M^*/pM^* \xrightarrow{p} M^*/p^2M^* \to M^*/pM^* \to 0$.

Proof. For each element $x \in M^k$ satisfying $dx \in pM^{k+1}$, let $[x] \in \mathrm{H}^k(M/pM)$ denote the cohomology class represented by x. Unwinding the definitions, we see that the map β satisfies the identity $\beta([x]) = [p^{-1}dx]$. It follows that for each $y \in (\eta_p M)^k$, we have

$$\beta(\overline{\gamma}(y)) = \beta([p^{-k}y]) = [p^{-k-1}dy] = \overline{\gamma}(dy),$$

so that $\overline{\gamma} : (\eta_p M)^* \to \mathrm{H}^*(M/pM)$ is a map of cochain complexes. It follows that γ is also a map of cochain complexes. It follows immediately from the definition of $(\eta_p M)^*$ that the map γ is surjective. Let K^* denote the kernel of γ. We will complete the proof by showing that the cochain complex K^* is acyclic. Unwinding the definitions, we can identify K^* with the subquotient of $M^*[p^{-1}]$ described by the formula

$$K^n = (p^{n+1}M^n + d(p^n M^{n-1})/(p^{n+1}M^n \cap d^{-1}(p^{n+2}M^{n+1})).$$

Suppose that $e \in K^n$ is a cocycle, represented by $p^{n+1}x + d(p^n y)$ for some $x \in M^n$ and $y \in M^{n-1}$. The equation $de = 0 \in K^{n+1}$ implies that $d(p^{n+1}x) \in p^{n+2}M^{n+1}$. It follows that e is also represented by $d(p^n y)$, and is therefore a coboundary in K^n. □

Corollary 2.4.5. *Let $f : M^* \to N^*$ be a map of cochain complexes of abelian groups. Assume that M^* and N^* are p-torsion-free. If the induced map $M^*/pM^* \to N^*/pN^*$ is a quasi-isomorphism, then the induced map*

$$(\eta_p M)^*/p(\eta_p M)^* \to (\eta_p N)^*/p(\eta_p N)^*$$

is also a quasi-isomorphism.

Proof. By virtue of Proposition 2.4.4, it suffices to show that the induced map $\mathrm{H}^*(M/pM) \to \mathrm{H}^*(N/pN)$ is a quasi-isomorphism (where both complexes are equipped with the differential given by the Bockstein). But this map is already an isomorphism, since the map $M^*/pM^* \to N^*/pN^*$ is a quasi-isomorphism. □

Proof of Theorem 2.4.2. Let (M^*, d, F) be a Dieudonné complex which is p-torsion-free. Then the saturation $\mathrm{Sat}(M^*)$ can be identified with the colimit of the sequence

$$M^* \xrightarrow{\alpha_F} (\eta_p M)^* \xrightarrow{\eta_p(\alpha_F)} (\eta_p \eta_p M)^* \xrightarrow{\eta_p(\eta_p(\alpha_F))} (\eta_p \eta_p \eta_p M)^* \to \cdots$$

Consequently, to show that the canonical map $M^*/pM^* \to \mathrm{Sat}(M^*)/p\,\mathrm{Sat}(M^*)$ is a quasi-isomorphism, it will suffice to show that each of the maps

$$(\eta_p^k M)^*/p(\eta_p^k M)^* \to (\eta_p^{k+1} M)^*/p(\eta_p^{k+1} M)^*$$

is a quasi-isomorphism. By virtue of Corollary 2.4.5, it suffices to verify this when $k = 0$. In this case, we have a commutative diagram of cochain complexes

$$\begin{array}{ccc} M^*/pM^* & \xrightarrow{\alpha_F} & (\eta_p M)^*/p(\eta_p M)^* \\ & \searrow \quad \swarrow_{\gamma} & \\ & \mathrm{H}^*(M/p), & \end{array}$$

where the map $M^*/pM^* \to H^*(M/p)$ is induced by the Frobenius. Since M^* is assumed to be of Cartier type, this map is an isomorphism. Since γ is a quasi-isomorphism by virtue of Proposition 2.4.4, it follows that the map $\alpha_F : M^*/pM^* \to (\eta_p M)^*/p(\eta_p M)^*$ is also a quasi-isomorphism, as desired. □

2.5. **Quotients of Saturated Dieudonné Complexes.**

Construction 2.5.1. Let M^* be a saturated Dieudonné complex. For each integer $r \geq 0$, we let $\mathcal{W}_r(M)^*$ denote the quotient of M^* by the subcomplex $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$, where V is the Verschiebung map of Remark 2.2.3. We denote the natural quotient maps $\mathcal{W}_{r+1}(M)^* \to \mathcal{W}_r(M)^*$ by $\mathrm{Res} : \mathcal{W}_{r+1}(M)^* \to \mathcal{W}_r(M)^*$.

Proposition 2.5.2. *Let M^* be a saturated Dieudonné complex and let r be a nonnegative integer. Then the map $F^r : M^* \to M^*$ induces an isomorphism of graded abelian groups $\mathcal{W}_r(M)^* \to \mathrm{H}^*(M^*/p^r M^*)$.*

Proof. The equality $d \circ F^r \simeq p^r(F^r \circ d)$ guarantees that F^r carries each element of M^* to a cycle in the quotient complex $M^*/p^r M^*$. Moreover, the identities

$$F^r V^r(x) = p^r x \qquad F^r dV^r(y) = dy$$

guarantee that F^r carries the subgroup $\mathrm{im}(V^r) + \mathrm{im}(dV^r) \subseteq M^*$ to the set of boundaries in the cochain complex $M^*/p^r M^*$. We therefore obtain a well-defined homomorphism $\theta : \mathcal{W}_r(M)^* \to \mathrm{H}^*(M^*/p^r M^*)$. The surjectivity of θ follows from Proposition 2.2.5. To prove injectivity, suppose that $x \in M^*$ has the property that $F^r x$ represents a boundary in the quotient complex $M^*/p^r M^*$, so that we can write $F^r x = p^r y + dz$ for some $y \in M^*$ and $z \in M^{*-1}$. Then $p^r x = V^r F^r x = V^r(p^r y + dz) = p^r V^r(y) + p^r dV^r(z)$, so that $x = V^r(y) + dV^r(z) \in \mathrm{im}(V^r) + \mathrm{im}(dV^r)$. □

Corollary 2.5.3. *Let (M^*, d, F) be a saturated Dieudonné complex. Then the quotient map $u : M^*/pM^* \to \mathcal{W}_1(M)^*$ is a quasi-isomorphism of cochain complexes.*

Proof. We have a commutative diagram of cochain complexes

$$\begin{array}{ccc} M^*/pM^* & \xrightarrow{\alpha_F} & (\eta_p M)^*/p(\eta_p M)^* \\ \downarrow{\scriptstyle u} & & \downarrow \\ \mathcal{W}_1(M)^* & \longrightarrow & \mathrm{H}^*(M/pM), \end{array}$$

where the bottom horizontal map is the isomorphism of Proposition 2.5.2, the right vertical map is the quasi-isomorphism of Proposition 2.4.4, and α_F is an isomorphism by virtue of our assumption that M^* is saturated. It follows that u is also a quasi-isomorphism. □

Remark 2.5.4. More generally, if M^* is a saturated Dieudonné complex, then the quotient map $M^*/p^r M^* \to \mathcal{W}_r(M)^*$ is a quasi-isomorphism for each $r \geq 0$. This can be proven by essentially the same argument, using a slight generalization of Proposition 2.4.4.

Corollary 2.5.5. *Let $f : M^* \to N^*$ be a morphism of saturated Dieudonné complexes. The following conditions are equivalent:*

(1) *The induced map $M^*/pM^* \to N^*/pN^*$ is a quasi-isomorphism of cochain complexes.*
(2) *The induced map $\mathcal{W}_1(M)^* \to \mathcal{W}_1(N)^*$ is an isomorphism of cochain complexes.*
(3) *For each $r \geq 0$, the induced map $M^*/p^r M^* \to N^*/p^r N^*$ is a quasi-isomorphism of cochain complexes.*
(4) *For each $r \geq 0$, the induced map $\mathcal{W}_r(M)^* \to \mathcal{W}_r(N)^*$ is an isomorphism of cochain complexes.*

Proof. The equivalences (1) ⇔ (2) and (3) ⇔ (4) follow from Proposition 2.5.2. The implication (3) ⇒ (1) is obvious, and the reverse implication follows by induction on r. □

2.6. The Completion of a Saturated Dieudonné Complex.

Remark 2.6.1 (Frobenius and Verschiebung at Finite Levels)**.** Let M^* be a saturated Dieudonné complex. Using the identities $F \circ d \circ V = d$ and $F \circ V = p\,\mathrm{id}$, we deduce that the Frobenius map $F : M^* \to M^*$ carries $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$ into $\mathrm{im}(V^{r-1}) + \mathrm{im}(dV^{r-1})$. It follows that there is a unique map of graded abelian groups $F : \mathcal{W}_r(M)^* \to \mathcal{W}_{r-1}(M)^*$ for which the diagram

$$\begin{array}{ccc} M^* & \xrightarrow{F} & M^* \\ \downarrow & & \downarrow \\ \mathcal{W}_r(M)^* & \xrightarrow{F} & \mathcal{W}_{r-1}(M)^* \end{array}$$

commutes.

Similarly, the identity $V \circ d = p(d \circ V)$ implies that V carries $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$ into $\mathrm{im}(V^{r+1}) + \mathrm{im}(dV^{r+1})$, so that there is a unique map $V : \mathcal{W}_r(M)^* \to \mathcal{W}_{r+1}(M)^*$

for which the diagram

$$\begin{array}{ccc} M^* & \xrightarrow{V} & M^* \\ \downarrow & & \downarrow \\ \mathcal{W}_r(M)^* & \xrightarrow{V} & \mathcal{W}_{r+1}(M)^* \end{array}$$

commutes.

Construction 2.6.2 (Completion)**.** Let M^* be a saturated Dieudonné complex. We let $\mathcal{W}(M)^*$ denote the inverse limit of the tower of cochain complexes

$$M^* \to \cdots \to \mathcal{W}_3(M)^* \xrightarrow{\text{Res}} \mathcal{W}_2(M)^* \xrightarrow{\text{Res}} \mathcal{W}_1(M)^* \xrightarrow{\text{Res}} \mathcal{W}_0(M)^* = 0.$$

We will refer to $\mathcal{W}(M)^*$ as the *completion* of the saturated Dieudonné complex M^*. We regard $\mathcal{W}(M)^*$ as a Dieudonné complex with respect to the Frobenius map $F : \mathcal{W}(M)^* \to \mathcal{W}(M)^*$ given by the inverse limit of the maps $\mathcal{W}_r(M)^* \to \mathcal{W}_{r-1}(M)^*$ appearing in Remark 2.6.1.

Remark 2.6.3. The formation of completions is functorial: if $f : M^* \to N^*$ is a map of saturated Dieudonné complexes, then f induces a map $\mathcal{W}(f) : \mathcal{W}(M)^* \to \mathcal{W}(N)^*$, given by the inverse limit of the tautological maps

$$M^*/(\text{im}(V^r) + \text{im}(dV^r)) \xrightarrow{f} N^*/(\text{im}(V^r) + \text{im}(dV^r)).$$

Our primary goal in this section is to prove the following:

Proposition 2.6.4. *Let M^* be a saturated Dieudonné complex. Then the completion $\mathcal{W}(M)^*$ is also a saturated Dieudonné complex.*

The proof of Proposition 2.6.4 will require some preliminaries. To begin with, we record some facts about the tower $\{\mathcal{W}_r(M)^*\}$.

Lemma 2.6.5. *Let M^* be a saturated Dieudonné complex and let $x \in M^*$ satisfy $d(V^r x) \in pM^{*+1}$. Then $x \in \text{im}(F)$.*

Proof. We have $dx = F^r d(V^r x) \in F^r pM^{*+1} \subseteq pM^{*+1}$, so the desired result follows from our assumption that M^* is saturated. □

Lemma 2.6.6. *Let M^* be a saturated Dieudonné complex and let $x \in M^*$. If the image of px vanishes in $\mathcal{W}_r(M)^*$, then the image of x vanishes in $\mathcal{W}_{r-1}(M)^*$.*

Proof. It suffices to show that if $x \in M^*$ and we have

$$px = V^{r+1}a + dV^{r+1}b, \tag{5}$$

then we can write

$$x = V^r\widetilde{a} + dV^r\widetilde{b}. \tag{6}$$

To do this, we observe that $dV^{r+1}a$ is divisible by p and then use Lemma 2.6.5 again write $a = F\widetilde{a}$ for some $\widetilde{a} \in M^*$. This in turn implies that $dV^{r+1}b$ is divisible by p, so that $b = F\widetilde{b}$ for some $\widetilde{b} \in M^*$. Then we can write

$$px = pV^r\widetilde{a} + dV^r(p\widetilde{b}),$$

and dividing by p, we obtain the desired claim. □

We can now prove Proposition 2.6.4. For future reference, it will be helpful to formulate the result in a slightly more general form. That is, we introduce a new category **TD**. This category will play an auxiliary role in the sequel at various points.

Definition 2.6.7. A **strict Dieudonné tower** is a tower $\{X_r^*, R : X_{r+1}^* \to X_r^*, r \geq 0\}$ of cochain complexes equipped with operators $F : X_{r+1}^* \to X_r^*$ and $V : X_r^* \to X_{r+1}^*$ in the category of graded abelian groups satisfying the following conditions.

(1) $X_0^* = 0$.
(2) The map $R : X_{r+1}^* \to X_r^*$ is surjective for each $r \geq 0$.
(3) The operator $F : X_{r+1}^* \to X_r^*$ satisfies $dF = pFd$ for each $r \geq 0$.
(4) The operators F, R, V commute with each other.
(5) We have $FV = VF = p$ on each X_r^*.
(6) Given any $x \in X_r^*$ such that dx is divisible by p, x belongs to the image of $F : X_{r+1}^* \to X_r^*$.
(7) The kernel of the map $R : X_{r+1}^* \to X_r^*$ is exactly the submodule $X_{r+1}^*[p] \subset X_{r+1}^*$ consisting of $x \in X_{r+1}^*$ such that $px = 0$.
(8) The kernel of $R : X_{r+1}^* \to X_r^*$ is the span of the images of $V^r, dV^r : X_1^* \to X_{r+1}^*$.

A morphism of strict Dieudonné towers is a morphism of towers of cochain complexes which are compatible with the F, V operators. We let **TD** be the category of strict Dieudonné towers.

We next show that the above definition axiomatizes the salient features of the construction $\{\mathcal{W}_r(M)^*\}$ for a saturated Dieudonné complex. That is, we consider the following construction.

Construction 2.6.8. Let M^* be a saturated Dieudonné complex. Then we consider the tower $\{\mathcal{W}_r(M)^*\}$ with $R = \mathrm{Res}$, equipped with the operators $F : \mathcal{W}_r(M)^* \to \mathcal{W}_{r-1}(M)^*$ and $V : \mathcal{W}_{r-1}(M)^* \to \mathcal{W}_r(M)^*$ induced by the analogous operators F, V on M^*.

Proposition 2.6.9. *If M^* is a saturated Dieudonné complex, then the tower $\{\mathcal{W}_r(M)^*\}$ with the operators F, V induced by Construction 2.6.8 is a strict Dieudonné tower, so that we obtain a functor* $\mathbf{KD}_{\mathrm{sat}} \to \mathbf{TD}$.

Proof. Axioms (1) through (5) and (8) are clear. To check (6), it is equivalent to showing that if $x \in M^*$ and we can write

$$dx = py + V^r a + dV^r b, \quad y, a, b \in M^*, \tag{7}$$

then we can find $z, a', b' \in M^*$ such that

$$x = Fz + V^r a' + dV^r b'. \tag{8}$$

In order to do this, we observe that (7) implies that $d(V^r a)$ is divisible by p, so by Lemma 2.6.5 we conclude that $a = F\widetilde{a}$ for some $\widetilde{a} \in M^*$. Substituting into (7), we thus have

$$d(x - V^r b) = p(y + V^{r-1}\widetilde{a}),$$

so that $x - V^r b = Fz$ for some $z \in M^*$ by saturation. Therefore, we can solve (8) with z as given, $a' = b, b' = 0$.

To check (7), we observe first that the kernel of $R : \mathcal{W}_{r+1}(M)^* \to \mathcal{W}_r(M)^*$ consists of simple p-torsion. In fact, this follows because if $x \in M^*$ belongs to $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$, then $px \in \mathrm{im}(V^{r+1}) + \mathrm{im}(dV^{r+1})$ because $p = FV$. Conversely, the map $R : \mathcal{W}_{r+1}(M)^* \to \mathcal{W}_r(M)^*$ annihilates simple p-torsion in view of Lemma 2.6.6. □

We now define a functor in the reverse direction, starting with a strict Dieudonné tower and producing a saturated Dieudonné complex.

Construction 2.6.10. Let $\{X_r^*\}$ be a strict Dieudonné tower. Let $X^* = \varprojlim_r X_r^*$ be the inverse limit of the tower, so that X^* is a cochain complex. Define the operators F, V on X^* as the inverse limit of the operators $F : X_r^* \to X_{r-1}^*$ and $V : X_{r-1}^* \to X_r^*$. Since we have $dF = pFd$, X^* defines an object of $\mathbf{KD}$.

Proposition 2.6.11. *If $\{X_r^*\} \in \mathbf{TD}$, then the inverse limit $X^* = \varprojlim_r X_r^*$ equipped with the operator $F : X^* \to X^*$ is a saturated Dieudonné complex.*

Proof. We observe first that X^* is p-torsion-free. Indeed, if $x \in X^*$, then x defines a system of elements $\{x_r \in X_r^*\}$ compatible under the restriction maps R. If $x \neq 0$ but $px = 0$, each of the $x_r \in X_r^*$ satisfy $px_r = 0$. However, now the assumption that the restriction maps annihilate simple p-torsion forces all the x_r to vanish.

We have $FV = VF = p$ on X^*, so F is injective. To show that X^* is saturated, it suffices now to show that if $x \in X^*$ and dx is divisible by p, then x belongs to the image of F. For each r, let $x_r \in X_r^*$ be the image of x. By assumption and (6), we can find $y_{r+1} \in X_{r+1}^*$ such that $Fy_{r+1} = x_r$. The system $\{y_r \in X_r^*\}$ is, however, not necessarily compatible under R. We have, however, that $R(y_{r+1}) - y_r$ is annihilated by F and thus by R, since R annihilates simple p-torsion and $p = VF$. It follows that the sequence $\{R(y_{r+1}) \in X_r^*\}$ is compatible under R and defines an element $y \in X^*$ with $Fy = x$. □

Proof of Proposition 2.6.4. Combine Propositions 2.6.9 and 2.6.11. □

2.7. **Strict Dieudonné Complexes.** For every saturated Dieudonné complex M^*, the tower of cochain complexes

$$M^* \to \cdots \to \mathcal{W}_3(M)^* \xrightarrow{\text{Res}} \mathcal{W}_2(M)^* \xrightarrow{\text{Res}} \mathcal{W}_1(M)^* \xrightarrow{\text{Res}} \mathcal{W}_0(M)^* = 0.$$

determines a tautological map $\rho_M : M^* \to \mathcal{W}(M)^*$. It follows immediately from the definitions that ρ_M is a map of (saturated) Dieudonné complexes, which depends functorially on M^*.

Proposition 2.7.1. *Let M^* be a saturated Dieudonné complex. Then the map $\rho_M : M^* \to \mathcal{W}(M)^*$ induces isomorphisms $\mathcal{W}_r(M)^* \to \mathcal{W}_r(\mathcal{W}(M))^*$ for $r \geq 0$.*

Proof. By virtue of Corollary 2.5.5, it will suffice to treat the case $r = 1$. Let $\gamma : \mathcal{W}_1(M)^* \to \mathcal{W}_1(\mathcal{W}(M))^*$ be the map determined by ρ_M. We first show that γ is injective. Choose an element $\overline{x} \in \mathcal{W}_1(M)^*$ satisfying $\gamma(\overline{x}) = 0$; we wish to show that $\overline{x} = 0$. Represent $\overline{x}$ by an element $x \in M^*$. The vanishing of $\gamma(\overline{x})$ then implies that we can write $\rho_M(x) = Vy + dVz$ for some $y \in \mathcal{W}(M)^*$ and $z \in \mathcal{W}(M)^{*-1}$. Then we can identify y and z with compatible sequences $\overline{y}_m \in \mathcal{W}_m(M)^*$ and $\overline{z}_m \in \mathcal{W}_m(M)^{*-1}$. The equality $\rho_M(x) = Vy + dVz$ then yields $\overline{x} = V(\overline{y}_0) + dV(\overline{z}_0) = 0$.

We now prove that γ is surjective. Choose an element $e \in \mathcal{W}_1(\mathcal{W}(M))^*$; we wish to show that e belongs to the image of γ. Let $x \in \mathcal{W}(M)^*$ be an element representing e, which we can identify with a compatible sequence of elements $\overline{x}_m \in \mathcal{W}_m(M)^*$ represented by elements $x_m \in M^*$. The compatibility of the sequence $\{\overline{x}_m\}_{m\geq 0}$ guarantees that we can write $x_{m+1} = x_m + V^m y_m + dV^m z_m$ for some elements $y_m \in M^*$ and $z_m \in M^{*-1}$. We then have the identity

$$x = \rho_M(x_1) + V(\sum_{m\geq 0} V^m y_{m+1}) + dV(\sum_{m\geq 0} V^m z_{m+1})$$

in $\mathcal{W}(M)^*$, which yields the identity $e = \gamma(\overline{x}_1)$ in $\mathcal{W}_1(\mathcal{W}(M))^*$. □

Definition 2.7.2. Let M^* be a Dieudonné complex. We will say that M^* is *strict* if it is saturated and the map $\rho_M : M^* \to \mathcal{W}(M)^*$ is an isomorphism. We let $\mathbf{KD}_{\text{str}}$ denote the full subcategory of $\mathbf{KD}$ spanned by the strict Dieudonné complexes.

Proposition 2.7.3. *Let M^* and N^* be saturated Dieudonné complexes, where N^* is strict. Then composition with the map ρ_M induces a bijection*

$$\theta : \text{Hom}_{\mathbf{KD}}(\mathcal{W}(M)^*, N^*) \to \text{Hom}_{\mathbf{KD}}(M^*, N^*).$$

Proof. We first show that θ is injective. Let $f : \mathcal{W}(M)^* \to N^*$ be a morphism of Dieudonné complexes such that $\theta(f) = f \circ \rho_M$ vanishes; we wish to show that f

vanishes. For each $r \geq 0$, we have a commutative diagram of cochain complexes

$$\begin{array}{ccccc} M^* & \xrightarrow{\rho_M} & \mathcal{W}(M)^* & \xrightarrow{f} & N^* \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{W}_r(M)^* & \xrightarrow{\mathcal{W}_r(\rho_M)} & \mathcal{W}_r(\mathcal{W}(M))^* & \xrightarrow{\mathcal{W}_r(f)} & \mathcal{W}_r(N)^* \end{array}$$

Since $\mathcal{W}_r(f) \circ \mathcal{W}_r(\rho_M) = \mathcal{W}_r(f \circ \rho_M)$ vanishes and $\mathcal{W}_r(\rho_M)$ is an isomorphism (Proposition 2.7.1), we conclude that $\mathcal{W}_r(f) = 0$. It follows that the composite map $\mathcal{W}(M)^* \xrightarrow{f} N^* \to \mathcal{W}_r(N)^*$ vanishes for all r. Invoking the completeness of N^*, we conclude that $f = 0$.

We now argue that θ is surjective. Suppose we are given a map of Dieudonné complexes $f_0 : M^* \to N^*$; we wish to show that f_0 belongs to the image of θ. We have a commutative diagram of Dieudonné complexes

$$\begin{array}{ccc} M^* & \xrightarrow{f_0} & N^* \\ \downarrow{\rho_M} & & \downarrow{\rho_N} \\ \mathcal{W}(M)^* & \xrightarrow{\mathcal{W}(f_0)} & \mathcal{W}(N)^*, \end{array}$$

where the right vertical map is an isomorphism (since N^* is complete). It follows that $f_0 = \theta(\rho_N^{-1} \circ \mathcal{W} f_0)$ belongs to the image of θ, as desired. □

Corollary 2.7.4. *The inclusion functor* $\mathbf{KD}_{\mathrm{str}} \hookrightarrow \mathbf{KD}_{\mathrm{sat}}$ *admits a left adjoint, given by the completion functor* $M^* \mapsto \mathcal{W}(M)^*$.

Notation 2.7.5. Let M^* be a Dieudonné complex. We let $\mathcal{W}\,\mathrm{Sat}(M^*)$ denote the completion of the saturation $\mathrm{Sat}(M^*)$.

Corollary 2.7.6. *The inclusion functor* $\mathbf{KD}_{\mathrm{str}} \hookrightarrow \mathbf{KD}$ *admits a left adjoint, given by the completed saturation functor* $\mathcal{W}\,\mathrm{Sat} : \mathbf{KD} \to \mathbf{KD}_{\mathrm{str}}$.

Proof. Combine Corollaries 2.3.2 and 2.8.3. □

Here again there is a form of these results for strict Dieudonné towers, which we record for future reference.

Proposition 2.7.7. *If* $\{X_r^*\} \in \mathbf{TD}$ *and* $X^* = \varprojlim X_r^* \in \mathbf{KD}_{\mathrm{sat}}$, *then we have a natural equivalence in* $\mathbf{TD}$, $\{\mathcal{W}_r(X)^*\} \simeq \{X_r^*\}$.

Proof. For each r, we have the map $X^* \to X_r^*$, which is clearly compatible with F, V. We claim that this map induces an isomorphism $\mathcal{W}_r(X)^* \simeq X_r^*$.

We claim first that the kernel of the restriction map $X_{r+k}^* \to X_r^*$ is the span of the images of $V^r, dV^r : X_k^* \to X_{r+k}^*$. We argue by induction on k; for $k = 1$, it is part of the assumption. If we have proven it for $k-1$, we consider the composite of surjections $X_{r+k}^* \to X_{r+k-1}^* \to X_r^*$. Since V^r, dV^r generate the kernel of the second map by the inductive hypothesis and V^{r+k-1}, dV^{r+k-1} generate the kernel of the first map by assumption, we find that V^r, dV^r generate the kernel of the composite.

Thus, for each k, we have an exact sequence

$$X_k^* \oplus X_k^{*-1} \xrightarrow{V^r, dV^r} X_{r+k}^* \to X_r^* \to 0.$$

The above is an exact sequence of inverse systems (under the restriction maps R) as k varies. The restriction maps in the first term are all surjective. As a result, when we take the inverse limit we find an exact sequence

$$X^* \oplus X^{*-1} \xrightarrow{V^r, dV^r} X^* \to X_r^* \to 0.$$

This implies that $X_r^* \simeq \mathcal{W}_r(X)^*$, as desired. □

Corollary 2.7.8. *The composite functor* $\mathbf{KD}_{\mathrm{str}} \hookrightarrow \mathbf{KD} \to \mathbf{TD}$ *sending a strict Dieudonné complex* X^* *to the strict Dieudonné tower* $\{\mathcal{W}_r(X)^*\}$*, is an equivalence of categories, whose inverse sends a tower* $\{X_r^*\}$ *to* $X^* = \varprojlim_r X_r^*$.

Proof. Combine Propositions 2.6.11, 2.7.7 and the above analysis. □

2.8. Comparison of M^* and $\mathcal{W}(M)^*$.

In this subsection, we record a few homological properties of the construction $\mathcal{W}(\cdot)$: in particular, that it does not change the mod p quasi-isomorphism type of a saturated Dieudonné complex.

Proposition 2.8.1. *Let* M^* *be a saturated Dieudonné complex and let* $\mathcal{W}(M)^*$ *be its completion. Then the tautological map* $\rho_M : M^* \to \mathcal{W}(M)^*$ *induces a quasi-isomorphism* $M^*/p^r M^* \to \mathcal{W}(M)^*/p^r \mathcal{W}(M)^*$ *for every nonnegative integer* r.

Proof. Combine Proposition 2.7.1 with Corollary 2.5.5. □

Corollary 2.8.2. *Let* M^* *be a saturated Dieudonné complex and let* $\mathcal{W}(M)^*$ *be its completion. Then the tautological map* $\rho_M : M^* \to \mathcal{W}(M)^*$ *exhibits* $\mathcal{W}(M)^*$ *as a* p*-completion of* M^* *in the derived category of abelian groups.*

Proof. By virtue of Proposition 2.8.1, it will suffice to show that $\mathcal{W}(M)^*$ is a p-complete object of the derived category. But in fact something stronger is true: each term in the cochain complex $\mathcal{W}(M)^*$ is a p-complete abelian group, since $\mathcal{W}(M)^n$ is given by an inverse limit $\varprojlim \mathcal{W}_r(M)^n$, where $\mathcal{W}_r(M)^n$ is annihilated by p^r. □

Corollary 2.8.3. *Let* M^* *be a saturated Dieudonné complex and let* $\mathcal{W}(M)^*$ *be its completion. The following conditions are equivalent:*

1. *The tautological map* $\rho_M : M^* \to \mathcal{W}(M)^*$ *is a quasi-isomorphism.*
2. *The underlying cochain complex* M^* *is a* p*-complete object of the derived category of abelian groups.*

Corollary 2.8.4. *Let M^* be a Dieudonné complex. Suppose that the abelian group M^k is p-complete and p-torsion free for each k, and that M^* is of Cartier type. Then the canonical map $u : M^* \to \mathcal{W}\operatorname{Sat}(M^*)$ is a quasi-isomorphism.*

Proof. Note that the domain and codomain of u are cochain complexes of abelian groups which are p-complete and p-torsion-free. Consequently, to show that u is a quasi-isomorphism, it will suffice to show that the induced map $M^*/pM^* \to \mathcal{W}\operatorname{Sat}(M^*)/p\mathcal{W}\operatorname{Sat}(M^*)$ is a quasi-isomorphism. This map factors as a composition

$$M^*/pM^* \xrightarrow{v} \operatorname{Sat}(M^*)/p\operatorname{Sat}(M^*) \xrightarrow{v'} \mathcal{W}\operatorname{Sat}(M^*)/p\mathcal{W}\operatorname{Sat}(M^*).$$

The map v is a quasi-isomorphism by virtue of Theorem 2.4.2, and the map v' is a quasi-isomorphism by Proposition 2.8.1. □

2.9. **Examples.**

Example 2.9.1. There is a strict Dieudonné complex X^* with $X^0 = \mathbf{Z}_p$, $X^i = 0$ for $i \neq 0$, and such that F is the identity. Given $Y^* \in \mathbf{KD}_{\mathrm{str}}$, $\operatorname{Hom}_{\mathbf{KD}}(X^*, Y^*)$ is given by the submodule of Y^0 given by those cycles fixed under F.

Example 2.9.2. Let A be the p-completion of a free $\mathbf{Z}$-module and let $\varphi : A \simeq A$ be an automorphism. Then there is a strict Dieudonné complex X^* concentrated in degree zero with $X^0 = A$ and $F = \varphi$.

Example 2.9.3. We construct the free strict Dieudonné complex X^* on a generator x as follows. First, X^0 is the set of formal infinite sums $\sum_{n \geq 0} a_n F^n x + \sum_{n>0} b_n V^n x$ with $a_n, b_n \in \mathbf{Z}_p$ and where $a_n \to 0$ p-adically as $n \to \infty$. Next, X^{-1} is the set of formal sums $\sum_{n\geq 0} c_n F^n dx + \sum_{n>0} dV^n x$ where $c_n, d_n \in \mathbf{Z}_p$ and $c_n \to 0$ p-adically as $n \to \infty$. The differential on X^* and the operator F are evident. We leave it to the reader to verify that X^* is a strict Dieudonné complex, and that for any $Y^* \in \mathbf{KD}_{\mathrm{str}}$, we have $\operatorname{Hom}_{\mathbf{KD}}(X^*, Y^*) = Y^0$.

We now give a much more general construction of strict Dieudonné complexes, which is a special case of the approach taken in Section 6 below.

Example 2.9.4. Let M be an abelian group. Suppose given a surjection $g : M \twoheadrightarrow M$ whose kernel is precisely the subgroup $M[p] \subset M$ consisting of (simple) p-torsion. Then we will construct a two-term strict Dieudonné complex X^* as follows. We build a strict Dieudonné tower $\{X_n^*\}$ such that:

(1) $X_n^0 = M/p^n, X_n^{-1} = M[p^n]$, and the differential $d : X_n^{-1} \to X_n^0$ is the composite $M[p^n] \subset M \to M/p^n$.
(2) The map $R : X_n^* \to X_{n-1}^*$ is obtained by applying the map g. Note that the hypotheses on g imply that g carries $M[p^n]$ into $M[p^{n-1}]$.
(3) $F : X_n^* \to X_{n-1}^*$ is given by the quotient map $M/p^n \twoheadrightarrow M/p^{n-1}$ and the map $M[p^n] \to M[p^{n-1}]$ given by multiplication by p.

(4) $V : X^*_{n-1} \to X^*_n$ is given by the inclusion $M[p^{n-1}] \to M[p^n]$ and the map $p : M/p^{n-1} \to M/p^n$.

One checks easily that the above gives a strict Dieudonné tower, and taking the inverse limit yields a strict Dieudonné complex.

3. Dieudonné Algebras

We next study the multiplicative version, and introduce the theory of Dieudonné algebras in this section. We show in particular that de Rham complexes of torsion-free rings equipped with a Frobenius lift give rise to Dieudonné algebras (Proposition 3.2.1) and that operations of saturation and strictifications interact well with the algebra structure.

3.1. Definitions.

Definition 3.1.1. A *Dieudonné algebra* is a triple (A^*, d, F), where (A^*, d) is a commutative differential graded algebra, $F : A^* \to A^*$ is an algebra homomorphism satisfying the following additional conditions:

- (i) The map F satisfies the identity $dF(x) = pF(dx)$ for $x \in A^*$.
- (ii) The groups A^n vanish for $n < 0$.
- (iii) For $x \in A^0$, we have $Fx \equiv x^p \pmod p$.

Let (A^*, d, F) and (A'^*, d', F') be Dieudonné algebras. A *morphism of Dieudonné algebras* from (A^*, d, F) to (A'^*, d', F') is a graded ring homomorphism $f : A^* \to A'^*$ satisfying $d' \circ f = f \circ d$ and $F' \circ f = f \circ F$. We let **AD** denote the category of whose objects are Dieudonné algebras and whose morphisms are Dieudonné algebra morphisms.

Remark 3.1.2. In the situation of Definition 3.1.1, we will generally abuse terminology and simply refer to A^* as a Dieudonné algebra.

Remark 3.1.3. There is an evident forgetful functor $\mathbf{AD} \to \mathbf{KD}$, which carries a Dieudonné algebra A^* to its underlying Dieudonné complex (obtained by neglecting the multiplication on A^*).

Remark 3.1.4. Let A^* be a commutative differential graded algebra, and suppose that A^* is p-torsion-free. Let $\eta_p A^* \subseteq A^*[p^{-1}]$ denote the subcomplex introduced in Remark 2.2.2. Then $\eta_p A^*$ is closed under multiplication, and therefore inherits the structure of a commutative differential graded algebra. Moreover, the dictionary of Remark 2.2.2 establishes a bijection between the following data:

- Maps of differential graded algebras $\alpha : A^* \to \eta_p A^*$
- Graded ring homomorphisms $F : A^* \to A^*$ satisfying $d \circ F = p(F \circ d)$.

Remark 3.1.5. Let (A^*, d, F) be a Dieudonné algebra, and suppose that A^* is p-torsion-free. Then we can regard $\eta_p A^*$ as a subalgebra of A^*, which is closed under the action of F. We claim that $(\eta_p A^*, d|_{\eta_p A^*}, F|_{\eta_p A^*})$ is also a Dieudonné algebra. The only nontrivial point is to check condition (iii) of Definition 3.1.1. Suppose we are given an element $x \in (\eta_p A)^0$: that is, an element $x \in A^0$ such that $dx = py$ for some $y \in A^1$. Since A^* satisfies condition (iii) of Definition 3.1.1, we can write $Fx = x^p + pz$ for some element $z \in A^0$. We wish to show that z also belongs to $(\eta_p A)^0$: that is, $dz \in pA^1$. This follows from the calculation

$$p(dz) = d(pz) = d(Fx - x^p) = pF(dx) - px^{p-1}dx = p^2F(y) - p^2x^{p-1}y,$$

since A^1 is p-torsion-free.

Example 3.1.6. Let R be a commutative $\mathbf{F}_p$-algebra and let $W(R)$ denote the ring of Witt vectors of R, regarded as a commutative differential graded algebra which is concentrated in degree zero. Then the Witt vector Frobenius $F : W(R) \to W(R)$ exhibits $W(R)$ as a Dieudonné algebra, in the sense of Definition 3.1.1. The Dieudonné algebra $W(R)$ is saturated if and only if the $\mathbf{F}_p$-algebra R is perfect; if this condition is satisfied, then $W(R)$ is also strict.

3.2. **Example: de Rham Complexes.**

Proposition 3.2.1. *Let R be a commutative ring which is p-torsion-free, and let $\varphi : R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$. Then there is a unique ring homomorphism $F : \Omega^*_R \to \Omega^*_R$ with the following properties:*

(1) *For each element $x \in R = \Omega^0_R$, we have $F(x) = \varphi(x)$.*
(2) *For each element $x \in R$, we have $F(dx) = x^{p-1}dx + d(\frac{\varphi(x)-x^p}{p})$*

Moreover, the triple (Ω^*_R, d, F) is a Dieudonné algebra.

Remark 3.2.2. In the situation of Proposition 3.2.1, suppose that the de Rham complex Ω^*_R is p-torsion-free, and regard $\eta_p\Omega^*_R$ as a (differential graded) subalgebra of Ω^*_R. The ring homomorphism φ extends uniquely to a map of commutative differential graded algebras $\alpha : \Omega^*_R \to \Omega^*_R$. The calculation

$$\alpha(dx) = d\alpha(x) = d\varphi(x) \equiv dx^p = px^{p-1}dx \equiv 0 \pmod{p}$$

shows that α carries Ω^n_R into $p^n\Omega^n_R$, and can therefore be regarded as a map of differential graded algebras. $\Omega^*_R \to \eta_p\Omega^*_R$. Applying Remark 3.1.4, we see that there is a unique ring homomorphism $F : \Omega^*_R \to \Omega^*_R$ which satisfies condition (1) of Proposition 3.2.1 and endows Ω^*_R with the structure of a Dieudonné algebra. Moreover, the map F automatically satisfies condition (2) of Proposition 3.2.1: this follows from the calculation

$$\begin{aligned} pF(dx) &= d(Fx) \\ &= d(\varphi(x)) \\ &= d(x^p + p\frac{\varphi(x) - x^p}{p}) \\ &= p(x^{p-1}dx + d\frac{\varphi(x) - x^p}{p}). \end{aligned}$$

Proof of Proposition 3.2.1. The uniqueness of the homomorphism F is clear, since the de Rham complex Ω^*_R is generated (as a graded ring) by elements of the form

x and dx, where $x \in R$. We now prove existence. Define $\theta : R \to R$ by the formula $\theta(x) = \frac{\varphi(x) - x^p}{p}$. A simple calculation then gives

$$\theta(x+y) = \theta(x) + \theta(y) - \sum_{0<i<p} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i} \tag{9}$$

$$\theta(xy) = \varphi(x)\theta(y) + \theta(x)\varphi(y) - p\theta(x)\theta(y). \tag{10}$$

Consider the map $\rho : R \to \Omega^1 R$ given by the formula $\rho(x) = x^{p-1}dx + d\theta(x)$. We first claim that ρ is a group homomorphism. This follows from the calculation

$$\begin{aligned}
\rho(x+y) &= (x+y)^{p-1} d(x+y) + d\theta(x+y) \\
&= (x+y)^{p-1} d(x+y) + d\theta(x) + d\theta(y) - d(\sum_{0<i<p} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}) \\
&= \rho(x) + \rho(y) + ((x+y)^{p-1} - x^{p-1} - \sum_{0<i<p} \frac{(p-1)!}{(i-1)!(p-i)!} x^{i-1} y^{p-i}) dx \\
&\quad + ((x+y)^{p-1} - y^{p-1} - \sum_{0<i<p} \frac{(p-1)!}{i!(p-i-1)!} x^i y^{p-i-1}) dy \\
&= \rho(x) + \rho(y).
\end{aligned}$$

We next claim that ρ is a φ-linear derivation from R into Ω^1_R: that is, it satisfies the identity $\rho(xy) = \varphi(y)\rho(x) + \varphi(x)\rho(y)$. This follows from the calculation

$$\begin{aligned}
\rho(xy) &= (xy)^{p-1} d(xy) + d\theta(xy) \\
&= (xy)^{p-1} d(xy) + d(\varphi(x)\theta(y) + \theta(x)\varphi(y) - p\theta(x)\theta(y)) \\
&= (x^{p-1} y^p dx + \theta(y) d\varphi(x) + \varphi(y) d\theta(x) - p\theta(y) d\theta(x)) \\
&\quad + (x^p y^{p-1} dy + \varphi(x) d\theta(y) + \theta(x) d\varphi(y) - p\theta(x) d\theta(y)) \\
&= ((y^p + p\theta(y)) x^{p-1} dx + \varphi(y) d\theta(x)) + ((x^p + p\theta(x)) y^{p-1} dy + \varphi(x) d\theta(y)) \\
&= \varphi(y)(x^{p-1} dx + d\theta(x)) + \varphi(x)(y^{p-1} dy + d\theta(y)) \\
&= \varphi(y)\rho(x) + \varphi(x)\rho(y).
\end{aligned}$$

Invoking the universal property of the module of Kähler differentials Ω^1_R, we deduce that there is a unique φ-semilinear map $F : \Omega^1_R \to \Omega^1_R$ satisfying $F(dx) = \lambda(x)$ for $x \in R$. For every element $\omega \in \Omega^1_R$, we have $F(\omega)^2 = 0$ in Ω^*_R (since $F(\omega)$ is a 1-form), so F extends to a ring homomorphism $\Omega^*_R \to \Omega^*_R$ satisfying (1) and (2).

To complete the proof, it will suffice to show that F endows Ω^*_R with the structure of a Dieudonné algebra. It follows (1) that $F(x) \equiv x^p \pmod{p}$ for $x \in \Omega^0_R$. It will therefore suffice to show that $d \circ F = p(F \circ d)$. Let $A \subseteq \Omega^*_R$ be the subset consisting of those elements ω which satisfy $dF(\omega) = pF(d\omega)$. It is clear that A is a graded subgroup of Ω^*_R. Moreover, A is closed under multiplication:

if $\omega \in \Omega^m_R$ and $\omega' \in \Omega^n_R$ belong to A, then we compute

$$\begin{aligned} dF(\omega \wedge \omega') &= d(F(\omega) \wedge F(\omega')) \\ &= ((dF\omega) \wedge F\omega') + (-1)^m (F\omega \wedge (dF\omega')) \\ &= p((Fd\omega) \wedge F\omega') + (-1)^m (F\omega \wedge (Fd\omega')) \\ &= pF(d\omega \wedge \omega' + (-1)^m \omega \wedge d\omega') \\ &= pFd(\omega \wedge \omega'). \end{aligned}$$

Consequently, to show that $A = \Omega^*_R$, it will suffice to show that A contains x and dx for each $x \in R$. The inclusion $dx \in A$ is clear (note that $Fddx = 0 = dF(dx)$, since $F(x) = x^{p-1}dx + d\theta(x))$ is a closed 1-form). The inclusion $x \in A$ follows from the calculation

$$\begin{aligned} dFx &= d\varphi(x) \\ &= d(x^p + p\theta(x)) \\ &= px^{p-1}dx + pd\theta(x) \\ &= p(x^{p-1}dx + d\theta(x)) \\ &= pF(dx). \end{aligned}$$

□

The Dieudonné algebra of Proposition 3.2.1 enjoys the following universal property:

Proposition 3.2.3. *Let R be a commutative ring which is p-torsion-free, let $\varphi : R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod p$, and regard the de Rham complex Ω^*_R as equipped with the Dieudonné algebra structure of Proposition 3.2.1. Let A^* be another Dieudonné algebra which is p-torsion-free. Then the restriction map*

$$\mathrm{Hom}_{\mathbf{AD}}(\Omega^*_R, A^*) \to \mathrm{Hom}(R, A^0)$$

is injective, and its image consists of those ring homomorphisms $f : R \to A^0$ satisfying $f(\varphi(x)) = F(f(x))$ for $x \in R$.

Proof. Let $f : R \to A^0$ be a ring homomorphism. Invoking the universal property of the de Rham complex Ω^*_R, we see that f extends uniquely to a homomorphism of differential graded algebras $\overline{f} : \Omega^*_R \to A^*$. To complete the proof, it will suffice to show that $\overline{f}$ is a map of Dieudonné algebras if and only if f satisfies the identity $f(\varphi(x)) = F(f(x))$ for each $x \in R$. The "only if" direction is trivial. Conversely, suppose that $f(\varphi(x)) = F(f(x))$ for $x \in R$; we wish to show that $\overline{f}(F\omega) = F\overline{f}(\omega)$ for each $\omega \in \Omega^*_R$. The collection of those elements $\omega \in \Omega^*_R$ which satisfy this identity span a subring of Ω^*_R. Consequently, we may assume without loss of generality that $\omega = x$ or $\omega = dx$, for some $x \in R$. In the first case, the desired result follows from our assumption. To handle the second, we compute

$$\begin{aligned} p\overline{f}(Fdx) &= p\overline{f}(x^{p-1}dx + d\frac{\varphi(x) - x^p}{p}) \\ &= \overline{f}(px^{p-1}dx + d(\varphi(x) - x^p)) \\ &= \overline{f}(d\varphi(x)) \\ &= d\overline{f}(\varphi(x)) \\ &= dFf(x) \\ &= pFd(f(x)). \end{aligned}$$

Since A^1 is p-torsion-free, it follows that $\overline{f}(Fdx) = Fd\overline{f}(x)$, as desired. □

Remark 3.2.4. Although we will not use this in the sequel, we observe that the above results can be generalized to the case where R has p-torsion. We sketch the details in the remainder of this subsection, which can be skipped without loss of continuity.

Recall first the notion of a *λ_p-ring* or *θ-algebra*. This is a p-typical notion of a λ-ring introduced by Joyal [17]. The theory of λ_p-rings is also developed using the language of *plethories* in [8]. A λ_p-ring is a commutative ring R equipped with a function $\theta : R \to R$ satisfying the equations (9), (10), with $\varphi : R \to R$ defined as $\varphi(x) = x^p + p\theta(x)$. In other words, $\varphi : R \to R$ is a ring homomorphism which is congruent modulo p to the Frobenius. However, we require a "witness" θ of the Frobenius congruence, which satisfies all the relations it needs to for φ to be a ring homomorphism that one can obtain by "dividing by p." Crucially, when R is p-torsion-free, a λ_p-ring is precisely a commutative ring with an endomorphism φ which lifts the Frobenius mod p; this is a p-typical version of the Wilkerson criterion [29] for λ-rings. The free λ_p-ring on one generator is given by $S = \mathbf{Z}[x, \theta(x), \theta^2(x), \dots]$, and the free λ_p-ring on a set of generators is a tensor product of copies of S.

Definition 3.2.5. A *λ_p-cdga* is the datum of:

(1) A commutative, differential graded algebra (X^*, d) such that $X^i = 0$ for $i < 0$.
(2) A self-map $F : X^* \to X^*$ of graded commutative rings such that F such that $dF = pFd$, i.e., F equips (X^*, d) with the structure of a Dieudonné complex.
(3) The structure of a λ_p-ring on X^0 such that $F(x) = \varphi(x) = x^p + p\theta(x)$ for $x \in X^0$.
(4) The relation $Fdx = x^{p-1}dx + d\theta(x)$ holds for any $x \in X^0$. Note that this relation is automatic if X^1 is p-torsion-free, since it holds after multiplying by p.

Note that a λ_p-cdga is a special type of Dieudonné algebra; a λ_p-cdga always defines a Dieudonné algebra, and a Dieudonné algebra with p-torsion free terms yields a λ_p-cdga.

Proposition 3.2.6. *Let R be a λ_p-ring. Then the de Rham complex Ω_R^* of R is naturally a λ_p-cdga extending the λ_p-structure on R. Moreover, for any λ_p-cdga X^* we have a canonical bijection*

$$\{\lambda_p - \text{cdga maps } \Omega_R^* \to X^*\} \simeq \left\{\lambda_p - \text{ring maps } R \to X^0\right\}.$$

In other words, the construction $R \mapsto \Omega_R^*$ is the left adjoint of the forgetful functor $X^* \mapsto X^0$ from λ_p-cdgas to λ_p-rings.

Proof. When R, Ω_R^1 and X^* are p-torsion-free, this is precisely Proposition 3.2.3. In general, suppose first that R, Ω_R^1 are p-torsion-free but do not assume any condition on X^*. Then, given a map of λ_p-rings $f : R \to X^0$, we obtain first a map of commutative differential graded algebras $\overline{f} : \Omega_R^* \to X^*$, using the universal property of the de Rham complex. It remains to check that $\overline{f}$ is a map of λ_p-cdgas, i.e., that it commutes with F and with θ in degree zero. Clearly $\overline{f}$ commutes with θ, since f does. Moreover, F is an algebra map, so (as before) it suffices to see that $\overline{f}(F dx) = F\overline{f}(dx), x \in R$. This follows from the assumptions.

Finally, the case of a general λ_p-ring reduces to the case where $\{R, \Omega_R^1\}$ are p-torsion-free as follows: the category of λ_p-rings is generated under reflexive coequalizers by the free algebras. Given any λ_p-ring R, we can write R as a reflexive coequalizer of $R_2 \rightrightarrows R_1$, where R_2, R_1 are free on sets of generators. Thus $R_1, R_2, \Omega_{R_1}^1, \Omega_{R_2}^1$ are p-torsion-free. Thus $\Omega_{R_1}^*, \Omega_{R_2}^*$ are λ_p-cdgas with the desired universal property. Taking reflexive coequalizers in λ_p-cdgas (which are computed at the level of graded abelian groups), we find that Ω_R^* acquires the structure of a λ_p-cdga with the desired universal property. □

3.3. **The Cartier Isomorphism.** We will need a variant of Proposition 3.2.1 for *completed* de Rham complexes.

Variant 3.3.1 (Completed de Rham Complexes)**.** Let R be a commutative ring. We let $\widehat{\Omega}_R^*$ denote the inverse limit

$$\varprojlim_n \Omega_R^*/p^n\Omega_R^* \simeq \varprojlim_n \Omega_{R/p^nR}^*.$$

We will refer to $\widehat{\Omega}_R^*$ as the *completed de Rham complex* of R. Note that there is a unique multiplication on $\widehat{\Omega}_R^*$ for which the tautological map $\Omega_R^* \to \widehat{\Omega}_R^*$ is a morphism of differential graded algebras.

Now suppose that R is p-torsion-free, and that we are given a ring homomorphism $\varphi : R \to R$ satisfying $\varphi(x) \equiv x^p \pmod p$. Then the homomorphism $F : \Omega_R^* \to \Omega_R^*$ of Proposition 3.2.1 induces a map $\widehat{\Omega}_R^* \to \widehat{\Omega}_R^*$, which we will also denote by F, which endows $\widehat{\Omega}^*_R$ with the structure of a Dieudonné algebra. Moreover, this Dieudonné algebra enjoys the following universal property: if A^* is any Dieudonné algebra which is p-adically complete and p-torsion-free, then the restriction map

$$\mathrm{Hom}_{\mathbf{AD}}(\widehat{\Omega}^*_R, A^*) \to \mathrm{Hom}(R, A^0)$$

is an injection, whose image is the collection of ring homomorphisms $f : R \to A^0$ satisfying $f \circ \varphi = F \circ f$. This follows immediately from Proposition 3.2.3.

Remark 3.3.2. Let R be a commutative ring. Then the quotient $\widehat{\Omega}^*_R / p^n \widehat{\Omega}^*_R$ can be identified with the de Rham complex $\Omega^*_{R/p^n R}$.

Remark 3.3.3. Let R be a commutative ring which is p-torsion-free. Suppose that there exists a perfect $\mathbf{F}_p$-algebra k such that R/p is a smooth k-algebra. Then each $R/p^n R$ is also a smooth $W_n(k)$-algebra, and our assumption that k is perfect guarantees that the quotient $\widehat{\Omega}^*_R / p^n \widehat{\Omega}^*_R$ can be identified with the de Rham complex of $R/p^n R$ *relative* to $W_n(k)$. It follows that each $\widehat{\Omega}^i_R / p^n \widehat{\Omega}^i_R$ is a projective $R/p^n R$-module of finite rank, and that the transition maps

$$(\widehat{\Omega}^i_R / p^n \widehat{\Omega}^i_R) \otimes_{\mathbf{Z}/p^n \mathbf{Z}} (\mathbf{Z}/p^{n-1}\mathbf{Z}) \to (\widehat{\Omega}^i_R / p^{n-1} \widehat{\Omega}^i_R)$$

are isomorphisms. In this case, we conclude that each $\widehat{\Omega}^i_R$ is a projective module of finite rank over the completion $\widehat{R} = \varprojlim R/p^n R$. In particular, $\widehat{\Omega}^*_R$ is p-torsion-free.

Remark 3.3.4. Let R be a commutative ring which is p-torsion-free, let $\varphi : R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$, and regard $\widehat{\Omega}^*_R$ as a Dieudonné algebra as in Variant 3.3.1. Then the Frobenius map $F : \widehat{\Omega}^*_R \to \widehat{\Omega}^*_R$ induces a map of graded rings

$$\mathrm{Cart} : \Omega^*_{R/p} \simeq \widehat{\Omega}^*_R / p \widehat{\Omega}^*_R \to \mathrm{H}^*(\widehat{\Omega}^*_R / p \widehat{\Omega}^*_R) \simeq \mathrm{H}^*(\Omega^*_{R/p}).$$

Using the formulae of Proposition 3.2.1, we see that this map is given concretely by the formula

$$\mathrm{Cart}(x_0 dx_1 \wedge \cdots dx_n) = [x_0^p (x_1^{p-1} dx_1) \wedge \cdots \wedge (x_n^{p-1} dx_n)],$$

where $[\omega]$ denotes the cohomology class represented by a cocycle ω. In particular, this formula shows that Cart is independent of the choice of φ.

We will need the following result. Compare [19, Th. 7.2] for a treatment.

Theorem 3.3.5 (Cartier Isomorphism)**.** *Let R be a commutative ring which is p-torsion-free and let $\varphi : R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$. Suppose that there exists a perfect $\mathbf{F}_p$-algebra k such that R/p is a smooth algebra over k. Then the map $\mathrm{Cart} : \Omega^*_{R/p} \to \mathrm{H}^*(\Omega^*_{R/p})$ is an isomorphism. That is, the Dieudonné complex $\widehat{\Omega}^*_R$ is of Cartier type in the sense of Definition 2.4.1.*

Corollary 3.3.6. *Let R be a commutative ring which is p-torsion-free and let $\varphi : R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod p$. Suppose that there exists a perfect $\mathbf{F}_p$-algebra k such that R/p is a smooth algebra over k. Then the canonical map $\widehat{\Omega}^*_R \to \mathcal{W}\operatorname{Sat}(\widehat{\Omega}^*_R)$ is a quasi-isomorphism of chain complexes.*

Proof. Combine Theorem 3.3.5 with Corollary 2.8.4. □

3.4. **Saturation of Dieudonné Algebras.**

Definition 3.4.1. Let A^* be a Dieudonné algebra. We will say that A^* is *saturated* if it is saturated when regarded as a Dieudonné complex: that is, if it is p-torsion-free and the map $F : A^* \to \{x \in A^* : dx \in pA^*\}$ is a bijection. We let $\mathbf{AD}_{\mathrm{sat}}$ denote the full subcategory of $\mathbf{AD}$ spanned by the saturated Dieudonné algebras.

In the setting of saturated Dieudonné algebras, axiom (iii) of Definition 3.1.1 can be slightly weakened:

Proposition 3.4.2. *Let A^* be a saturated Dieudonné complex satisfying $A^* = 0$ for $* < 0$. Suppose that A^* is equipped with the structure of a commutative differential graded algebra for which the Frobenius map $F : A^* \to A^*$ is a ring homomorphism. The following conditions are equivalent:*

- (a) *The complex A^* is a Dieudonné algebra: that is, each element $x \in A^0$ satisfies $Fx \equiv x^p \pmod p$.*
- (b) *Each element $x \in A^0$ satisfies $Fx \equiv x^p \pmod{VA^0}$.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. For the converse, suppose that (b) is satisfied and let $x \in A^0$. Then we can write $Fx = x^p + Vy$ for some $y \in A^0$. Applying the differential d, we obtain $d(Vy) = d(Fx - x^p) = p(F(dx) - x^{p-1}dx) \in pA^1$. Invoking Lemma 2.6.5, we can write $y = Fz$ for some $z \in A^0$, so that $Fx = x^p + Vy = x^p + VFz = x^p + pz \equiv x^p \pmod p$. □

Let $f : A^* \to B^*$ be a morphism of Dieudonné algebras. We will say that f *exhibits B^* as a saturation of A^** if B^* is saturated and, for every saturated Dieudonné algebra C^*, composition with f induces a bijection

$$\operatorname{Hom}_{\mathbf{AD}}(B^*, C^*) \to \operatorname{Hom}_{\mathbf{AD}}(A^*, C^*).$$

Proposition 3.4.3. *Let A^* be a Dieudonné algebra. Then there exists a map of Dieudonné algebras $f : A^* \to B^*$ which exhibits B^* as a saturation of A^*. Moreover, f induces an isomorphism of Dieudonné complexes $\operatorname{Sat}(A^*) \to B^*$, where $\operatorname{Sat}(A^*)$ is the saturation of §2.3.*

Remark 3.4.4. We can summarize Proposition 3.4.3 more informally as follows: if A^* is a Dieudonné algebra and $\operatorname{Sat}(A^*)$ is the saturation of A^* in the category Dieudonné complexes, then $\operatorname{Sat}(A^*)$ inherits the structure of a Dieudonné algebra, and is also a saturation of A^* in the category of Dieudonné algebras.

Proof of Proposition 3.4.3. Replacing A^* by the quotient $A^*/A^*[p^\infty]$, we can reduce to the case where A^* is p-torsion-free. In this case, the Frobenius on A determines a map of Dieudonné algebras $\alpha_F : A^* \to (\eta_p A)^*$ (Remark 3.1.5), and A^* is saturated if and only if α_F is an isomorphism. The saturation of A can then be computed as the direct limit of the sequence

$$A^* \xrightarrow{\alpha_F} (\eta_p A)^* \xrightarrow{\eta_p(\alpha_F)} (\eta_p \eta_p A)^* \xrightarrow{\eta_p(\eta_p(\alpha_F))} (\eta_p \eta_p \eta_p A)^* \to \cdots,$$

which also computes the saturation of A^* as a Dieudonné complex. □

3.5. Completions of Dieudonné Algebras.

Proposition 3.5.1. *Let A^* be a saturated Dieudonné algebra, and let $V : A^* \to A^*$ be the Verschiebung map of Remark 2.2.3. Then:*

- (i) *The map V satisfies the projection formula $xV(y) = V(F(x)y)$.*
- (ii) *For each $r \geq 0$, the sum $\mathrm{im}(V^r) + \mathrm{im}(dV^r) \subseteq A^*$ is a (differential graded) ideal.*

Proof. We first prove (i). Given $x, y \in A^*$, we compute

$$F(xV(y)) = F(x)F(V(y)) = pF(x)y = F(V(F(x)y)).$$

Since the map F is injective, we conclude that $xV(y) = V(F(x)y)$.

We now prove (ii). Let x be an element of A^*; we wish to show that multiplication by x carries $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$ into itself. This follows from the identities

$$xV^r(y) = V^r(F^r(x)y)$$

$$\pm x d(V^r y) = d(xV^r(y)) - (dx)V^r(y) = dV^r(F^r(x)y) - V^r(F^r(dx)y).$$

□

Corollary 3.5.2. *Let A^* be a saturated Dieudonné algebra. For each $r \geq 0$, there is a unique ring structure on the quotient $\mathcal{W}_r(A)^*$ for which the projection map $A^* \to \mathcal{W}_r(A)^*$ is a ring homomorphism. Moreover, this ring structure exhibits $\mathcal{W}_r(A)^*$ as a commutative differential graded algebra.*

Remark 3.5.3. Let A^* be a saturated Dieudonné algebra. Since $A^{-1} = 0$, we have $\mathcal{W}_r(A)^0 = A^0/V^r A^0$. In particular, we have $\mathcal{W}_1(A)^0 = A^0/VA^0$.

Construction 3.5.4. Let A^* be a saturated Dieudonné algebra. We let $\mathcal{W}(A)^*$ denote the completion of A^* as a (saturated) Dieudonné complex, given by the inverse limit of the tower

$$\cdots \to \mathcal{W}_3(A)^* \to \mathcal{W}_2(A)^* \to \mathcal{W}_1(A)^* \to \mathcal{W}_0(A)^* \simeq 0.$$

Since each $\mathcal{W}_n(A)^*$ has the structure of a commutative differential graded algebra, the inverse limit $\mathcal{W}(A)^*$ inherits the structure of a commutative differential graded algebra.

Proposition 3.5.5. *Let A^* be a saturated Dieudonné algebra. Then the completion $\mathcal{W}(A)^*$ is also a saturated Dieudonné algebra.*

Proof. It follows from Proposition 2.6.4 that $\mathcal{W}(A)^*$ is a saturated Dieudonné complex. Moreover, the Frobenius map $F : \mathcal{W}(A)^* \to \mathcal{W}(A)^*$ is an inverse limit of the maps $F : \mathcal{W}_r(A)^* \to \mathcal{W}_{r-1}(A)^*$ appearing in Remark 2.6.1), each of which is a ring homomorphism (since the Frobenius on A^* is a ring homomorphism). It follows that $F : \mathcal{W}(A)^* \to \mathcal{W}(A)^*$ is a ring homomorphism. The vanishing of $\mathcal{W}(A)^*$ for $* < 0$ follows immediately from the analogous property of A^*. To complete the proof, it will suffice to show that each element $x \in \mathcal{W}(A)^0$ satisfies $Fx \equiv x^p \pmod{V\,\mathcal{W}(A)^0}$ (Proposition 3.4.2). In other words, we must show that F induces the usual Frobenius map on the $\mathbf{F}_p$-algebra $\mathcal{W}_1(\mathcal{W}(A)^0) = \mathcal{W}(A)^0/V\,\mathcal{W}(A)^0$. This follows from the analogous property for A^*, since the tautological map $\mathcal{W}_1(A)^* \to \mathcal{W}_1(\mathcal{W}(A))^*$ is an isomorphism by Proposition 2.7.1. □

Let A^* be a saturated Dieudonné algebra. Then the tautological map $\rho_A : A^* \to \mathcal{W}(A)^*$ of §2.7 is a map of (saturated) Dieudonné algebras.

Definition 3.5.6. Let A^* be a saturated Dieudonné algebra. We will say that A^* is *strict* if the map $\rho_A : A^* \to \mathcal{W}(A)^*$ is an isomorphism of Dieudonné algebras. We let $\mathbf{AD}_{\mathrm{str}}$ denote the full subcategory of $\mathbf{AD}_{\mathrm{sat}}$ spanned by the strict Dieudonné algebras.

Remark 3.5.7. A saturated Dieudonné algebra A^* is complete if it is complete when regarded as a saturated Dieudonné complex.

We have the following nonlinear version of Proposition 2.7.3:

Proposition 3.5.8. *Let A^* and B^* be saturated Dieudonné algebras, where B^* is strict. Then composition with the map ρ_A induces a bijection*

$$\theta : \mathrm{Hom}_{\mathbf{AD}}(\mathcal{W}(A)^*, B^*) \to \mathrm{Hom}_{\mathbf{AD}}(A^*, B^*).$$

Proof. By virtue of Proposition 2.7.3, it will suffice to show that if $f : \mathcal{W}(A)^* \to B^*$ is a map of Dieudonné complexes for which $f \circ \rho_A$ is a ring homomorphism, then f is a ring homomorphism. Since B^* is complete, it will suffice to show that each of the composite maps $\mathcal{W}(A)^* \to B^* \to \mathcal{W}_r(B)^*$ is a ring homomorphism. This follows by inspecting the commutative diagram

$$\begin{array}{ccccc} A^* & \xrightarrow{\rho_A} & \mathcal{W}(A)^* & \xrightarrow{f} & B^* \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{W}_r(A)^* & \xrightarrow{\beta} & \mathcal{W}_r(\mathcal{W}(A))^* & \longrightarrow & \mathcal{W}_r(B)^*, \end{array}$$

since the map β bijective (Proposition 2.7.1). □

Corollary 3.5.9. *The inclusion functor* $\mathbf{AD}_{\mathrm{str}} \hookrightarrow \mathbf{AD}_{\mathrm{sat}}$ *admits a left adjoint, given by the formation of completions* $A^* \to \mathcal{W}(A)^*$.

Corollary 3.5.10. *The inclusion functor* $\mathbf{AD}_{\mathrm{str}} \hookrightarrow \mathbf{AD}$ *admits a left adjoint, given by the completed saturation functor* $\mathcal{W}\,\mathrm{Sat} : \mathbf{AD} \to \mathbf{AD}_{\mathrm{str}}$.

Proof. Combine Proposition 3.4.3 with Corollary 3.5.9. □

3.6. **Comparison with Witt Vectors.**

Lemma 3.6.1. *Let* A^* *be a saturated Dieudonné algebra. Then the quotient* $R = A^0/VA^0$ *is a reduced* $\mathbf{F}_p$*-algebra.*

Proof. It follows from Remark 3.5.3 that VA^0 is an ideal in A^0, so that we can regard R as a commutative ring. Note that the ideal VA^0 contains $V(1) = V(F(1)) = p$, so that R is an $\mathbf{F}_p$-algebra. To show that R is reduced, it will suffice to show that if $\overline{x}$ is an element of R satisfying $\overline{x}^p = 0$, then $\overline{x} = 0$. Choose an element $x \in A^0$ representing $\overline{x}$. The condition $\overline{x}^p = 0$ implies that $x^p \in VA^0$, so that $F(x) \equiv x^p \pmod p$ also belongs to VA^0. We can therefore write $Fx = Vy$ for some $y \in A^0$. Applying the differential d, we obtain $d(Vy) = d(Fx) = pF(dx) \in pA^1$. Invoking Lemma 2.6.5, we can write $y = Fz$ for some element $z \in A^0$. Then $Fx = VFz = FVz$, so that $x = Vz \in VA^0$ and therefore $\overline{x} = 0$, as desired. □

Proposition 3.6.2. *Let* A^* *be a strict Dieudonné algebra and let* $R = A^0/VA^0$. *Then there is a unique ring isomorphism* $u : A^0 \to W(R)$ *with the following properties:*

(*i*) *The diagram*

$$\begin{array}{ccc} A^0 & \xrightarrow{u} & W(R) \\ \downarrow & & \downarrow \\ A^0/VA^0 & \xrightarrow{\mathrm{id}} & R \end{array}$$

commutes (where the vertical maps are the natural projections).

(*ii*) *The diagram*

$$\begin{array}{ccc} A^0 & \xrightarrow{u} & W(R) \\ \downarrow F & & \downarrow F \\ A^0 & \xrightarrow{u} & W(R) \end{array}$$

commutes (where the right vertical map is the Witt vector Frobenius).

Proof. Since the ring R is reduced (Lemma 3.6.1), the ring of Witt vectors $W(R)$ is p-torsion free as it embeds inside $W(R_{\mathrm{perf}})$ where R_{perf} denotes the *perfection* $\varinjlim_{x \mapsto x^p} R$. Applying the universal property of $W(R)$ [17], we deduce that there exists a unique ring homomorphism $u : A^0 \to W(R)$ which satisfies conditions (*i*) and (*ii*). To complete the proof, it will suffice to show that u is an isomorphism.

We first note that u satisfies the identity $u(Vx) = Vu(x)$ for each $x \in A^0$ (where the second V denotes the usual Witt vector Verschiebung). To prove this, we begin with the identity

$$F(u(Vx)) = u(FVx) = u(px) = pu(x) = FVu(x)$$

and then invoke the fact that the Frobenius map $F : W(R) \to W(R)$ is injective (by virtue of the fact that R is reduced). It follows that u carries $\mathcal{W}_r(A)^0$ into $V^r W(R)$, and therefore induces a ring homomorphism $u_r : A^0/\mathcal{W}_r(A)^0 \to W_r(R)$ for every nonnegative integer r. Since A^* is complete, we can identify u with the inverse limit of the tower of maps $\{u_r\}_{r\geq 0}$. Consequently, to show that u is an isomorphism, it will suffice to show that each u_r is an isomorphism. We now proceed by induction on r, the case $r = 0$ being vacuous. We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0/VA^0 & \xrightarrow{V^{r-1}} & A^0/V^rA^0 & \longrightarrow & A^0/V^{r-1}A^0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_r} & & \downarrow{\scriptstyle u_{r-1}} & & \\ 0 & \longrightarrow & R & \xrightarrow{V^{r-1}} & W_r(R) & \longrightarrow & W_{r-1}(R) & \longrightarrow & 0 \end{array}$$

where u_1 is the identity map from $A^0/VA^0 = R$ to itself, and u_{r-1} is an isomorphism by the inductive hypothesis. It follows that u_r is also an isomorphism, as desired. □

Combining Proposition 3.6.2 with the universal property of the Witt vectors $W(R)$, we obtain the following result:

Corollary 3.6.3. *Let B be a commutative ring which is p-torsion-free, and let $\varphi : B \to B$ be a ring homomorphism satisfying $\varphi(b) \equiv b^p \pmod p$ for each $b \in B$. Let A^* be a strict Dieudonné algebra. Then every ring homomorphism $f_0 : B \to A^0/VA^0$ admits an essentially unique lift to a ring homomorphism $f : B \to A^0$ which satisfies $f \circ \varphi = F \circ f$.*

We will need a slight variation on Corollary 3.6.3.

Proposition 3.6.4. *Let A^* be a strict Dieudonné algebra and let R be a commutative $\mathbf{F}_p$-algebra. For every ring homomorphism $f_0 : R \to A^0/VA^0$, there is a unique ring homomorphism $f : W(R) \to A^0$ satisfying the following pair of conditions:*

(*i*) *The diagram*

$$\begin{array}{ccc} W(R) & \xrightarrow{f} & A^0 \\ \downarrow & & \downarrow \\ R & \xrightarrow{f_0} & A^0/VA^0 \end{array}$$

commutes (where the vertical maps are the natural projections).

(*ii*) *The diagram*

$$\begin{array}{ccc} W(R) & \xrightarrow{f} & A^0 \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ W(R) & \xrightarrow{f} & A^0 \end{array}$$

commutes (where the left vertical map is the Witt vector Frobenius).

Remark 3.6.5. In the special case where R is reduced, the ring of Witt vectors $W(R)$ is p-torsion-free, so that Proposition 3.6.4 follows from Corollary 3.6.3.

In the situation of Proposition 3.6.4, we can apply Proposition 3.6.2 to identify A^0 with the ring of Witt vectors $W(A^0/VA^0)$. Consequently, the *existence* of the map $f : W(R) \to A^0$ is clear: we simply apply the Witt vector functor to the ring homomorphism $f_0 : R \to A^0/VA^0$. The uniqueness of f is a consequence of the following elementary assertion:

Lemma 3.6.6. *Let $f : R \to R'$ be a homomorphism of commutative $\mathbf{F}_p$-algebras, and let $g : W(R) \to W(R')$ be a ring homomorphism. Suppose that the diagrams*

$$\begin{array}{ccc} W(R) & \xrightarrow{g} & W(R') \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & R' \end{array} \qquad \begin{array}{ccc} W(R) & \xrightarrow{g} & W(R') \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ W(R) & \xrightarrow{g} & W(R') \end{array}$$

commute. If R' is reduced, then $g = W(f)$ coincides with the map determined by f (and the functoriality of the construction $A \mapsto W(A)$).

Proof. For each element $a \in R$, let $[a] \in W(R)$ denote its Teichmüller representative, and define $[b] \in W(R')$ for $b \in R'$ similarly. We first claim that g commutes is compatible with the formation of Teichmüller representatives: that is, we have $g([a]) = [f(a)]$ for each $a \in R$. For this, it will suffice to show that $g([a]) \equiv [f(a)] \pmod{V^r}$ for each $r \geq 0$. We proceed by induction on r. The case $r = 1$ follows from our compatibility assumption concerning f and g. To carry out the inductive step, let us assume that we have an identity of the form $g([a]) = [f(a)] + V^r b$ for some $x \in W(R')$. We then compute

$$\begin{aligned} Fg([a]) &= g(F[a]) \\ &= g([a]^p) \\ &= g([a])^p \\ &= ([f(a)] + V^r b)^p \\ &\equiv [f(a)]^p \pmod{pV^r} \\ &\equiv F[f(a)] \pmod{FV^{r+1}}. \end{aligned}$$

Since R' is reduced, the Frobenius map $F : W(R') \to W(R')$ is injective, so we deduce that $g([a]) \equiv [f(a)] \pmod{V^{r+1}}$, thereby completing the induction.

For each $x \in W(R)$, we have the identity

$$F(g(Vx)) = g(FVx) = g(px) = pg(x) = FVg(x).$$

Using the injectivity of F again, we conclude that $g(Vx) = Vg(x)$: that is, the map g commutes with Verschiebung.

Combining these observations, we see that for any element $x \in W(R)$ with Teichmüller expansion $\sum_{n \geq 0} V^n[a_n]$, we have a congruence

$$\begin{aligned} g(x) &\equiv g([a_0] + V[a_1] + \cdots + V^n[a_n]) \pmod{V^{n+1}} \\ &\equiv g([a_0]) + Vg([a_1]) + \cdots + V^n g([a_n]) \pmod{V^{n+1}} \\ &\equiv [f(a_0)] + V[f(a_1)] + \cdots + V^n[f(a_n)] \pmod{V^{n+1}} \\ &\equiv W(f)(x) \pmod{V^{n+1}.} \end{aligned}$$

Allowing n to vary, we deduce that $g(x) = W(f)(x)$, as desired. □

4. The de Rham-Witt Complex

In this section, we introduce the de Rham-Witt complex of an $\mathbf{F}_p$-algebra (Definition 4.1.1) as the strict Dieudonné algebra satisfying a particular universal property analogous to that of the de Rham complex. An essential result (Theorem 4.2.3) shows that this is quasi-isomorphic to the completed de Rham complex of a lift in case the algebra is smooth over a perfect ring. In Theorem 4.4.12, we show that the de Rham-Witt complex agrees (in the regular case) with the classical de Rham-Witt complex constructed in [15].

4.1. **Construction.**

Definition 4.1.1. Let A^* be a strict Dieudonné algebra and suppose we are given an $\mathbf{F}_p$-algebra homomorphism $f : R \to A^0/VA^0$. We will say that f *exhibits* A^* *as a de Rham-Witt complex of* R if it satisfies the following universal property: for every strict Dieudonné algebra B^*, composition with f induces a bijection

$$\mathrm{Hom}_{\mathbf{AD}}(A^*, B^*) \to \mathrm{Hom}(R, B^0/VB^0).$$

Notation 4.1.2. Let R be an $\mathbf{F}_p$-algebra. It follows immediately from the definitions that if there exists a strict Dieudonné algebra A^* and a map $f : R \to A^0/VA^0$ which exhibits A^* as a de Rham-Witt complex of R, then A^* (and the map f) are determined up to unique isomorphism. We will indicate this dependence by writing A^* as $\mathcal{W}\Omega^*_R$, and referring to A^* as *the de Rham-Witt complex of* R.

Warning 4.1.3. The de Rham Witt complex $\mathcal{W}\Omega^*_R$ of Notation 4.1.2 is generally not the same as the de Rham-Witt complex defined in [15] (though they agree for a large class of $\mathbf{F}_p$-algebras: see Theorem 4.4.12). For example, it follows from Lemma 3.6.1 that our de Rham-Witt complex $\mathcal{W}\Omega^*_R$ is the same as the de Rham-Witt complex $\mathcal{W}\Omega^*_{R^{\mathrm{red}}}$, where R^{red} denotes the quotient of R by its nilradical. The de Rham-Witt complex of Deligne and Illusie does not have this property. In fact, with a bit more work, one can also find reduced rings R where the two complexes differ (Remark 7.1.2).

For existence, we have the following:

Proposition 4.1.4. *Let* R *be an* $\mathbf{F}_p$*-algebra. Then there exists strict Dieudonné algebra* A^* *and a map* $f : R \to A^0/VA^0$ *which exhibits* A^* *as a de Rham-Witt complex of* R*.*

Proof. By virtue of Lemma 3.6.1, we may assume without loss of generality that R is reduced. Let $W(R)$ denote the ring of Witt vectors of R, and let $\varphi : W(R) \to W(R)$ be the Witt vector Frobenius. Since R is reduced, $W(R)$ is p-torsion-free. Proposition 3.2.1 now equips the de Rham complex $\Omega^*_{W(R)}$ with the structure of a Dieudonné algebra. Let A^* be the completed saturation $\mathcal{W}\mathrm{Sat}(\Omega^*_{W(R)})$. Combining the universal properties descibed in Proposition

3.5.8, Proposition 3.4.3, Proposition 3.2.3, and Proposition 3.6.4, we see that for every strict Dieudonné algebra B^*, we have natural bijections

$$\begin{aligned}\mathrm{Hom}_{\mathbf{AD}_{\mathrm{str}}}(A^*, B^*) &\simeq \mathrm{Hom}_{\mathbf{AD}_{\mathrm{sat}}}(\mathrm{Sat}(\Omega^*_{W(R)}, B^*) \\ &\simeq \mathrm{Hom}_{\mathbf{AD}}(\Omega^*_{W(R)}, B^*) \\ &\simeq \mathrm{Hom}_F(W(R), B) \\ &\simeq \mathrm{Hom}(R, B^0/VB^0);\end{aligned}$$

here $\mathrm{Hom}_F(W(R), B)$ denotes the set of ring homomorphisms from $f : W(R) \to B$ satisfying $f \circ \varphi = F \circ f$. Taking $B^* = A^*$, the image of the identity map id_{A^*} under the composite bijection determines a ring homomorphism $R \to A^0/VA^0$ with the desired property. □

Let $\mathrm{CAlg}_{\mathbf{F}_p}$ denote the category of commutative $\mathbf{F}_p$-algebras. Then the construction $A^* \mapsto A^0/VA^0$ determines a functor $\mathbf{AD}_{\mathrm{str}} \to \mathrm{CAlg}_{\mathbf{F}_p}$.

Corollary 4.1.5. *The functor $A^* \mapsto A^0/VA^0$ described above admits a left adjoint $\mathcal{W}\Omega^* : \mathrm{CAlg}_{\mathbf{F}_p} \to \mathbf{AD}_{\mathrm{str}}$, given by the formation of de Rham-Witt complexes $R \mapsto \mathcal{W}\Omega^*_R$.*

Notation 4.1.6. Let R be an $\mathbf{F}_p$-algebra. For each $r \geq 0$, we let $\mathcal{W}_r\Omega^*_R$ denote the quotient $\mathcal{W}\Omega^*_R/(\mathrm{im}(V^r) + \mathrm{im}(dV^r))$. By construction, we have a tautological map $e : R \to \mathcal{W}_1\Omega^*_R$.

4.2. Comparison with a Smooth Lift.

Proposition 4.2.1. *Let R be a commutative ring which is p-torsion-free, and let $\varphi : R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$ for $x \in R$. Let $\widehat{\Omega}^*_R$ be the completed de Rham complex of Variant 3.3.1, and let A^* be a strict Dieudonné algebra. Then the evident restriction map*

$$\mathrm{Hom}_{\mathbf{AD}}(\widehat{\Omega}^*_R, A^*) \to \mathrm{Hom}(R, A^0/VA^0)$$

is bijective. In other words, every ring homomorphism $R \to A^0/VA^0$ can be lifted uniquely to a map of Dieudonné algebras $\widehat{\Omega}^*_R \to A^*$.

Proof. Combine the universal properties of Variant 3.3.1 and Corollary 3.6.3 (noting that if A^* is a strict Dieudonné algebra, then each of the abelian groups A^n is p-adically complete). □

Corollary 4.2.2. *Let R be a commutative ring which is p-torsion-free, and let $\varphi : R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$ for $x \in R$. Then there is a unique map of Dieudonné algebras $\mu : \widehat{\Omega}^*_R \to \mathcal{W}\Omega^*_{R/p}$ for which the* *diagram*

$$\begin{array}{ccccc} R & \longrightarrow & \widehat{\Omega}^0_R & \xrightarrow{\mu} & \mathcal{W}_1\,\Omega^0_{R/p} \\ \downarrow & & & & \downarrow \\ R/p & & \xrightarrow{\quad e \quad} & & \mathcal{W}_1\,\Omega^0_{R/p} \end{array}$$

commutes (here e is the map appearing in Notation 4.1.6). Moreover, μ induces an isomorphism of Dieudonné algebras $\mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_R) \to \mathcal{W}\,\Omega^*_{R/p}$.

Proof. The existence and uniqueness of μ follow from Proposition 4.2.1. To prove the last assertion, it will suffice to show that for every strict Dieudonné algebra A^*, composition with μ induces an isomorphism $\theta : \mathrm{Hom}_{\mathbf{AD}}(\mathcal{W}\,\Omega^*_{R/p}, A^*) \to \mathrm{Hom}_{\mathbf{AD}}(\widehat{\Omega}^*_R, A^*)$. Using Proposition 4.2.1 and the definition of the de Rham-Witt complex, we can identify θ with the evident map $\mathrm{Hom}(R/p, A^0/VA^0) \to \mathrm{Hom}(R, A^0/VA^0)$. This map is bijective, since A^0/VA^0 is an $\mathbf{F}_p$-algebra. □

Theorem 4.2.3. *Let R be a commutative ring which is p-torsion-free, and let $\varphi : R \to R$ satisfying $\varphi(x) \equiv x^p \pmod p$. Suppose that there exists a perfect ring k of characteristic p such that R/p is a smooth k-algebra. Then the map $\mu : \widehat{\Omega}^*_R \to \mathcal{W}\,\Omega^*_{R/p}$ of Corollary 4.2.2 is a quasi-isomorphism.*

Proof. By virtue of Corollary 4.2.2, it will suffice to show that the tautological map $\widehat{\Omega}^*_R \to \mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_R)$ is a quasi-isomorphism, which follows from Corollary 3.3.6. □

Example 4.2.4. We record an example of the saturation process and give an explicit description of the de Rham-Witt complex of a product of copies of $\mathbf{G}_m$, recovering the discussion in [15, Sec. I.2].

Consider the ring $R = \mathbf{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ equipped with self-map $\varphi : R \to R$ sending $x_i \mapsto x_i^p$. The de Rham complex of R, Ω^*_R, is as a graded ring the exterior algebra on R on the forms $d\log x_i$ for $1 \leq i \leq n$, and the differential is determined by $d(x_i^n) = n x_i^n d\log x_i$. The map φ induces the structure of a Dieudonné algebra on Ω^*_R thanks to Proposition 3.2.1. A computation shows $F(d\log x_i) = d\log x_i$.

Now we use the description of the saturation $\mathrm{Sat}(\Omega^*_R)$. The graded algebra $\Omega^*_R[F^{-1}]$ is the exterior algebra over the ring $R_\infty = \mathbf{Z}[x_1^{\pm 1/p^\infty}, \dots, x_n^{\pm 1/p^\infty}]$ on the same classes $\{d\log x_i, 1 \leq i \leq n\}$ and the differential $d : \Omega^*_R[F^{-1}] \to \Omega^*_R[F^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[1/p]$ sends, for example, $x_i^{a/p^n} \mapsto (a/p^n) x_i^{a/p^n} d\log x_i$. It follows that $\mathrm{Sat}(\Omega^*_R)$ is the subalgebra of the exterior algebra over R_∞ on $\{d\log x_i\}$ consisting of elements ω such that $d\omega$ is still integral. Furthermore, the de Rham-Witt complex of $\mathbf{F}_p[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is obtained as the completion $\mathcal{W}\mathrm{Sat}(\Omega^*_R)$. This recovers the description via *integral forms* in [15, Sec. I.2].

4.3. **Comparison with the de Rham Complex.**

Construction 4.3.1. Let R be a commutative $\mathbf{F}_p$-algebra, let $\mathcal{W}\Omega^*_R$ be the de Rham-Witt complex of R, and let $\mathcal{W}_1\Omega^*_R$ be the quotient of $\mathcal{W}\Omega^*_R$ of Notation 4.1.6. Then the tautological ring homomorphism $e: R \to \mathcal{W}_1\Omega^0_R$ admits a unique extension to a map of differential graded algebras $\nu : \Omega^*_R \to \mathcal{W}_1\Omega^*_R$.

Theorem 4.3.2. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the map ν : $\Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is an isomorphism.*

We first treat the special case where R is a smooth algebra over a perfect field k. In fact, our argument works more generally if k is a perfect ring:

Proposition 4.3.3. *Let k be a perfect $\mathbf{F}_p$-algebra and let R be a smooth algebra over k. Then the map $\nu : \Omega^*_R \to \mathcal{W}_1\Omega^* R$ is an isomorphism.*

Proof. Let A be a smooth $W(k)$-algebra equipped with an isomorphism $A/pA \simeq R$, and let $\widehat{A}$ denote the p-adic completion of A. Using the smoothness of A, we see that the ring homomorphism $A \to A/pA \simeq R \xrightarrow{x \mapsto x^p} R$ can be lifted to a map $A \to \widehat{A}$. Passing to the completion, we obtain a ring homomorphism $\varphi : \widehat{A} \to \widehat{A}$ satisfying $\varphi(x) \equiv x^p \pmod p$. Applying Corollary 4.2.2, we see that there is a unique morphism of Dieudonné algebras $\mu : \widehat{\Omega}^*_{\widehat{A}} \to \mathcal{W}\Omega^*_R$ for which the diagram

$$\begin{array}{ccccc} \widehat{A} & \xrightarrow{\sim} & \widehat{\Omega}^0_{\widehat{A}} & \xrightarrow{\mu} & \mathcal{W}\Omega^0_R \\ \downarrow & & & & \downarrow \\ \widehat{A}/p\widehat{A} & \xrightarrow{\sim} & R & \xrightarrow{e} & \mathcal{W}_1\Omega^0_R \end{array}$$

commutes. From this commutativity, we see that ν can be identified with the composition

$$\Omega^*_R \simeq \widehat{\Omega}^*_{\widehat{A}}/p\widehat{\Omega}^*_{\widehat{A}} \xrightarrow{\mu} \mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R \to \mathcal{W}_1\Omega^*_R.$$

We now have a commutative diagram of graded abelian groups

$$\begin{array}{ccccc} \widehat{\Omega}^*_{\widehat{A}}/p\widehat{\Omega}^*_{\widehat{A}} & & \xrightarrow{\mathrm{Cart}} & & \mathrm{H}^*(\widehat{\Omega}^*_{\widehat{A}}/p\widehat{\Omega}^*_{\widehat{A}}) \\ \downarrow & \searrow^{\nu} & & & \downarrow \\ \mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R & \longrightarrow & \mathcal{W}_1\Omega^*_R & \longrightarrow & \mathrm{H}^*(\mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R), \end{array}$$

where the vertical maps are induced by μ, and the top horizontal and bottom right horizontal maps are induced by the Frobenius on $\widehat{\Omega}^*_{\widehat{A}}$ and $\mathcal{W}\Omega^*_R$, respectively. We now observe that the top horizontal map is the Cartier isomorphism (Theorem 3.3.5), the right vertical map is an isomorphism by virtue of Theorem 4.2.3, and the bottom right horizontal map is an isomorphism by Proposition 2.5.2. It follows that ν is also an isomorphism. □

To handle the general case, we will need the following:

Proposition 4.3.4. *The category* $\mathbf{AD}_{\mathrm{str}}$ *admits small filtered colimits, which are preserved by the functor* $A^* \mapsto \mathcal{W}_r(A)^* = A^*/(\mathrm{im}(V^r) + \mathrm{im}(dV^r))$.

Proof. We first observe that the category $\mathbf{AD}$ of Dieudonné algebras admits filtered colimits, which can be computed at the level of the underlying graded abelian groups. Moreover, the subcategory $\mathbf{AD}_{\mathrm{sat}} \subseteq \mathbf{AD}$ is closed under filtered colimits. In particular, the construction $A^* \mapsto \mathcal{W}_r(A)^*$ preserves filtered colimits when regarded as a functor from the category $\mathbf{AD}_{\mathrm{sat}}$ to the category of graded abelian groups. It follows from Corollary 3.5.9 that the category $\mathbf{AD}_{\mathrm{str}}$ also admits small filtered colimits, which are computed by first taking a colimit in the larger category $\mathbf{AD}_{\mathrm{sat}}$ and then applying the completion construction $A^* \mapsto \mathcal{W}(A)^*$. Consequently, to show that restriction of $\mathcal{W}_r$ to $\mathbf{AD}_{\mathrm{str}}$ commutes with filtered colimits, it suffices to show that the canonical map $\mathcal{W}_r(\varinjlim A_\alpha)^* \to \mathcal{W}_r(\mathcal{W}(\varinjlim A_\alpha))^*$ is an isomorphism for every filtered diagram $\{A^*_\alpha\}$ in $\mathbf{AD}_{\mathrm{str}}$. This is a special case of Proposition 2.7.1. □

Remark 4.3.5. In the statement and proof of Proposition 4.3.4, we can replace Dieudonné algebras by Dieudonné complexes.

Corollary 4.3.6. *The construction* $R \mapsto \mathcal{W}_r\Omega^*_R$ *commutes with filtered colimits (when regarded as a functor from the category of* $\mathbf{F}_p$*-algebras to the category of graded abelian groups).*

Proof. Combine Proposition 4.3.4 with the observation that the functor $R \mapsto \mathcal{W}\Omega^*_R$ preserves colimits (since it is defined as the left adjoint to the functor $A^* \mapsto A^0/VA^0$). □

Proof of Theorem 4.3.2. Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Applying Popescu's smoothing theorem (***ref***), we can write R as a filtered colimit $\varinjlim R_\alpha$, where each R_α is a smooth $\mathbf{F}_p$-algebra. It follows from Corollary 4.3.6 that the canonical map $\nu : \Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is a filtered colimit of maps $\nu_\alpha : \Omega^*_{R_\alpha} \to \mathcal{W}_1\Omega_{R_\alpha}$. It will therefore suffice to show that each of the maps ν_α is an isomorphism, which follows from Proposition 4.3.3. □

Remark 4.3.7. Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Composing the tautological map $\mathcal{W}\Omega^*_R \to \mathcal{W}_1\Omega^*_R$ with the inverse of the isomorphism $\nu : \Omega^*_R \to \mathcal{W}\Omega^*_R$, we obtain a surjective map of differential graded algebras $\mathcal{W}\Omega^*_R \to \Omega^*_R$. It follows from Theorem 4.3.2 and Corollary 2.5.3 that the induced map $\mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R \to \Omega^*_R$ is a quasi-isomorphism.

4.4. **Comparison with Illusie.** We now consider the relationship between the de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of Notation 4.1.2 and the de Rham-Witt complex $W\Omega^*_R$ of Bloch-Deligne-Illusie. We begin with some definitions.

Definition 4.4.1. Let R be a commutative $\mathbf{F}_p$-algebra. An *R-framed V-pro-complex* consists of the following data:

(1) An inverse system
$$\cdots \to A_4^* \to A_3^* \to A_2^* \to A_1^* \to A_0^*$$
of commutative differential graded algebras. We will denote each of the transition maps in this inverse system by $\mathrm{Res} : A_{r+1}^* \to A_r^*$, and refer to them as *restriction maps*. We let A_∞^* denote the inverse limit $\varprojlim_r A_r^*$.

(2) A collection of maps $V : A_r^* \to A_{r+1}^*$ in the category of graded abelian groups, which we will refer to as *Verschiebung maps*.

(3) A ring homomorphism $\beta : W(R) \to A_\infty^0$. For each $r \geq 0$, we let β_r denote the composite map $W(R) \xrightarrow{\beta} A_\infty^0 \to A_r^0$.

These data are required to satisfy the following axioms:

(a) The groups A_r^* vanish for $* < 0$ and for $r = 0$.

(b) The Verschiebung and restriction maps are compatible with one another: that is, for every $r > 0$, we have a commutative diagram
$$\begin{array}{ccc} A_r^* & \xrightarrow{V} & A_{r+1}^* \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ A_{r-1}^* & \xrightarrow{V} & A_r^*. \end{array}$$
It follows that the Verschiebung maps induce a homomorphism of graded abelian groups $V : A_\infty^* \to A_\infty^*$, which we will also denote by V.

(c) The map $\beta : W(R) \to A_\infty^0$ is compatible with Verschiebung: that is, we have a commutative diagram
$$\begin{array}{ccc} W(R) & \xrightarrow{\beta} & A_\infty^0 \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} \\ W(R) & \xrightarrow{\beta} & A_\infty^0. \end{array}$$

(d) The Verschiebung maps $V : A_r^* \to A_{r+1}^*$ satisfy the identity $V(xdy) = V(x)dV(y)$.

(e) Let λ be an element of R and let $[\lambda] \in W(R)$ denote its Teichmüller representative. Then, for each $x \in A_r^*$, we have an identity
$$(Vx)d\beta_{r+1}([\lambda]) = V(\beta_r([\lambda])^{p-1} x d\beta_r[\lambda])$$
in A_{r+1}^*.

Let $(\{A_r^*\}_{r\geq 0}, V, \beta)$ and $(\{A_r'^*\}_{r\geq 0}, V', \beta')$ be R-framed V-pro-complexes. A *morphism* of R-framed V-pro-complexes from $(\{A_r^*\}_{r\geq 0}, V, \beta)$ to $(\{A_r'^*\}_{r\geq 0}, V', \beta')$

is a collection of differential graded algebra homomorphisms $f_r : A_r^* \to A_r'^*$ for which the diagrams

$$\begin{array}{ccc} A_r^* & \xrightarrow{f_r} & A_r'^* \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ A_{r-1}^* & \xrightarrow{f_{r-1}} & A_{r-1}'^* \end{array} \qquad \begin{array}{ccc} A_r^* & \xrightarrow{f_r} & A_r'^* \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V'} \\ A_{r+1}^* & \longrightarrow & A_{r+1}'^* \end{array}$$

$$\begin{array}{ccc} & W(R) & \\ {\scriptstyle \beta_r}\swarrow & & \searrow{\scriptstyle \beta_r'} \\ A_r^0 & \xrightarrow{f_r} & A_r'^0 \end{array}$$

are commutative. We let $\mathbf{VPC}_R$ denote the category whose objects are R-framed V-pro-complexes and whose morphisms are morphisms of R-framed V-pro-complexes.

Remark 4.4.2. In the situation of Definition 4.4.1, we will generally abuse terminology by simply referring to the inverse system $\{A_r^*\}_{r\geq 0}$ as an *R-framed V-pro-complex*; in this case, we are implicitly assuming that Verschiebung maps $V : A_r^* \to A_{r+1}^*$ and a map $\beta : W(R) \to A_\infty^0$ have also been specified.

Remark 4.4.3. Let R be a commutative $\mathbf{F}_p$-algebra and let $\{A_r^*\}$ be an R-framed V-pro-complex. It follows from conditions (a) and (b) of Definition 4.4.1 that, for each $r \geq 0$, the composite map

$$A_r^* \xrightarrow{V^r} A_{2r}^* \xrightarrow{\mathrm{Res}^r} A_r^*$$

vanishes. Consequently, condition (c) of Definition 4.4.1 implies that the map $\beta_r : W(R) \to A_r^0$ annihilates the subgroup $V^r W(R) \subseteq W(R)$, and therefore factors through the quotient $W_r(R) \simeq W(R)/V^r W(R)$.

Proposition 4.4.4. *Let R be a commutative $\mathbf{F}_p$-algebra. Then the category $\mathbf{VPC}_R$ has an initial object.*

Proof Sketch. For each $r \geq 0$, let $\Omega^*_{W_r(R)}$ denote the de Rham complex of $W_r(R)$ (relative to $\mathbf{Z}$). We will say that a collection of differential graded ideals $\{I_r^* \subseteq \Omega^*_{W_r(R)}\}_{r\geq 0}$ is *good* if the following conditions are satisfied:

- (i) Each of the restriction maps $W_{r+1}(R) \to W_r(R)$ determines a map of de Rham complexes $\Omega^*_{W_{r+1}(R)} \to \Omega^*_{W_r(R)}$ which carries I_{r+1}^* into I_r^*, and therefore induces a map of differential graded algebras $\Omega^*_{W_{r+1}(R)}/I_{r+1}^* \to \Omega^*_{W_r(R)}/I_r^*$.
- (ii) For each $r \geq 0$, there exists a map $V : \Omega^*_{W_r(R)}/I_r^* \to \Omega^*_{W_{r+1}(R)}/I_{r+1}^*$ which satisfies the identity

$$V(x_0 dx_1 \wedge \cdots \wedge dx_n) = V(x_0) dV(x_1) \wedge \cdots \wedge dV(x_n);$$

here we abuse notation by identifying an element of $\Omega^*_{W_r(R)}$ with its image in the quotient $\Omega^*_{W_r(R)}/I^*_r$. Note that such a map V is automatically unique.

(*iii*) For each $x \in \Omega^*_{W_r(R)}/I^*_r$ and each $\lambda \in R$, the difference $Vxd[\lambda] - V([\lambda]^{p-1}x)dV[\lambda]$ belongs to I^{*+1}_{r+1} (here we abuse notation by identifying the Teichmüller representative $[\lambda]$ with its image each $\Omega^*_{W_s(R)}/I^*_s$).

It is not difficult to see that there exists a smallest good collection of differential graded ideals $\{I^*_r\}_{r\geq 0}$, and that the inverse system $\{\Omega^*_{W_r(R)}/I^*_r\}_{r\geq 0}$ (together with with the Verschiebung maps defined in (*ii*) and the evident structural map $\beta : W(R) \to \varprojlim_r \Omega^0_{W_r(R)}/I^0_r$) is an initial object of $\mathbf{VPC}_R$. □

Remark 4.4.5. In the situation of the proof of Proposition 4.4.4, the differential graded ideal I^*_1 vanishes: that is, the tautological map $\Omega^*_R \to W_1\Omega^*_R$ is always an isomorphism.

Definition 4.4.6. Let R be a commutative $\mathbf{F}_p$-algebra and let $\{W_r\Omega^*_R\}_{r\geq 0}$ denote an initial object of the category of $\mathbf{VPC}_R$. We let $W\Omega^*_R$ denote the inverse limit $\varprojlim_r W_r\Omega^*_R$. We will refer to $W\Omega^*_R$ as the *classical de Rham-Witt complex of* R.

Remark 4.4.7. The proof of Proposition 4.4.4 provides an explicit model for the inverse system $\{W_r\Omega^*_R\}$: each $W_r\Omega^*_R$ can be regarded as a quotient of the absolute de Rham complex $\Omega^*_{W_r(R)}$ by a differential graded ideal I^*_r, which is dictated by the axiomatics of Definition 4.4.1.

Warning 4.4.8. Our definition of the classical de Rham-Witt complex $W\Omega^*_R$ differs slightly from the definition appearing in [15]. In [15], the inverse system $\{W_r\Omega^*_R\}_{r\geq 0}$ is defined to be an initial object of a category $\mathbf{VPC}'_R$, which can be identified with the full subcategory of $\mathbf{VPC}_R$ spanned by those R-framed V-procomplexes $\{A^*_r\}$ for which $\beta : W(R) \to A^0_\infty$ induces isomorphisms $W_r(R) \simeq A^0_r$ for $r \geq 0$. However, it is not difficult to see that the initial object of $\mathbf{VPC}_R$ belongs to this subcategory (this follows by inspecting the proof of Proposition 4.4.4), so that $W\Omega^*_R$ is isomorphic to the de Rham-Witt complex which appears in [15].

Remark 4.4.9. Let R be a commutative $\mathbf{F}_p$-algebra and let $W\Omega^*_R$ be the classical de Rham-Witt complex of R. The proof of Proposition 4.4.4 shows that the restriction maps $W_{r+1}\Omega^*_R \to W_r\Omega^*_R$ are surjective. It follows that each $W_r\Omega^*_R$ can be viewed as a quotient of $W\Omega^*_R$.

Our next goal is to compare the de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of Notation 4.1.2 with the classical de Rham-Witt complex $W\Omega^*_R$. We begin with the following observation:

Proposition 4.4.10. *Let R be a commutative $\mathbf{F}_p$-algebra, let A^* be a strict Dieudonné algebra, and let $\beta_1 : R \to A^0/VA^0$ be a ring homomorphism. Then:*

(1) *The map* β_1 *admits a unique lift to a ring homomorphism* $\beta : W(R) \to A^0$ *satisfying* $\beta \circ F = F \circ \beta$.
(2) *The inverse system* $\{\mathcal{W}_r(A)^*\}_{r\geq 0}$ *is an* R*-framed* V*-pro-complex (when equipped with the Verschiebung maps* $V : \mathcal{W}_r(A)^* \to \mathcal{W}_{r+1}(A)^*$ *of Remark 2.6.1 and the map* β *described in* (1)*).*

Proof. Assertion (1) is the content of Proposition 3.6.4. To prove (2), we must verify that β and V satisfy axioms (a) through (e) of Definition 4.4.1. Axioms (a) and (b) are obvious. To prove (c), we compute

$$F\beta(Vx) = \beta(FVx) = \beta(px) = p\beta(x) = FV\beta(x),$$

so that $\beta(Vx) = V\beta(x)$ by virtue of the fact that the Frobenius map $F : A^0 \to A^0$ is injective. The verification of (d) is similar: for $x, y \in A^*$, we have

$$FV(xdy) = (px)(dy) = (FVx)(FdVy) = F((Vx)(dVy));$$

invoking the injectivity of F again, we obtain $V(xdy) = (Vx)(dVy)$. To prove (e), we compute

$$\begin{aligned}
F((Vx)d\beta([\lambda])) &= F(Vx)F(d\beta([\lambda])) \\
&= pxF(d\beta([\lambda)]) \\
&= xd(F\beta([\lambda])) \\
&= xd(\beta(F[\lambda])) \\
&= xd\beta([\lambda]^p) \\
&= xd\beta([\lambda])^p \\
&= px\beta([\lambda])^{p-1}d\beta([\lambda]) \\
&= FV(x\beta([\lambda])^{p-1}d\beta([\lambda]).
\end{aligned}$$

Canceling F, we obtain the desired identity $Vxd\beta([\lambda]) = V(x\beta([\lambda])^{p-1}d\beta([\lambda]))$. □

Applying Proposition 4.4.10 in the case $A^* = \mathcal{W}\Omega^*_R$ and invoking the universal property of the classical de Rham-Witt complex, we obtain the following:

Corollary 4.4.11. *Let* R *be a commutative* $\mathbf{F}_p$*-algebra. Then there is a unique homomorphism of differential graded algebras* $\gamma : W\Omega^*_R \to \mathcal{W}\Omega^*_R$ *with the following properties:*

(i) *For each* $r \geq 0$*, the composite map*

$$W\Omega^*_R \xrightarrow{\gamma} \mathcal{W}\Omega^*_R \to \mathcal{W}_r\Omega^*_R$$

factors through $W_r\Omega^*_R$ *(necessarily uniquely, by virtue of Remark 4.4.9); consequently,* γ *can be realized as the inverse limit of a tower of differential graded algebra homomorphisms* $\gamma_r : W_r\Omega^*_R \to \mathcal{W}_r\Omega^*_R$.

(*ii*) *The diagrams*

$$\begin{array}{ccc} W\Omega^*_R & \xrightarrow{V} & W\Omega^*_R \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ \mathcal{W}\Omega^*_R & \xrightarrow{V} & \mathcal{W}\Omega^*_R \end{array} \qquad \begin{array}{ccc} W(R) & \longrightarrow & W\Omega^0_R \\ \downarrow & & \downarrow{\scriptstyle\gamma} \\ R & \longrightarrow & \mathcal{W}\Omega^0_R / V\mathcal{W}\Omega^0_R \end{array}$$

are commutative.

We can now formulate the main result of this section:

Theorem 4.4.12. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the map $\gamma : W\Omega^*_R \to \mathcal{W}\Omega^*_R$ of Corollary 4.4.11 is an isomorphism.*

Proof. The map γ can be realized as the inverse limit of a tower of maps $\gamma_r : W_r\Omega^*_R \to \mathcal{W}_r\Omega^*_R$; it will therefore suffice to show that each γ_r is an isomorphism. Since the constructions $R \mapsto W_r\Omega^*_R$ and $R \mapsto \mathcal{W}_r\Omega^*_R$ commute with filtered colimits, we can use Popescu's smoothing theorem to reduce to the case where R is a smooth $\mathbf{F}_p$-algebra. Using Theorem I.2.17 and Proposition I.2.18 of [15], we see that there exists a map $F : W\Omega^*_R \to W\Omega^*_R$ which endows $W\Omega^*_R$ with the structure of a Dieudonné algebra and satisfies the identities $FV = VF = p$. Since R is smooth over $\mathbf{F}_p$, Remark I.3.21.1 of [15] guarantees that the Dieudonné algebra $W\Omega^*_R$ is saturated. Moreover, Proposition I.3.2 implies that the kernel of the projection map $W\Omega^*_R \to W_r\Omega^*_R$ is equal to $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$, so that

$$W\Omega^*_R \simeq \varprojlim W_r\Omega^*_R \simeq \varprojlim W\Omega^*_R/(\mathrm{im}(V^r) + \mathrm{im}(dV^r))$$

is a strict Dieudonné algebra. For each $x \in W\Omega^*_R$, we have

$$V\gamma(Fx) = \gamma(VFx) = \gamma(px) = p\gamma(x) = VF\gamma(x).$$

Since the map $V : \mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_R$ is injective, it follows that $\gamma(Fx) = F\gamma(x)$: that is, γ is a morphism of (strict) Dieudonné algebras. We wish to show $\gamma_r = \mathcal{W}_r(\gamma)$ is an isomorphism for each $r \geq 0$. By virtue of Corollary 2.5.5, it suffices to prove this in the case $r = 1$. Invoking Remark 4.4.5, we are reduced to proving that the tautological map $\Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is an isomorphism, which follows from Theorem 4.3.2. □

5. Globalization

In this section, we globalize the construction of the de Rham-Witt complex to schemes. In particular, we show that the de Rham-Witt complex behaves very well with respect to an étale base change. The general result is slightly tricky and relies on the interaction of the Witt vector functor and étaleness, so we begin with the case of a Zariski localization, which enables one to define the de Rham-Witt complex of a scheme. We then treat the case of étale localization. In both cases, we find that if X is an $\mathbf{F}_p$-scheme, then $\mathcal{W}_r(\mathcal{O}_X)$ is a quasi-coherent module on the Witt scheme $W_r(X)$.

5.1. Localizations of Dieudonné Algebras.

Proposition 5.1.1. *Let A^* be a Dieudonné algebra and suppose we are given a multiplicatively closed subset $S \subseteq A^0$ such that $F(S) \subseteq S$. Then the graded ring $A^*[S^{-1}]$ inherits the structure of a Dieudonné algebra, which is uniquely determined by the requirement that the tautological map $A^* \to A^*[S^{-1}]$ is a morphism of Dieudonné algebras.*

Proof. We first observe that there is a unique differential on the ring $A^*[S^{-1}]$ for which the map $A^* \to A^*[S^{-1}]$ is a morphism of differential graded algebras, given concretely by the formula $d(\frac{x}{s}) = \frac{dx}{s} - \frac{xds}{s^2}$. Similarly, the assumption that $F(S) \subseteq S$ guarantees that there is a unique homomorphism of graded rings $F : A^*[S^{-1}] \to A^*[S^{-1}]$ for which the diagram

$$\begin{array}{ccc} A^* & \longrightarrow & A^*[S^{-1}] \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ A^* & \longrightarrow & A^*[S^{-1}] \end{array}$$

is commutative, given concretely by the formula $F(\frac{x}{s}) = \frac{F(x)}{F(s)}$. To complete the proof, it will suffice to show that the triple $(A^*[S^{-1}], d, F)$ is a Dieudonné algebra: that is, that is satisfies the axioms of Definition 3.1.1. Axiom (ii) is immediate, and axiom (i) follows from the calculation.

$$\begin{aligned} dF(\frac{x}{s}) &= d\frac{F(x)}{F(s)} \\ &= \frac{dF(x)}{F(s)} - \frac{F(x)dF(s)}{F(s)^2} \\ &= \frac{pF(dx)}{F(s)} - \frac{pF(x)F(ds)}{F(s)^2} \\ &= pF(\frac{dx}{s} - \frac{xds}{s^2}) \\ &= pF(d\frac{x}{s}). \end{aligned}$$

To prove (iii), choose any $x \in A^0$ and any $s \in S$. Since A^* is a Dieudonné algebra, we have

$$F(x)s^p \equiv x^p s^p \equiv x^p F(s) \pmod p,$$

so that we can write $F(x)s^p = x^p F(s) + py$ for some $y \in A^0$. It follows that

$$F(\frac{x}{s}) = \frac{F(x)}{F(s)} = \frac{x^p}{s^p} + p\frac{y}{s^p F(s)}$$

in the commutative ring $A^0[S^{-1}]$, so that $F(\frac{x}{s}) \equiv (\frac{x}{s})^p \pmod p$. □

Remark 5.1.2. In the situation of Proposition 5.1.1, the Dieudonné algebra $A^*[S^{-1}]$ is characterized by the following universal property: for any Dieudonné algebra B^*, precomposition with the tautological map $A^* \to A^*[S^{-1}]$ induces a monomorphism of sets

$$\mathrm{Hom}_{\mathbf{AD}}(A^*[S^{-1}], B^*) \to \mathrm{Hom}_{\mathbf{AD}}(A^*, B^*),$$

which image consists of those morphisms of Dieudonné algebras $f : A^* \to B^*$ with the property that $f(s)$ is an invertible element of B^0, for each $s \in S$.

Example 5.1.3. Let A^* be a Dieudonné algebra and let s be an element of A^0 which satisfies the equation $F(s) = s^p$. Applying Proposition 5.1.1 to the set $S = \{1, s, s^2, \ldots\}$, we conclude that the localization $A^*[s^{-1}]$ inherits the structure of a Dieudonné algebra.

Proposition 5.1.4. *Let A^* be a Dieudonné algebra, and let $s \in A^0$ be an element satisfying the equation $F(s) = s^p$. If A^* is saturated, then the localization $A^*[s^{-1}]$ is also saturated.*

Proof. We first show that the Frobenius map $F : A^*[s^{-1}] \to A^*[s^{-1}]$ is a monomorphism. Suppose that $F(\frac{x}{s^k}) = 0$ for some $x \in A^*$. Then we have an equality

$s^n F(x) = 0$ in A^0 for some $n \gg 0$. Enlarging n if necessary, we can assume that $n = pm$ for some integer $m \geq 0$, so that

$$0 = s^n F(x) = F(s^m)F(x) = F(s^m x).$$

Since the Frobenius map F is a monomorphism on A^*, it follows that $s^m x = 0$, so that $\frac{x}{s^k}$ vanishes in $A^*[s^{-1}]$.

Now suppose that we are given an element $\frac{y}{s^n} \in A^*[S^{-1}]$ such that $d(\frac{y}{s^n})$ is divisible by p in $A^*[S^{-1}]$; we wish to show that $\frac{y}{s^n}$ belongs to the image of F. Enlarging n if necessary, we can assume that $n = pm$ for some $m \geq 0$, so that

$$d(\frac{y}{s^n}) = \frac{dy}{s^n} - pm\frac{yds}{s^{n+1}} \equiv \frac{dy}{s^n} \pmod{p}.$$

It follows that we can choose $k \gg 0$ for which $s^{pk}dy$ is divisible by p in A^*. We then have

$$d(s^{pk}y) = s^{pk}dy + pks^{pk-1}yds \equiv s^{pk}dy \equiv 0 \pmod{p}.$$

Invoking our assumption that A^* is saturated, we can write $s^{pk}y = Fz$ for some $z \in A^*$. It follows that $\frac{y}{s^n} = \frac{s^{pk}y}{s^{pm}} = F(\frac{z}{s^m})$, so that $\frac{y}{s^n}$ belongs to the image of F as desired. □

In the situation of Proposition 5.1.4, the Dieudonné algebra $A^*[s^{-1}]$ is usually not strict, even if A^* is assumed to be strict. However, the strictification of $A^*[s^{-1}]$ is easy to describe, by virtue of the following result:

Proposition 5.1.5. *Let A^* be a saturated Dieudonné algebra and let $s \in A^0$ be an element satisfying $F(s) = s^p$. Let $r \geq 0$ and let $\overline{s}$ denote the image of s in the quotient $\mathcal{W}_r(A)^0$. Then the canonical map $\psi : \mathcal{W}_r(A)^*[\overline{s}^{-1}] \to \mathcal{W}_r(A[s^{-1}])^*$ is an isomorphism (of differential graded algebras).*

Proof. The surjectivity of ψ is immediate from the definitions. To prove injectivity, we must show that the inclusion

$$(V^r A^* + dV^r A^*)[s^{-1}] \subseteq V^r(A^*[s^{-1}]) + dV^r(A^*[s^{-1}])$$

Since the left hand side is a subcomplex of $A^*[s^{-1}]$, it will suffice to show that the left hand side contains $V^r(A^*[s^{-1}])$. This follows immediately from the identity

$$V^r(s^{-n}x) = V^r(F^r(s^{-n})s^{n(p^r-1)}x) = s^{-n}V^r(s^{n(p^r-1)}x).$$

□

Corollary 5.1.6. *Let R be a commutative $\mathbf{F}_p$-algebra, let A^* be a Dieudonné algebra, and let $f : R \to A^0/VA^0$ be a ring homomorphism which exhibits A^* as a de Rham-Witt complex of R (in the sense of Definition 4.1.1). Let $s \in R$ be an element, and let us abuse notation by identifying the Teichmüller representative $[s] \in W(R)$ with its image in A^0. Set $B^* = A^*[[s]^{-1}]$. Then the map*

$$R[s^{-1}] \xrightarrow{f} (A^0/VA^0)[s^{-1}] \simeq B^0/VB^0 = \mathcal{W}(B)^*/V\mathcal{W}(B)^*$$

exhibits $\mathcal{W}(B)^*$ *as a de Rham-Witt complex of* $R[s^{-1}]$*.*

Proof. For any strict Dieudonné algebra C^*, we have a commutative diagram of sets

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{AD}}(\mathcal{W}(B)^*, C^*) & \longrightarrow & \mathrm{Hom}_{\mathbf{AD}}(A^*, C^*) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(R[s^{-1}], C^0/VC^0) & \longrightarrow & \mathrm{Hom}(R, C^0/VC^0). \end{array}$$

To prove Corollary 5.1.6, it will suffice to show that this diagram is a pullback square. Invoking the universal property of B^* supplied by Remark 5.1.2, we can reformulate this assertion as follows:

- $(*)$ Let $f : A^* \to C^*$ be a morphism of Dieudonné algebras. Then $f([s])$ is an invertible element of C^0 if and only if its image in C^0/VC^0 is invertible.

The "only if" direction is obvious. For the converse, suppose that $f([s])$ admits an inverse in C^0/VC^0, which is represented by an element $y \in C^0$. Then we can write $yf([s]) = 1 - Vz$ for some $z \in C^0$. Since C^0 is V-adically complete, the element $f([s])$ has an inverse in C^0, given by the product $y(1+Vz+(Vz)^2+\cdots)$. □

5.2. **The de Rham-Witt Complex of an $\mathbf{F}_p$-Scheme.** We now apply Corollary 5.1.6 to globalize the construction of Definition 4.1.1.

Construction 5.2.1. Let $X = (X, \mathscr{O}_X)$ be an $\mathbf{F}_p$-scheme, and let $\mathcal{U}_{\mathrm{aff}}(X)$ denote the collection of all affine open subsets of X. We define a functor $\mathcal{W}\Omega^*_X : \mathcal{U}_{\mathrm{aff}}(X)^{\mathrm{op}} \to \mathbf{AD}$ by the formula

$$\mathcal{W}\Omega^*_X(U) = \mathcal{W}\Omega^*_{\mathscr{O}_X(U)}.$$

We regard $\mathcal{W}\Omega^*_X(U)$ as a presheaf on X (defined only on affine subsets of X) with values in the category of Dieudonné algebras.

Theorem 5.2.2. *Let X be an $\mathbf{F}_p$-scheme. Then the presheaf $\mathcal{W}\Omega^*_X$ of Construction 5.2.1 is a sheaf (with respect to the topology on $\mathcal{U}_{\mathrm{aff}}(X)$ given by open coverings).*

Proof. We will show that, for every integer $d \geq 0$, the presheaf $\mathcal{W}\Omega^d_X$ is a sheaf of abelian groups on X. Unwinding the definitions, we can write $\mathcal{W}\Omega^d_X$ as the inverse limit of a tower of presheaves

$$\cdots \to \mathcal{W}_3\Omega^d_X \to \mathcal{W}_2\Omega^d_X \to \mathcal{W}_1\Omega^d_X,$$

where $\mathcal{W}_r\Omega^d_X$ is given by the formula $\mathcal{W}_r\Omega^d_X(U) = \mathcal{W}_r\Omega^d_{\mathscr{O}_X(U)}$. It will therefore suffice to show that each $\mathcal{W}_r\Omega^d_X$ is a sheaf of abelian groups. In other words, we must show that for every affine open subset $U \subseteq X$ and every covering of U by affine open subsets U_α, the sequence

$$0 \to \mathcal{W}_r\Omega^d_X(U) \to \prod_\alpha \mathcal{W}_r\Omega^d_X(U_\alpha) \to \prod_{\alpha,\beta} \mathcal{W}_r\Omega^d_X(U_\alpha \cap U_\beta)$$

is exact. Without loss of generality, we may replace X by U and thereby reduce to the case where $X = \operatorname{Spec}(R)$. By a standard argument, we can further reduce to the case where each U_α is the complement of the vanishing locus of some element $s_\alpha \in R$. In this case, we wish to show that the sequence

$$0 \to \mathcal{W}_r \Omega^d_R \to \prod_\alpha \mathcal{W}_r \Omega^d_{R[s_\alpha^{-1}]} \to \prod_{\alpha,\beta} \mathcal{W}_r \Omega^d_{R[s_\alpha^{-1}, s_\beta^{-1}]}$$

is exact. Setting $M = \mathcal{W}_r \Omega^d_R$, we can apply Corollary 5.1.6 and Proposition 5.1.5 to rewrite this sequence as

$$0 \to M \to \prod_\alpha M[\overline{s}_\alpha^{-1}] \to \prod_{\alpha,\beta} M[\overline{s}_\alpha^{-1}, \overline{s}_\beta^{-1}],$$

where $\overline{s}_\alpha$ denotes the image in $W_r(R)$ of the Teichmüller representative $[s_\alpha] \in W(R)$. The desired result now follows from the observation that the elements $\overline{s}_\alpha$ generate the unit ideal in $W_r(R)$. □

It follows that from Theorem 5.2.2 that for any $\mathbf{F}_p$-scheme X, the presheaf $\mathcal{W}\Omega^*_X$ can be extended uniquely to a sheaf of Dieudonné algebras on the collection of all open subsets of X. We will denote this sheaf also by $\mathcal{W}\Omega^*_X$ and refer to it as the *de Rham-Witt complex of* X.

Remark 5.2.3. The proof of Theorem 5.2.2 shows that each $\mathcal{W}_r \Omega^d_X$ can be regarded as a *quasi-coherent* sheaf on the $(\mathbf{Z}/p^r\mathbf{Z})$-scheme $(X, W_r(\mathscr{O}_X))$; here $W_r(\mathscr{O}_X)$ denotes the sheaf of commutative rings on X given on affine open sets U by the formula $W_r(\mathscr{O}_X)(U) = W_r(\mathscr{O}_X(U))$.

Proposition 5.2.4. *Let R be a commutative $\mathbf{F}_p$-algebra and let $X = \operatorname{Spec} R$ be the associated affine scheme. Then, for every integer d, the canonical map $\mathcal{W}\Omega^d_R \to \mathrm{H}^0(X; \mathcal{W}\Omega^d_X)$ is an isomorphism and the cohomology groups $\mathrm{H}^n(X; \mathcal{W}\Omega^d_X)$ vanish for $n > 0$.*

Proof. Let $\mathscr{F}^\bullet$ denote the homotopy limit of the diagram

$$\cdots \to \mathcal{W}_3 \Omega^d_X \to \mathcal{W}_2 \Omega^d_X \to \mathcal{W}_1 \Omega^d_X,$$

in the derived category of abelian sheaves on X. For every affine open subset $U \subseteq X$, the hypercohomology $\mathrm{R}\Gamma(U; \mathscr{F}^\bullet)$ can be identified with the homotopy limit of the diagram

$$\cdots \to \mathrm{R}\Gamma(U; \mathcal{W}_3 \Omega^d_X) \to \mathrm{R}\Gamma(U; \mathcal{W}_2 \Omega^d_X) \to \mathrm{R}\Gamma(U; \mathcal{W}_1 \Omega^d_X).$$

It follows from Remark 5.2.3 that each $\mathcal{W}_r \Omega^d_X$ can be regarded as a quasi-coherent sheaf on $(X, W_r(\mathscr{O}_X))$, so we can identify the preceding diagram with the tower of abelian groups

$$\cdots \to \mathcal{W}_3 \Omega^d_{\mathscr{O}_X(U)} \to \mathcal{W}_2 \Omega^d_{\mathscr{O}_X(U)} \to \mathcal{W}_1 \Omega^d_{\mathscr{O}_X(U)}.$$

This diagram has surjective transition maps, so its homotopy limit can be identified with the abelian group $\mathcal{W}\Omega^d_{\mathscr{O}_X(U)}$ (regarded as an abelian group, concentrated in degree zero). It follows that $\mathscr{F}^\bullet$ can be identified with $\mathcal{W}\Omega^d_X$ (regarded as a chain complex concentrated in degree zero), so the preceding calculation gives $\mathrm{R}\Gamma(U;\mathcal{W}\Omega^d_X) \simeq \mathcal{W}\Omega^d_{\mathscr{O}_X(U)}$ for each affine open subset $U \subseteq X$. Proposition 5.2.4 now follows by taking $U = X$. □

5.3. **Étale localization.** In this subsection, we will generalize the previous construction of Zariski localization to étale localization. The construction will require slightly more technology involving the Witt vectors.

In this subsection, we will often abbreviate "commutative differential graded algebra under A^*" simply to "A^*-algebra."

Definition 5.3.1. Let $f : A^* \to B^*$ be a map of commutative, differential graded algebras. We will say that f is *étale* (or that B^* is an *étale A^*-algebra*) if $f : A^0 \to B^0$ is étale and the map of graded algebras $A^* \otimes_{A^0} B^0 \to B^*$ is an isomorphism.

We observe now that any étale A^0-algebra can be realized as an étale commutative, differential graded algebra.

Proposition 5.3.2. *The forgetful functor $B^* \mapsto B^0$ establishes an equivalence between the category of étale A^*-algebras and the category of étale A^0-algebras.*

Proof. The functor that sends an étale A^0-algebra B^0 to the A^*-algebra $\Omega^*_{B^0} \otimes_{\Omega^*_{A^0}} A^*$ provides a quasi-inverse to the forgetful functor from étale A^*-algebras to étale A^0-algebras. In fact, we observe that the map of commutative, differential graded algebras $\Omega^*_{A^0} \to \Omega^*_{B^0}$ is étale as $A^0 \to B^0$ is étale and therefore $\Omega_{B^0/\mathbf{Z}} \simeq B^0 \otimes_{A^0} \Omega_{A^0/\mathbf{Z}}$. By base-change, this implies that $\Omega^*_{B^0} \otimes_{\Omega^*_{A^0}} A^*$ is étale over A^*.

The above description provides a universal property: to give a map of A^*-algebras $\Omega^*_{B^0} \otimes_{\Omega^*_{A^0}} A^*$ into an A^*-algebra C^* is simply to give a map of commutative rings $B^0 \to C^0$. Given an étale A^*-algebra C^*, we find a natural map of A^*-algebras $g : \Omega^*_{C^0} \otimes_{\Omega^*_{A^0}} A^* \to C^*$. Since both sides are étale over A^* and agree in degree zero, it follows that the map g is an isomorphism. It follows that the two functors are quasi-inverse to each other as desired. □

The main result of this section (Theorem 5.3.13 below) is an analog in the setting of strict Dieudonné algebras. In order to prove this result, we will need a basic compatibility of the Witt vectors with étaleness. This result (in various forms, and not only for $\mathbf{F}_p$-algebras) has been proved by Langer-Zink [20, Prop. A.8, A.11], van der Kallen [28, Th. 2.4], Borger [7, Th. 9.2]. We will need the following case, parts of which also appear in the original work of Illusie [15, Prop. 1.5.8].

Proposition 5.3.3. *Let $A \to B$ be an étale map of $\mathbf{F}_p$-algebras. Then the maps $W_n(A) \to W_n(B)$ are étale. Furthermore, the diagrams of rings*

$$\begin{array}{ccc} W_n(A) & \longrightarrow & W_n(B) \\ \downarrow R & & \downarrow R \\ W_{n-1}(A) & \longrightarrow & W_{n-1}(B) \end{array} \quad , \quad \begin{array}{ccc} W_n(A) & \longrightarrow & W_n(B) \\ \downarrow F & & \downarrow F \\ W_{n-1}(A) & \longrightarrow & W_{n-1}(B) \end{array}$$

are cocartesian.

Definition 5.3.4. Let A be an $\mathbf{F}_p$-algebra and let M be a $W(A)$-module. We will say that M is *nilpotent* if M is annihilated by the ideal $V^n W(A) \subset W(A)$ for some n, i.e., if M is actually a $W_n(A)$-module for some n.

Definition 5.3.5. Let $A \to B$ be an étale map of $\mathbf{F}_p$-algebras. If M is a nilpotent $W(A)$-module, so that M is a $W_n(A)$-module for some n, then we write M_B for the $W(B)$-module $M \otimes_{W_n(A)} W_n(B)$. By Proposition 5.3.3, the construction of M_B does not depend on the choice of n.

The construction $M \mapsto M_B$ is clearly the left adjoint of the forgetful functor from nilpotent $W(B)$-modules to nilpotent $W(A)$-modules. In addition, the functor $M \mapsto M_B$ is exact because $W_n(A) \to W_n(B)$ is étale, hence flat.

Definition 5.3.6. Given a $W(A)$-module M and $a \in \mathbf{Z}_{\geq 0}$, we let $M_{(a)}$ denote the new $W(A)$-module obtained by restriction of scalars of M along $F^a : W(A) \to W(A)$.

Suppose M is a nilpotent $W(A)$-module; then $M_{(a)}$ is also clearly nilpotent. In fact, if M is a $W_n(A)$-module for some n, then so is $M_{(a)}$.

Proposition 5.3.7. *Given an étale map $A \to B$ of $\mathbf{F}_p$-algebras and a nilpotent $W(A)$-module M, we have $(M_B)_{(a)} \simeq (M_{(a)})_B$.*

Proof. Without loss of generality, assume that M is a $W_n(A)$-module. We clearly have a natural map $(M_{(a)})_B \to (M_B)_{(a)}$, and since both sides are exact functors on the category of $W_n(A)$-modules which commute with filtered colimits, it suffices to see that the map is an equivalence when $M = W_n(A)$ itself. However, this follows from Proposition 5.3.3. □

Definition 5.3.8. Let M, N be nilpotent $W(A)$-modules. We will say that a homomorphism $f : M \to N$ of abelian groups is (F^a, F^b)-linear if for all $x \in W(A)$ and $m \in M$, we have

$$f((F^a x)m) = (F^b x) f(x) \in N.$$

Equivalently, f defines a map of $W(A)$-modules $M_{(a)} \to N_{(b)}$.

Corollary 5.3.9. *Fix an étale map $A \to B$. Let M be a nilpotent $W(A)$-module and let N be a nilpotent $W(B)$-module, considered as a nilpotent $W(A)$-module* *via restriction of scalars. Then any (F^a, F^b)-linear map $f : M \to N$ extends uniquely to a (F^a, F^b)-linear map $M_B \to N$.*

Proof. This follows from Proposition 5.3.7. □

Example 5.3.10. Let M, N, P be nilpotent $W(R)$-modules. Let $f : M \to N$ and $g : N \to P$ be maps which are (F^a, F^b)-linear and (F^c, F^d)-linear respectively. Suppose that $M \xrightarrow{f} N \xrightarrow{g} P$ is exact in the category of abelian groups. Then $M_B \xrightarrow{f_B} N_B \xrightarrow{g_B} P_B$ is exact as well. To see this, by replacing (a, b) by $(a+k, b+k)$ and (c, d) by $(c+l, d+l)$ for some $k, l \geq 0$, we can assume that $b = c$. Thus we have a short exact sequence of nilpotent $W(R)$-modules $M_{(a)} \to N_{(b)} \to P_{(d)}$, and base-changing we find a short exact sequence $(M_B)_{(a)} \to (N_B)_{(b)} \to (P_B)_{(d)}$.

As an example of this picture, we unwind the construction of Proposition 5.3.2 in this case. First, we need the following lemma.

Lemma 5.3.11. *Let R be a ring. Let M be a $W_n(R)$-module such that $p^k M = 0$ and let $d : W_n(R) \to M$ be a derivation. Then the composite $d \circ F^k : W_{n+k}(R) \to M$ vanishes.*

Proof. It suffices to take $n \to \infty$ and show that any derivation $d : W(R) \to N$ for a $W(R)$-module N has the property that $d \circ F^k$ is divisible by p^k. This follows from consideration of the universal case $N = \Omega^1_{W(N)}$ and the fact that $\Omega^*_{W(R)}$ is a Dieudonné complex. If $W(R)$ does not have p-torsion (e.g., R a reduced $\mathbf{F}_p$-algebra), this is Proposition 3.2.1 and in the general case this follows from Remark 3.2.4. Alternatively, one proceeds inductively by observing that for any $x \in W(R)$, we have $F^n x \in \sum_{i=0}^n p^{n-i} R^{p^i}$ where $R^{p^i} \subset R$ denotes the subset consisting of pth powers. □

Let R^* be a commutative, differential graded algebra and suppose given a map of rings $W(A) \to R^0$ which factors through $W_n(R)$ for some n. Let B be an étale A-algebra.

First, we can form $R^*_B = R^* \otimes_{W_n(A)} W_n(B)$ as a graded algebra by base-change. We cannot directly tensor the differential up along $W_n(A) \to W_n(B)$, since the differential is not $W_n(A)$-linear. However, it follows now that the differential $d : R^* \to R^*$ is (F^n, F^n)-linear. Therefore, we obtain a unique differential d on R^*_B which is (F^n, F^n)-linear over $W(B)$ and which agrees with the differential on R^*. It follows that d is the derivation on R^*_B given by Proposition 5.3.2: namely, the derivation given by Proposition 5.3.2 has the aforementioned properties, by Lemma 5.3.11 again.

We now adapt the definition to strict Dieudonné algebras, where we will require étaleness only at each finite level.

Definition 5.3.12. Let $f : A^* \to B^*$ be a map of strict Dieudonné algebras. We will say that f is *V-adically étale* if, for each n, $\mathcal{W}_n(f) : \mathcal{W}_n(A)^* \to \mathcal{W}_n(B)^*$ is étale as a map of commutative, differential graded algebras.

The main result of this subsection is the following.

Theorem 5.3.13. *Let A^* be a strict Dieudonné algebra. The functor $B^* \mapsto B^0/VB^0$ establishes an equivalence of categories between the category of V-adically étale strict Dieudonné algebras under A^* and the category of étale A^0/VA^0-algebras.*

First, we check full faithfulness. Here one has a slightly stronger result; namely, V-adically étale maps are characterized by an analogous universal property.

Proposition 5.3.14. *Let B^* be a strict Dieudonné algebra which is V-adically étale over A^* and let C^* be another strict Dieudonné algebra over A^*. Then maps of strict Dieudonné algebras $B^* \to C^*$ over A^* are in bijection with maps of A^0/VA^0-algebras $B^0/VB^0 \to C^0/VC^0$. That is, we have an isomorphism*

$$\mathrm{Hom}_{\mathbf{AD}_{A^*}}(B^*, C^*) = \mathrm{Hom}_{A^0/VA^0}(B^0/VB^0, C^0/VC^0). \tag{11}$$

Proof. Indeed, it follows from strictness that maps of strict Dieudonné algebras $f : B^* \to C^*$ are in bijection with systems of maps of differential graded algebras $f_n : \mathcal{W}_n(B)^* \to \mathcal{W}_n(C)^*$ such that we have commutative diagrams of graded algebras

$$\begin{array}{ccc} \mathcal{W}_n(B)^* & \xrightarrow{f_n} & \mathcal{W}_n(C)^* \\ \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\ \mathcal{W}_{n-1}(B)^* & \xrightarrow{f_{n-1}} & \mathcal{W}_{n-1}(C)^* \end{array} \quad , \quad \begin{array}{ccc} \mathcal{W}_n(B)^* & \xrightarrow{f_n} & \mathcal{W}_n(C)^* \\ \downarrow F & & \downarrow F \\ \mathcal{W}_{n-1}(B)^* & \xrightarrow{f_{n-1}} & \mathcal{W}_{n-1}(C)^* \end{array} .$$

Now suppose given a map $B^0/VB^0 \to C^0/VC^0$ under A^0/VA^0, which also induces maps on the Witt vectors $W_n(B^0/VB^0) \to W_n(C^0/VC^0)$, under $W_n(A^0/VA^0)$. Then the maps of differential graded algebras $\mathcal{W}_n(A)^* \to \mathcal{W}_n(B)^*$ are étale for each n, so we obtain the (unique) maps f_n from Proposition 5.3.2. The commutativity of the desired diagrams (which only depends on the underlying graded algebras) also follows, since the analogous diagrams commute when one restricts from $\mathcal{W}_n(B)^*$ to $\mathcal{W}_n(A)^*$ and the diagrams commute in degree zero by functoriality of the Witt vectors. □

Proof of Theorem 5.3.13. Suppose given an étale A^0/VA^0-algebra S. We now construct a V-adically étale strict Dieudonné algebra C^* under A^* together with an identification $C^0/VC^0 \simeq S$.

In order to do this, we construct the associated strict Dieudonné tower $\{\mathcal{W}_n(C^*)\}$. Consider first the strict Dieudonné tower $\mathcal{W}_n(A)^*$. Each term is a nilpotent $W(A^0/VA^0)$-module. We define $C_n^* = (\mathcal{W}_n(A)^*)_S$ to be the associated nilpotent $W(S)$-module.

We claim that the $\{C_n^*\}$ naturally assemble into a strict Dieudonné tower. To see this, we use systematically Corollary 5.3.9. First, we construct the differentials. The differential d on $\mathcal{W}_n(A)^*$ is (F^n, F^n)-linear by Lemma 5.3.11, so we obtain a differential on C_n^* which is (F^n, F^n)-linear. Next, the restriction maps $R : \mathcal{W}_n(A)^* \to \mathcal{W}_{n-1}(A)^*$ are $W(A^0/VA^0)$-linear, so we obtain restriction maps $R : C_n^* \to C_{n-1}^*$ which are $W(S)$-linear. The Frobenius maps $F : \mathcal{W}_n(A)^* \to \mathcal{W}_{n-1}(A)^*$ are (F^0, F^1)-linear, so we obtain Frobenius maps $F : C_n^* \to C_{n-1}^*$ which are (F^0, F^1)-linear as maps of $W(S)$-modules. Finally, the Verschiebung maps $V : \mathcal{W}_{n-1}(A)^* \to \mathcal{W}_n(A)^*$ are (F^1, F^0)-linear, so they extend to maps $C_{n-1}^* \to C_n^*$ as well. All the relevant identities involving R, F, d, V hold for the $\{\mathcal{W}_n(A)^*\}$, so they hold for $\{C_n^*\}$ by base-change.

It remains to check that the $\{C_n^*\}$ form indeed a strict Dieudonné tower in the sense of Definition 2.6.7. All these follow similarly from base-change, since by Example 5.3.10 exact sequences are preserved by base-change. For instance, $R : C_n^* \to C_{n-1}^*$ is a base-change of $\mathcal{W}_n(A)^* \to \mathcal{W}_{n-1}(A)^*$ and is therefore surjective as well. To see that the kernel of R is precisely the p-torsion, we base-change the exact sequences

$$0 \to \mathcal{W}_n(A)^*[p] \to \mathcal{W}_n(A)^* \xrightarrow{R} \mathcal{W}_{n-1}(A)^* \to 0, \quad 0 \to \mathcal{W}_n(A)^*[p] \to \mathcal{W}_n(A)^* \xrightarrow{p} \mathcal{W}_n(A)^*$$

along $W_n(A^0/VA^0) \to W_n(S)$ to obtain a short exact sequence

$$0 \to C_n^*[p] \to C_n^* \xrightarrow{R} C_{n-1}^* \to 0.$$

Similarly, we have the exact sequence

$$\mathcal{W}_{n+1}(A)^* \xrightarrow{F} \mathcal{W}_n(A)^* \xrightarrow{d} \mathcal{W}_n(A)^{*+1}/p,$$

which upon base-change (note that the map d is (F^n, F^n)-linear) induces an exact sequence

$$C_{n+1}^* \xrightarrow{F} C_n^* \xrightarrow{d} C_n^{*+1}/p,$$

showing that the image of $F : C_{n+1}^* \to C_n^*$ consists of those elements x with dx divisible by p, as desired.

Finally, as in the discussion preceding Definition 5.3.13, the C_n^* naturally acquire the structure of commutative, differential graded algebras compatibly in n, with $C_n^0 = W_n(S)$. Taking the inverse limit, we find using Corollary 2.7.8 that $C^* = \varprojlim_n C_n^*$ is a strict Dieudonné algebra with $\mathcal{W}_n(C)^* = C_n^*$. □

Proposition 5.3.15. *Let $R \to S$ be an étale map of $\mathbf{F}_p$-algebras. Then the map $\mathcal{W}\Omega_R^* \to \mathcal{W}\Omega_S^*$ of strict Dieudonné complexes is V-adically étale. Furthermore, for each n, we have an isomorphism*

$$\mathcal{W}_n\Omega_R^* \otimes_{W_n(R)} W_n(S) \simeq \mathcal{W}_n\Omega_S^*.$$

Proof. Let $\mathcal{W}\Omega_R^* \in \mathbf{AD}$ be the de Rham-Witt complex of R, so that we have a map $R \to \mathcal{W}_1\Omega_R^0$. By Theorem 5.3.13, there exists a V-adically étale strict Dieudonné algebra X^* under $\mathcal{W}\Omega_R^*$ such that $\mathcal{W}_1(X)^0 = \mathcal{W}_1\Omega_R^0 \otimes_R S$ and therefore $\mathcal{W}_n(X)^* = \mathcal{W}_n\Omega_R^* \otimes_{W_n(R)} W_n(S)$. We claim that the natural map $S \to \mathcal{W}_1(X)^0$ exhibits X^* as a de Rham-Witt complex of S. Indeed, to see this, we use (11) to conclude that X^* has the desired universal property. In fact, if $Y^* \in \mathbf{AD}$ is a strict Dieudonné algebra, we conclude from the universal property (4) of the de Rham-Witt complex together with (11) to find:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{AD}}(X^*, Y^*) &= \mathrm{Hom}_{\mathbf{AD}}(\mathcal{W}\Omega^*_R, Y^*) \times_{\mathrm{Hom}(R, Y^0/VY^0)} \mathrm{Hom}(S, Y^0/VY^0) \\ &= \mathrm{Hom}(S, Y^0/VY^0), \end{aligned}$$

as desired. □

6. The Nygaard filtration

Fix a smooth $\mathbf{F}_p$-algebra R equipped with a p-adically complete flat lift $\widetilde{R}$ to $\mathbf{Z}_p$ as well as a lift $\varphi : \widetilde{R} \to \widetilde{R}$ of the Frobenius on R. In this situation, the p-adic completion $\widehat{\Omega}^*_{\widetilde{R}}$ of the de Rham complex of $\widetilde{R}$ is naturally equipped with the structure of a Dieudonné complex, as in Variant 3.3.1. One can then consider the following subcomplexes:

$$\mathcal{N}^k \widehat{\Omega}^*_{\widetilde{R}} := p^k \widetilde{R} \to p^{k-1}\widehat{\Omega}^1_{\widetilde{R}} \xrightarrow{d} \dots \xrightarrow{d} p\widehat{\Omega}^{k-1}_{\widetilde{R}} \xrightarrow{d} \widehat{\Omega}^k_{\widetilde{R}} \xrightarrow{d} \widehat{\Omega}^{k+1}_{\widetilde{R}} \xrightarrow{d} \dots. \tag{12}$$

As k varies, we obtain a descending filtration $\{\mathcal{N}^k \widehat{\Omega}^*_{\widetilde{R}}\}_{k\geq 0}$ of $\widehat{\Omega}^*_{\widetilde{R}}$. This filtration has the following key features:

(1) The Frobenius $\varphi^* : \widehat{\Omega}^*_{\widetilde{R}} \to \widehat{\Omega}^*_{\widetilde{R}}$ is divisible by p^k when restricted to $\mathcal{N}^k \widehat{\Omega}^*_{\widetilde{R}}$ as φ^* is divisible by p^i on i-forms,

(2) The maps $\mathcal{N}^i \widehat{\Omega}^*_{\widetilde{R}} \xrightarrow{\varphi^*} p^i\widehat{\Omega}^*_{\widetilde{R}}$ induced by (1) for $i = k, k+1$ induce an isomorphism

$$\mathrm{gr}^k_{\mathcal{N}}\widehat{\Omega}^*_{\widetilde{R}} \simeq \tau^{\leq k}\big(p^k\widehat{\Omega}^*_{\widetilde{R}}/p^{k+1}\widehat{\Omega}^*_{\widetilde{R}}\big) \simeq \tau^{\leq k}\Omega^*_R,$$

where the last isomorphism is obtained by dividing by p^k.

This construction evidently depends on the chosen lifts $\widetilde{R}$ and φ. Nevertheless, it was observed by Nygaard (and elaborated by Illusie-Raynaud [16, §III.3]) that a filtration $\{\mathcal{N}^k W\Omega^*_R\}_{k\geq 0}$ with similar properties can be constructed directly on the de Rham-Witt complex $W\Omega^*_R$ intrinsically in terms of R; this so-called Nygaard filtration matches up with the filtration $\{\mathcal{N}^k \widehat{\Omega}^*_{\widetilde{R}}\}_{k\geq 0}$ of $\widehat{\Omega}^*_{\widetilde{R}}$ constructed above in the derived category under the canonical quasi-isomorphism $\widehat{\Omega}^*_{\widetilde{R}} \simeq W\Omega^*_R$ from Theorem 4.2.3, thus proving that the filtration $\{\mathcal{N}^k \widehat{\Omega}^*_{\widetilde{R}}\}_{k\geq 0}$ is independent of choices in the derived category.

In this section, we shall construct a similar "Nygaard" filtration for any saturated Dieudonné complex M^* that lives in non-negative degrees (Construction 6.1.1). Having constructed it, we observe in Remark 6.2.3 that the existence of this filtration can also be seen as a formal consequence of the quasiisomorphism $M \simeq L\eta_p M$ using the language of filtered derived categories; the latter borrows from [5] which exploits this observation for $M := R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$.

6.1. Constructing the Nygaard filtration.

For the rest of this section, let (M^*, d, F) be a saturated Dieudonné complex with $M^i = 0$ for $i < 0$.

Construction 6.1.1 (The Nygaard filtration)**.** For $i, k \geq 0$, define $\mathcal{N}^k M^i$ to be $p^{k-i-1}VM^i$ when $i < k$, and M^i for $i \geq k$. Using the identity $pdV = Vd$, it is easy to check that $\mathcal{N}^k M^* \subset M^*$ is a subcomplex. We view it as

$$\mathcal{N}^k M^* := p^{k-1}VM^0 \xrightarrow{d} p^{k-2}VM^1 \xrightarrow{d} \dots \xrightarrow{d} VM^{k-1} \xrightarrow{d} M^k \xrightarrow{d} M^{k+1}\dots.$$

Note that $\mathcal{N}^{k+1} M^* \subset \mathcal{N}^k M^*$, so we can view $\{\mathcal{N}^k M^*\}_{k\geq 0}$ as a descending filtration on M^*; we call this the *Nygaard filtration* on M^*, and write

$$\mathrm{gr}^k_{\mathcal{N}} M^* := \mathcal{N}^k M^* / \mathcal{N}^{k+1} M^*$$

for the corresponding graded pieces.

Remark 6.1.2. Note that $p^{k-i}M^i \subset \mathcal{N}^k M^i \subset p^{k-i-1}M^i$ for all $i < k$: the second containment is obvious, while the first follows as $p^{k-i}M^i = p^{k-i-1}V(FM^i)$. Thus, M^* is Nygaard complete (i.e., that $M^i \simeq \varprojlim_k M^i/\mathcal{N}^k M^i$ for all i) if and only if each M^i is p-adically complete.

Example 6.1.3. Say R is a perfect ring of characteristic p, and let $M = W(R)$, as in Example 3.1.6. Then $\mathcal{N}^k W(R) = p^k W(R)$ as F is an automorphism of $W(R)$ (since R is perfect) and $V = F^{-1}p$ (since $FV = p$).

Remark 6.1.4. Say R is a smooth algebra over a perfect field of characteristic p, so its de Rham-Witt complex $W\Omega^*_R$ is a saturated Dieudonné complex. The image of $W\Omega^*_R$ in the derived category $D(\mathbf{Z}_p)$ can be described as the absolute cyrstalline cohomology of $\operatorname{Spec} R$, i.e., as $R\Gamma((\operatorname{Spec} R)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}})$ where $\mathcal{O}_{\mathrm{crys}}$ is the structure sheaf on the crystalline site of $\operatorname{Spec} R$. The Nygaard filtration $\mathcal{N}^k W\Omega^*_R$ also admits a crystalline interpretation when $k < p$, as explained by Langer-Zink in [21, Theorem 4.6]: the image of $\mathcal{N}^k W\Omega^*_R$ in the derived category $D(\mathbf{Z}_p)$ is identified with $R\Gamma((\operatorname{Spec} R)_{\mathrm{crys}}, \mathcal{I}^{[k]}_{\mathrm{crys}})$, where $\mathcal{I}^{[k]}_{\mathrm{crys}} \subset \mathcal{O}_{\mathrm{crys}}$ is the k-th divided power of the universal pd-ideal sheaf $\mathcal{I}_{\mathrm{crys}} \subset \mathcal{O}_{\mathrm{crys}}$ on the crystalline site. It is also clear that this description cannot work for $k \geq p$: taking $p = 2$ and R perfect already gives a counterexample as $(2^{[k]}) = (2) \neq (2^k)$ for $k \geq 2$. We are not aware of a crystalline description of the Nygaard filtration for $k \geq p$.

As $M^i = 0$ for $i < 0$, we have an induced map

$$\varphi_M : M^* \xrightarrow{\alpha_F} \eta_p M^* \subset M^*$$

of complexes. Explicitly, this map is $p^i F$ on the degree i term M^i. In terms of this map, $\mathcal{N}^k M^* \subset M^*$ is the maximal subcomplex of M^* on which φ_M is "obviously" divisible by p^k. One can make this into a more precise assertion:

Proposition 6.1.5. *Fix $k \geq 0$.*

(1) *The terms of the complex $\mathrm{gr}^k_{\mathcal{N}} M^*$ vanish in degrees $> k$.*
(2) *The map φ_M carries $\mathcal{N}^k M^*$ into $p^k M^*$. Moreover, $\varphi_M^{-1}(p^k M^*) = \mathcal{N}^k M^*$.*
(3) *The map φ_M induces a map $\mathrm{gr}^k_{\mathcal{N}} M^* \to \tau^{\leq k}\big(p^k M^*/p^{k+1} M^*\big)$ by (1) and (2); this is an isomorphism of complexes.*
(4) *Dividing φ_M by p^k gives an isomorphism $\mathrm{gr}^k_{\mathcal{N}} M^* \simeq \tau^{\leq k}\big(M^*/pM^*\big)$.*

Proof. (1) Both $\mathcal{N}^k M^i$ and $\mathcal{N}^{k+1} M^i$ agree with M^i for $i \geq k + 1$.

(2) As φ_M equals $p^i F$ on M^i, the assertion is trivially true in degrees $i \geq k$. For smaller degrees, we simply observe that

$$\varphi_M(p^{k-i-1}VM^i) = p^i F p^{k-i-1} VM^i = p^{k-1} FVM^i = p^k M^i,$$

which proves the first assertion. For the second, fix $x \in M^i$ with $\varphi_M(x) \in p^k M^i$. We must show $x \in \mathcal{N}^k M^i$. We may clearly assume $i < k$. Dividing by p^i, our hypothesis gives $F(x) = p^{k-i} y$ for some $y \in M^i$. As $i < k$, this gives $F(x) = p^{k-i-1}(FV)(y) = p^{k-i-1}F(V(y)) = F(p^{k-i-1}V(y))$. As F is injective, we get $x = p^{k-i-1}V(y)$, which clearly lies in $\mathcal{N}^k M^i$, as wanted.

(3) In degrees $i < k$, the map under consideration is given by

$$p^{k-i-1}VM^i / p^{k-i}VM^i \xrightarrow{p^i F} p^k M^i / p^{k-1} M^i.$$

It is thus sufficient to check that $p^i F$ gives a bijection $p^{k-i-1}VM^i \to p^k M^i$ for $i < k$. Since p is a nonzerodivisor, this reduces to checking that F gives a bijection $VM^i \to pM^i$, which is clear: F is injective and $FV = p$.

In degree k, the map under consideration is given by

$$M^k / VM^k \xrightarrow{p^k F} \left(p^k M^k \cap d^{-1}(p^{k+1} M^{k+1})\right) / p^{k+1} M^k.$$

To show this is an isomorphism, we may divide by p^k to reduce to checking that

$$M^k / VM^k \xrightarrow{F} d^{-1}(pM^{k+1}) / pM^k$$

is a bijection. We claim that the above map is separately an isomorphism for the "numerator" and the "denominator". Indeed, Lemma 2.6.5 and the injectivity of F give an identification $M^k \overset{F}{\simeq} d^{-1}(pM^{k+1})$, while the injectivity of F and the formula $FV = p$ give an identification $VM^k \overset{F}{\simeq} pM^k$.

(4) This is formal from (3). □

Let (M^*, d, F) be a p-torsion-free Dieudonné complex such that $M^i = 0$ for $i < 0$. Suppose that M^* satisfies the Cartier criterion of Theorem 2.4.2: that is, the map $M^*/pM^* \to H^*(M^*/pM^*)$ sending $m \mapsto [F(m)]$ is an isomorphism. In this case, we know that the map $M^* \to \mathrm{Sat}(M^*)$ induces a quasi-isomorphism mod p. We observe now that up to quasi-isomorphism, we can also recover the Nygaard filtration on $\mathrm{Sat}(M^*)$.

For each $k \geq 0$, we let $\mathcal{N}_u^k M^* \subset M^*$ be the subcomplex $p^k M^0 \to p^{k-1} M^1 \to \cdots \to pM^{k-1} \to M^k \to M^{k+1} \to \ldots$. This defines a descending filtration $M^* \supset \mathcal{N}_u^1 M^* \supset \ldots$ of cochain complexes. Note that the natural map $\mathcal{N}_u^k M^* \subset M^* \to \mathrm{Sat}(M^*)$ factors through $\mathcal{N}^k \mathrm{Sat}(M^*)$, i.e., this filtration is compatible with the Nygaard filtration on $\mathrm{Sat}(M^*)$. Our next result shows that the natural map gives a quasi-isomorphism modulo p, and gives an abstraction of the construction in (12) for the completed de Rham complex.

Proposition 6.1.6. *Suppose M^* satisfies the Cartier criterion and $M^i = 0$ for $i < 0$. Then for each $k \geq 0$, the map*

$$\mathcal{N}_u^k M^* \to \mathcal{N}^k \mathrm{Sat}(M^*)$$

induces a quasi-isomorphism $\mathcal{N}_u^k M^*/p \to \mathcal{N}^k \mathrm{Sat}(M^*)/p$.

Proof. We already know the result for $k = 0$ (Theorem 2.4.2). To deduce the result for arbitrary k, by induction it suffices to show that on associated gradeds we obtain a quasi-isomorphism

$$\mathcal{N}_u^k M^*/\mathcal{N}_u^{k+1} M^* \to \mathcal{N}^k \mathrm{Sat}(M^*)/\mathcal{N}^{k+1}\mathrm{Sat}(M^*).$$

Note first that the differential on the right-hand-side vanishes, since $d(\mathcal{N}_u^k M^*) \subset \mathcal{N}_u^{k+1} M^*$. In particular, the cohomology of the right-hand-side is given by M^i/p for $0 \leq i \leq k$ and 0 for $i > k$. To determine the cohomology on the other side, we use Proposition 6.1.5 which gives us an isomorphism $\mathcal{N}^k\mathrm{Sat}(M^*)/\mathcal{N}^{k+1}\mathrm{Sat}(M^*) \to \tau^{\leq k}(\mathrm{Sat}(M)^*/p)$ given by $\varphi_{\mathrm{Sat}(M^*)}/p^k$. Unwinding the definitions, the composite map $M^i/p \to H^i(\tau^{\leq k}(\mathrm{Sat}(M)^*/p)) \simeq H^i(\tau^{\leq k}(M^*/p))$ (where the last equivalence uses the Cartier criterion) and this map is an isomorphism since M^* satisfies the Cartier criterion. □

6.2. **The Nygaard filtration and $L\eta_p$ via filtered derived categories.** Construction 6.1.1 is very explicit, but does not give a conceptual understanding of the Nygaard filtration. The latter can be obtained to some extent by using Proposition 6.1.5 to reinterpret the Nygaard filtration in terms of filtered derived categories, as we briefly explain. In particular, the discussion below makes clear that the Nygaard filtration is an intrinsic feature of any fixed point of $L\eta_p$ on $D(\mathbf{Z})$ in the sense of §8.3.

Let us first recall some definitions and facts about the filtered derived category $DF(\mathbf{Z})$ and the realization of the $L\eta_p$ operator as a truncation operator we refer to [5] for proofs.

Construction 6.2.1 (Filtered derived categories)**.** One defines $DF(\mathbf{Z})$ as the category of descending $\mathbf{Z}$-indexed filtrations $\{F^k\}_{k\in\mathbf{Z}}$ on objects of $D(\mathbf{Z})$; if everything is interpreted ∞-categorically, we can simply define $DF(\mathbf{Z}) := \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, D(\mathbf{Z}))$ as the ∞-category of $\mathbf{Z}^{\mathrm{op}}$-indexed diagrams. For an object $F := \{F^k\}_{k\in\mathbf{Z}} \in DF(\mathbf{Z})$, we call $F^{-\infty} := \varinjlim_{k\to-\infty} F^k$ the underlying nonfiltered complex. A diagram $\{F^k\}_{k\in\mathbf{Z}}$ is called $\mathbf{N}$-filtered if $\mathrm{gr}^k(F) = 0$ for $k < 0$; we shall write such objects as $\{F^k\}_{k\geq 0}$. The category $DF(\mathbf{Z})$ comes equipped with the Beilinson t-structure which is prescribed the following requirement: the connective part $DF^{\leq 0}(\mathbf{Z}) \subset DF(\mathbf{Z})$ comprises those filtered objects $F := \{F^k\}_{k\in\mathbf{Z}}$ such that $\mathrm{gr}^k(F) := F^k/F^{k+1}$ lies in $D^{\leq k}(\mathbf{Z})$ for all k. A defining feature of the connective cover map $\tau_B^{\leq 0}N \to N$ in this t-structure is that $\mathrm{gr}^k\tau_B^{\leq 0}N \simeq \tau^{\leq k}\mathrm{gr}^k N$ via the natural map. In particular, if N is $\mathbf{N}^{\mathrm{op}}$-indexed, so is $\tau_B^{\leq 0}N$.

Construction 6.2.2 (The $L\eta_p$ operator as a truncation)**.** We continue using the notation in Construction 6.2.1. Fix an $M \in D(\mathbf{Z})$ with $H^i(M) = 0$ for $i < 0$ and $H^0(M)$ being p-torsionfree. Consider the $\mathbf{N}^{\mathrm{op}}$-filtered object $\widetilde{M} := \{(p^k) \otimes^L_{\mathbf{Z}} M\}_{k\geq 0} \in DF(\mathbf{Z})$ given by the p-adic filtration on M. Then one can show that the natural map $L\eta_p(M) \to M$ is the underlying map on nonfiltered complexes for the the connective cover $\tau^{\leq 0}_B \widetilde{M} \to \widetilde{M}$ in the Beilinson t-structure on $DF(\mathbf{Z})$. In particular, $L\eta_p(M) \in D(\mathbf{Z})$ comes equipped with a preferred lift to the filtered derived category, and hence with a natural descending filtration $\{\mathrm{Fil}^k\}_{k\geq 0}$ that maps to the p-adic filtration $\{(p^k) \otimes^L_{\mathbf{Z}} M\}_{k\geq 0}$ under the canonical map $L\eta_p M \to M$. The defining feature of this construction is that $\mathrm{gr}^k L\eta_p(M)$ identifies with $\tau^{\leq k}(p^k M/p^{k+1}M)$ via the canonical map $L\eta_p(M) \to M$. In other words, given any filtered object $\widetilde{N} \in DF(\mathbf{Z})$ with a map $f : \widetilde{N} \to \widetilde{M}$, if $\mathrm{gr}^k(f)$ identifies $\mathrm{gr}^k(\widetilde{N})$ with $\tau^{\leq k}\mathrm{gr}^k(\widetilde{N})$, then f factors through a unique isomorphism $\widetilde{N} \simeq \tau^{\leq 0}_N \widetilde{M}$.

We can now give the promised description.

Remark 6.2.3 (The Nygaard filtration via filtered derived categories)**.** Fix a saturated Dieudonné complex (M^*, d, F) with $M^i = 0$ for $i < 0$. Then we have a given isomorphism $\alpha_F : M^* \simeq \eta_p M^*$ of complexes. Transfering the discussion of Construction 6.2.2 along this isomorphism, the image $M \in D(\mathbf{Z})$ of M^* comes equipped with a preferred filtration $\{\mathrm{Fil}^k\}_{k\geq 0}$ with the following features:

(1) The Frobenius map $\varphi_M : M \overset{\alpha_F}{\simeq} L\eta_p(M) \xrightarrow{can} M$ carries Fil^k into $p^k M$.
(2) The induced map $\mathrm{gr}^k M \to p^k M/p^{k+1}M$ identifies the left side with $\tau^{\leq k}$ of the right side.

One can then show using Proposition 6.1.5 that the Nygaard filtration $\{\mathcal{N}^k M^*\}_{k\geq 0}$ considered in this section provides a lift of $\{\mathrm{Fil}^k\}_{k\geq 0}$ to the category of chain complexes. In fact, since the above features characterize $\{\mathrm{Fil}^k\}_{k\geq 0}$ in $DF(\mathbf{Z})$, this discussion also recovers Proposition 6.1.6.

7. Relation to seminormality

In this section, we briefly explore the behaviour of the de Rham-Witt complexes introduced in this paper in the case of singularities. More precisely, in Proposition 7.2.4, we shall identify the functor $R \mapsto \mathcal{W}\Omega^0_R / V\mathcal{W}\Omega^0_R$ with the seminormalization functor $R \mapsto R^{sn}$; the key calculation that goes into this is that of a cusp, recorded in Example 7.1.1.

7.1. The example of a cusp.

The easiest example of a non-seminormal ring arises as the ring of functions on the cuspidal cubic. In this subsection, we calculate the $\mathcal{W}\Omega^*$-theory for this example, and explain why it differs from both Illusie's de Rham-Witt complex as well as crystalline cohomology.

Example 7.1.1 (The cusp)**.** Assume $p \geq 5$. Consider the map

$$\overline{R} = \mathbf{F}_p[x,y]/(x^2 - y^3) \simeq \mathbf{F}_p[t^2, t^3] \longrightarrow \overline{S} := \mathbf{F}_p[t]$$

of $\mathbf{F}_p$-algebras, where $x = t^3$ and $y = t^2$. Let us explain why this map induces an isomorphism on the de Rham-Witt complexes $\mathcal{W}\Omega^*$ of Notation 4.1.2.

We shall compute our de Rham-Witt complexes using Corollary 4.2.2. To apply this result, we use the obvious lifts $R := \mathbf{Z}_p[x,y]/(x^2 - y^3)$ and $S := \mathbf{Z}_p[t]$ of $\overline{R}$ and $\overline{S}$ respectively. There is an obvious inclusion $g : R \to S$ lifting the given inclusion modulo p. Moreover, raising the variables to their p-th powers defines Frobenius lifts φ_R and φ_S on R and S that are compatible under g. This gives rise to a map $\Omega^*_R \to \Omega^*_S$ of Dieudonné complexes. We shall check that this map induces an isomorphism $\mathrm{Sat}(\Omega^*_R) \to \mathrm{Sat}(\Omega^*_S)$.

First, note that Ω^*_S has no p-torsion, and that Ω^*_R and its maximal p-torsionfree quotient $\overline{\Omega^*_R}$ have the same saturation as saturated Dieudonné complexes are p-torsionfree by definition. It is thus enough to check that $\overline{\Omega^*_R} \to \Omega^*_S$ is an isomorphism on saturation. Next, we observe that the Frobenius lifts on R and S factor over the inclusion g: the element $\varphi_S(t) = t^p \in S := \mathbf{Z}_p[t]$ lies in the subring $R := \mathbf{Z}_p[t^2, t^3]$. Thus, we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{g} & S \\ {\scriptstyle \varphi_R}\downarrow & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle \varphi_S} \\ R & \xrightarrow{g} & S. \end{array}$$

This implies that the kernel K^* of $\overline{\Omega^*_R} \to \Omega^*_S$ is killed by the Dieudonné complex structure map $\alpha_F : \overline{\Omega^*_R} \to \eta_p \overline{\Omega^*_R}$ induced by φ_R. In particular, the Dieudonné complex structure on K^* induced by the formula $K^* = \ker(\overline{\Omega^*_R} \to \Omega_S)$ has trivial saturation. Thus, if we write Q^* for the image of $\overline{\Omega^*_R} \to \Omega^*_S$, then it is enough[1] to prove that $Q^* \subset \Omega^*_S$ induces an isomorphism on saturation. As $Q^0 = R$ and

[1]Here we implicitly use that $\mathrm{Sat}(-)$ is a left-adjoint, and hence it carries pushouts in $\mathbf{KD}$ to pushouts in $\mathbf{KD}_{\mathrm{sat}}$. Even though pushouts in the latter category are not easy to describe

$Q^1 = \mathrm{im}(\Omega^1_R \to \Omega^1_S)$, it is not difficult to see using the $\mathbf{N}$-grading of everything in sight that Ω^*_S/Q^* is given by the acyclic complex $\mathbf{Z}_p \cdot t \xrightarrow{d} \mathbf{Z}_p \cdot dt$: this is clear in degree 0, and in degree 1 it amounts to observing that the map

$$\Omega^1_R := \mathbf{Z}_p[t^2, t^3] \cdot t \cdot dt \oplus \mathbf{Z}_p[t^2, t^3] \cdot t^2 \cdot dt \to \Omega^1_S = \mathbf{Z}_p[t] \cdot dt$$

hits $t^i dt$ for all $i \neq 0$ (where we use $p \neq 2, 3$ to simplify the left hand side above). But then $Q^* \to \Omega^*_S$ is a quasi-isomorphism, and thus it remains so after applying η_{p^n} for all $n \geq 0$ by Proposition 8.2.1. Our claim now follows by the construction of the saturation functor in Proposition 2.3.1.

Let us use this calculation to explain why the $\mathcal{W}\Omega^*$ introduced in this paper differs from Illusie's de Rham-Witt complex $W\Omega^*$ for cusp.

Remark 7.1.2. Example 7.1.1 gives an example of a reduced $\mathbf{F}_p$-algebra $A := \mathbf{F}_p[x, y]/(x^2 - y^3)$ where the classical de Rham-Witt complex $W\Omega^*_A$ differs from the de Rham-Witt complex $\mathcal{W}\Omega^*_A$ introduced in this paper. More precisely, the comparison map

$$c_A : W\Omega^*_A \to \mathcal{W}\Omega^*_A$$

from Corollary 4.4.11 is not an isomorphism of complexes. Indeed, by Example 7.1.1, we know that $\mathcal{W}\Omega^2_A = 0$. On the other hand, Remark 4.4.5 shows that $W\Omega^*_A$ admits Ω^*_A as a quotient, and hence $W\Omega^2_A \neq 0$.

In fact, the comparison map c_A is not even a quasi-isomorphism. To see this, note that that we have a commutative diagram in the derived category of the form

$$\begin{array}{ccc} W\Omega^*_A & \longrightarrow & \mathcal{W}\Omega^*_A \\ \downarrow & & \downarrow \\ W\Omega^*_A \otimes^L_{\mathbf{Z}} \mathbf{F}_p & \longrightarrow & \mathcal{W}\Omega^*_A \otimes^L_{\mathbf{Z}} \mathbf{F}_p \\ \downarrow & & \downarrow \\ \Omega^*_A & \longrightarrow & \mathcal{W}_1\Omega^*_A. \end{array}$$

If the first horizontal map were a quasi-isomorphism, then the second one would be so as well. On the the other hand, the second vertical map on the right is a quasi-isomorphism by Corollary 2.5.3, so the diagram above would imply that the bottom horizontal map is surjective on H^*. By Example 7.1.1, the complex $\mathcal{W}_1\Omega^*_A$ identifies with Ω^*_B where $B = \mathbf{F}_p[t]$ is the seminormalization of A. In summary, we have shown that if c_A were a quasi-isomorphism, then $\Omega^*_A \to \Omega^*_B$ would be

explicitly in general, there is a simplification in our case: $\mathrm{Sat}(K^*) = 0$, so we formally have $\mathrm{Sat}(\overline{\Omega^*_R}) \simeq \mathrm{Sat}(Q^*)$.

surjective on H^*. Let us show this is false for H^0; we give the computation when $p = 5$, leaving the general case to the reader. We must show that the map

$$\ker(A \xrightarrow{d} \Omega^1_A) \to \ker(B \xrightarrow{d} \Omega^1_B)$$

is not surjective. By the Cartier isomorphism, the target identifies with $B^p = \mathbf{F}_p[t^5]$. Now $t^5 = xy$ under our inclusion $A \subset B$ (which was determined by $x = t^3$ and $y = t^2$). So it is enough to check that $xy \in A$ is not killed by the de Rham differential. But $d(xy) = x \cdot dy + y \cdot dx$. As $\Omega^1_A = \Big(A \cdot dx \oplus A \cdot dy\Big)/R \cdot (2x \cdot dx - 3y^2 \cdot dy)$, one checks using the $\mathbf{N}$-grading that $d(xy) \neq 0$.

Next, we wish to explain why the $\mathcal{W}\Omega^*$ introduced in this paper differs from classical crystalline cohomology. To make sense of this, we first explain why the latter maps naturally to former for a large class of $\mathbf{F}_p$-algebras. To this end, we give a new construction of the complex $\mathcal{W}\Omega^*_A \in \mathbf{KD}_{\mathrm{str}}$, via the equivalence in Theorem 8.4.11, in terms of derived de Rham-Witt complexes. Our discussion here is not self-contained, and we refer to other sources for proofs.

Remark 7.1.3 (Reconstructing $\mathcal{W}\Omega^*$ via derived de Rham-Witt complexes)**.** Let A be an $\mathbf{F}_p$-algebra. In this case, by left Kan extension from smooth $\mathbf{F}_p$-algebras to all simplicial commutative $\mathbf{F}_p$-algebras, one can define the *derived de Rham-Witt complex* $LW\Omega_A \in D(\mathbf{Z}_p)$ (as discussed in [5] for instance). When A admits the structure of an lci k-algebra, this agrees with the crystalline cohomology $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(A))$ of A by [3, Theorem 1.5]. In general, by reduction to the case of polynomial $\mathbf{F}_p$-algebras, one obtains a canonical map $LW\Omega_A \to L\eta_p LW\Omega_A$. Define its "strictification" $\mathcal{W}(LW\Omega_A)$ as the p-completion of the colimit of

$$LW\Omega_A \to L\eta_p LW\Omega_A \to L\eta_{p^2} LW\Omega_A \to \dots.$$

Then $\mathcal{W}(LW\Omega_A)$ is naturally an object of the ∞-category $\widehat{\mathcal{D}(\mathbf{Z}_p)_p}^{L\eta_p}$ of §8.4. Moreover, the equivalence

$$\widehat{\mathcal{D}(\mathbf{Z}_p)_p}^{L\eta_p} \simeq \mathbf{KD}_{\mathrm{str}}$$

coming from Theorem 8.4.11 carries $\mathcal{W}(LW\Omega_A)$ to $W\Omega^*_A$: this assertion holds true on the category of smooth $\mathbf{F}_p$-algebras by Theorem 4.4.12, and thus holds true in general as both functors commute with sifted colimits. In particular, there is a canonical comparison map

$$d_A : LW\Omega_A \to W\Omega^*_A$$

in $D(\mathbf{Z}_p)$ that is compatible with the Frobenius maps. If A admits the structure of an lci k-algebra for a perfect field k, we may also view d_A as a map

$$d_A : R\Gamma_{\mathrm{crys}}(A) \to W\Omega^*_A$$

in $D(\mathbf{Z}_p)$ that is compatible with the Frobenius maps.

Remark 7.1.4. Let $A = \mathbf{F}_p[x,y]/(x^2 - y^3)$. By Remark 7.1.3 and because A is an lci $\mathbf{F}_p$-algebra, there is a canonical comparison map

$$d_A : R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(A)) \to W\Omega^*_A$$

in $D(\mathbf{Z}_p)$. We shall explain why this is not an isomorphism. If it were an isomorphism, the $d_A \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$ would also be an isomorphism. By Example 7.1.1, we know that $H^2(W\Omega^*_A \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p) \neq 0$. On the other hand, $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(A)) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$ identifies with the derived de Rham complex $L\Omega_A$ as A is lci (see [3, Theorem 1.5]). As A admits a lift to $\mathbf{Z}/p^2$ together with a lift of Frobenius, there is a natural identification

$$\oplus_i \wedge^i L_{A/\mathbf{F}_p}[-i] \simeq L\Omega_A$$

by *loc. cit.*. In particular, since $H^2(\wedge^2 L_{A/\mathbf{F}_p}[-2]) \simeq H^0(\Omega^2_{A/\mathbf{F}_p}) \neq 0$, we learn that $H^2(R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(A)) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p) \neq 0$ as well, so $H^2(d_A \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p)$ cannot be an isomorphism.

7.2. **Realizing the seminormalization via the de Rham-Witt complex.** Using Example 7.1.1, we shall show that $\mathcal{W}\Omega^*_{(-)}$ is insensitive to seminormalization. We recall the relevant terminology and results from [26] first. A ring R is *seminormal* if it is reduced and satisfies the following condition: given $x, y \in R$ with $x^2 = y^3$, we can find $t \in R$ with $x = t^3$ and $y = t^2$. Any commutative ring R admits a universal map $R \to R^{sn}$ to a seminormal ring by [26, Theorem 4.1]; we refer to R^{sn} as the *seminormalization* of R.

Proposition 7.2.1. *Let $R \to S$ be a map of $\mathbf{F}_p$-algebras. Assume that $R \to S$ gives an isomorphism on seminormalization. Then $\mathcal{W}\Omega^*_R \simeq \mathcal{W}\Omega^*_S$.*

Proof. Since a ring and its reduction have the same seminormalization (by definition) and isomorphic de Rham-Witt complexes (Lemma 3.6.1), we may assume R and S are reduced. Let us introduce some terminology following [26, page 213]: given an injective map $S_1 \to S_2$ of reduced $\mathbf{F}_p$-algebras, we say it is *elementary extension* if S_2 is obtained by adjoining a single element $t \in S_2$ to S_1 with the property that $t^2, t^3 \in S_1$; any such map induces an isomorphism on seminormalizations. It is known that any extension $R \to S$ of reduced rings that gives an isomorphism on seminormalizations is a filtering union of R-subalgebras $S_i \subset S$ such that each $R \to S_i$ is obtained by finitely many elementary extensions (see [26, Lemma 2.6]). It is easy to show that the category $\mathbf{AD}_{\mathrm{str}}$ contains filtered colimits[2] and that the functor $\mathcal{W}\Omega^*_{(-)}$ commutes with filtered colimits (Corollary 4.1.5). We may thus

[2]It is clear that $\mathbf{AD}_{\mathrm{sat}}$ admits filtered colimits, and these are computed at the level of complexes, i.e., in $\mathbf{KD}_{\mathrm{sat}}$. To see that $\mathbf{AD}_{\mathrm{str}}$ admits filtered colimits, we simply apply the completion functor $\mathcal{W}(-)$ to the colimit of the corresponding diagram in $\mathbf{AD}_{\mathrm{sat}}$; this works thanks to Proposition 3.5.8.

inductively reduce to the case where $R \to S$ is an elementary extension. In this case, by definition, we can find a pushout square

$$\begin{array}{ccc} \mathbf{F}_p[t^2, t^3] & \longrightarrow & \mathbf{F}_p[t] \\ \downarrow & & \downarrow \\ R & \longrightarrow & \widetilde{S}, \end{array}$$

of commutative rings together with a surjective map $\widetilde{S} \to S$ factoring the original elementary extension. We shall check that both $R \to \widetilde{S}$ and $\widetilde{S} \to S$ induce isomorphisms on $\mathcal{W}\Omega^*_{(-)}$. For the map $R \to \widetilde{R}$: the map $\mathcal{W}\Omega^*_{(-)}$ induced by the top horizontal arrow in the square above is an isomorphism (Example 7.1.1), so the same holds true for the one induced by the bottom horizontal arrow as $\mathcal{W}\Omega^*_{(-)}$ preserves pushouts (Corollary 4.1.5). For $\widetilde{S} \to S$: as both $R \to \widetilde{S}$ and $R \to S$ give homeomorphisms on $\mathrm{Spec}(-)$, the same holds true for the surjective map $\widetilde{S} \to S$, so S is the reduction of $\widetilde{S}$, and then the claim follows from Lemma 3.6.1 as before. □

To identify our functor $R \mapsto \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R$ with the seminormalization functor $R \mapsto R^{sn}$, it is convenient to identify a class of maps inverted by the seminormalization functor (as in [26, Lemma 2.1]): a map $R \to S$ of commutative rings is *subintegral* if it is an integral map and $\mathrm{Spec}(S)(k) \to \mathrm{Spec}(R)(k)$ is bijective for every field k. This condition obviously only depends on the underlying map of reduced rings. Moreover, it is easy to see that any subintegral map $R \to S$ with R reduced must be injective: R embeds into a product of fields, and each such embedding factors uniquely over S. For any ring R, the map $R \to R^{sn}$ is subintegral by [26, Corollary 4.2]. Conversely:

Lemma 7.2.2. *If $R \to S$ is subintegral, then $R^{sn} \simeq S^{sn}$.*

Proof. We may assume R and S are reduced. But then $R \to S$ is injective by subintegrality. Then $R^{sn} \to S^{sn}$ is also an injective subintegral map by [26, Theorem 4.1, Corollary 4.2]. This forces $R^{sn} \simeq S^{sn}$ by the seminormality of R^{sn} and [26, Lemma 2.6]. □

There next criterion to test subintegrality will be useful for us:

Lemma 7.2.3. *Let $R \to S$ be a map of commutative rings. Then $R \to S$ is subintegral if and only if $\mathrm{Spec}(S)(V) \to \mathrm{Spec}(R)(V)$ is bijective for every valuation ring V. When R is noetherian, it suffices to work with discrete valuation rings in the previous sentence.*

Proof. We may assume that both R and S are reduced

For the forward direction: as a subintegral map is integral, it satisfies the valuative criterion for properness. The forward direction now follows as $\mathrm{Spec}(S)(K) \to \mathrm{Spec}(R)(K)$ is bijective for every field K.

Conversely, assume that every map $R \to V$ with V a valuation ring factors uniquely through S. Applying this condition for fields V immediately shows $\mathrm{Spec}(S)(k) \to \mathrm{Spec}(S)(k)$ is bijective for any field k. Applying this condition for valuation rings then shows that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ satisfies the valuative criterion for properness, and hence $R \to S$ is integral.

The simplification in the noetherian case follows as one only needs to use discrete valuation rings when testing the valuative criterion for noetherian rings. □

We can now prove the promised result:

Proposition 7.2.4. *Let R be an $\mathbf{F}_p$-algebra, and let $R' = \mathcal{W}\Omega^0_R / V\mathcal{W}\Omega^0_R$. Then the natural map $R \to R'$ exhibits R' as the seminormalization of R.*

Proof. Consider the functor $F(R) = \mathcal{W}\Omega^0_R / V\mathcal{W}\Omega^0_R$ on $\mathbf{F}_p$-algebras; this functor only depends on R_{red}, takes values in reduced $\mathbf{F}_p$-algebras by Lemma‘3.6.1, and there is a natural map $\eta_R : R \to F(R)$. Similarly, the seminormalization R^{sn} of any $\mathbf{F}_p$-algebra R only depends on R_{red}, is always reduced (by definition), and there is a natural map $R \to R^{sn}$. It thus suffices to identify $F(R)$ with R^{sn} for any reduced $\mathbf{F}_p$-algebra compatibly with the structure map from R. In fact, as both $F(-)$ and $(-)^{sn}$ commute with filtered colimits, we may even assume that R is a finite type $\mathbf{F}_p$-algebra that is reduced.

Let us show that any map $R \to S$ with S regular factors uniquely over η_R; by Lemma 7.2.3 and the noetherianity of R, this will imply that $R \to F(R)$ induces an isomorphism on seminormalizations. For existence: given such a map $R \to S$, naturality of η gives a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\eta_R} & F(R) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} \\ S & \xrightarrow{\eta_S} & F(S). \end{array}$$

By Theorem 4.3.2, the map η_S is an isomorphism when S is regular, so the square above gives the desired factorization. In fact, the same reasoning also gives uniqueness: given $a, b : F(R) \to S$ such that $F(a) \circ \eta_R = F(b) \circ \eta_R =: g$, composition with η_S and the above diagram shows that $\eta_S \circ a = \eta_S \circ b$ as they both equal $F(g)$, whence $a = b$ as η_S is an isomorphism.

The previous paragraph shows that $R \to F(R)$ is a subintegral map, so it induces an isomorphism on seminormalizations (Lemma 7.2.2). In particular, we get an injective R-algebra map $a : F(R) \to R^{sn}$ (where injectivity follows from reducedness of $F(R)$). On the other hand, the map $R \to R^{sn}$ is a subintegral map, and hence induces an isomorphism on applying $F(-)$ by Proposition 7.2.1. This gives an injective R-algebra map $b : R^{sn} \to F(R)$ (where injectivity follows from [26, Theorem 4.1]). By the universal property of seminormalization, the map

$a \circ b : R^{sn} \to R^{sn}$ must agree with the identity (as it does so on restriction to R). As both a and b are injective, this immediately implies that a and b are mutually inverse isomorphisms, so a gives the desired isomorphism $F(R) \simeq R^{sn}$. $\square$

8. Homological Algebra

In this section, we will describe the category of strict Dieudonné complexes (and similarly strict Dieudonné algebras) entirely in terms of the *derived category* and the functor $L\eta_p$ on it, introduced by Berthelot-Ogus [1]. A major application will be to give a short proof of the crystalline comparison of the cohomology theory $A\Omega$ of Bhatt-Morrow-Scholze [4].

Namely, we show (Theorem 8.3.11) that the category of strict Dieudonné complexes is equivalent to the *fixed points* of the functor $L\eta_p$ on the p-complete derived category $\widehat{D(\mathbf{Z})}_p$, i.e., the category of objects $K \in \widehat{D(\mathbf{Z})}_p$ together with an isomorphism $\varphi : K \simeq L\eta_p K$. In particular, any cochain complex with such an isomorphism (an abstraction of the Cartier isomorphism) admits a *canonical representative* or *strictification.* From this point of view, a fundamental feature of crystalline cohomology $R\Gamma_{\mathrm{crys}}(\mathrm{Spec} A)$ of a smooth affine algebra over a perfect ring k is the isomorphism it admits with its own $L\eta_p$, which leads precisely to its strictification, i.e., the de Rham-Witt complex. We will also show that this picture does not depend whether one uses the derived category or its natural ∞-categorical enhancement.

8.1. Recollections on the derived category.

We recall the following classical definition. Compare [18, Ch. 13–14] for a modern textbook reference.

Definition 8.1.1. The derived category $D(\mathbf{Z})$ of the integers can be defined in two equivalent ways:

(1) $D(\mathbf{Z})$ is the localization of the category $\mathrm{CoCh}(\mathbf{Z})$ of cochain complexes of abelian groups at the collection of quasi-isomorphisms.

(2) $D(\mathbf{Z})$ is the *homotopy category* of the subcategory $\mathrm{CoCh}(\mathbf{Z})^{\mathrm{free}} \subset \mathrm{CoCh}(\mathbf{Z})$ of cochain complexes of free abelian groups: that is, given $X^*, Y^* \in \mathrm{CoCh}(\mathbf{Z})^{\mathrm{free}}$, $\mathrm{Hom}_{D(\mathbf{Z})}(X, Y)$ is the abelian group of chain homotopy classes of maps of cochain complexes $X^* \to Y^*$.

Using the second description, we regard $D(\mathbf{Z})$ as a symmetric monoidal category under the tensor product of cochain complexes of free abelian groups.

Variant 8.1.2. If one uses the first definition of $D(\mathbf{Z})$ (as a localization), then one can replace the category of all cochain complexes $\mathrm{CoCh}(\mathbf{Z})$ with various subcategories. For example, $D(\mathbf{Z})$ is also the localization of the subcategory $\mathrm{CoCh}(\mathbf{Z})^{\mathrm{free}}$ at the quasi-isomorphisms. It will be helpful for us to know that $D(\mathbf{Z})$ is the localization of the intermediate subcategory $\mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z}) \subset \mathrm{CoCh}(\mathbf{Z})$ consisting of cochain complexes of *torsion-free* abelian groups.

As a matter of notation, we will typically use the superscript $*$ to refer to cochain complexes, and omit the $*$ to refer to the underlying object of the derived category.

Remark 8.1.3. If $\mathbf{Z}$ were replaced by a different ring R (which is not of finite homological dimension), the description of the derived category as the homotopy category of cochain complexes of free abelian groups should be modified: one should instead take the homotopy category of *K-projective* complexes in the sense of [25]. See also [13, Ch. 2] for a presentation as a model category.

Next, we recall elements of the theory of *p-complete* objects in $D(\mathbf{Z})$, as in [10] and going back to ideas of Bousfield [9].

Definition 8.1.4. Let $X \in D(\mathbf{Z})$. We will say that X is *p-complete* if for every $Y \in D(\mathbf{Z})$ such that $p : Y \to Y$ is an isomorphism in $D(\mathbf{Z})$, we have $\mathrm{Hom}_{D(\mathbf{Z})}(Y, X) = 0$. We let $\widehat{D(\mathbf{Z})}_p \subset D(\mathbf{Z})$ be the full subcategory spanned by the p-complete objects.

Definition 8.1.5. An abelian group X is called *derived p-complete* if, when considered as an object of $D(\mathbf{Z})$ concentrated in a single degree, it is p-complete.

An object of $D(\mathbf{Z})$ is p-complete if and only if its cohomology groups are derived p-complete [10, Prop. 5.2]. The category of derived p-complete abelian groups is itself abelian, and includes all p-complete abelian groups. A detailed treatment of the theory of derived complete (also called *L-complete*) abelian groups and modules is given in [14, Appendix A].

Example 8.1.6. Let X^* be a cochain complex. Suppose all the terms $X^i, i \in \mathbf{Z}$ are derived p-complete. Then the underlying object $X \in D(\mathbf{Z})$ is derived p-complete, because the cohomology groups are derived p-complete.

Definition 8.1.7. An derived p-complete abelian group A is called *pro-free* if the following equivalent conditions hold:

(1) A is p-torsion-free.
(2) A is isomorphic to the p-adic completion of a free abelian group.

Example 8.1.8. Let A be a pro-free abelian group and let $B \subset A$ be a derived p-complete subgroup. Then B is pro-free.

The main result of this subsection that we will need is that for cochain complexes of pro-free p-complete abelian groups, maps in the derived category can be described as chain homotopy classes of maps of complexes. Closely related results appear in appear in [24, 27] (where a model category structure is produced) and [23, Sec. 7.3.7]; for convenience of the reader, we spell out the details in this special case.

Proposition 8.1.9. *Let X^* be a cochain complex of free abelian groups. Suppose the underlying object $X \in D(\mathbf{Z})$ is p-complete. Let $\widehat{X^*}_p$ be the (levelwise) p-completion of the cochain complex X^*. Then $X^* \to \widehat{X^*}_p$ is a quasi-isomorphism.*

Proof. In fact, since both sides are p-complete objects of $D(\mathbf{Z})$, it suffices to show that the map $X^* \to \widehat{X^*}_p$ becomes a quasi-isomorphism after taking the derived tensor product with $\mathbf{Z}/p$. But the map is in fact an isomorphism modulo p, and both sides are p-torsion-free. □

Proposition 8.1.10. *$\widehat{D(\mathbf{Z})_p}$ is equivalent to the homotopy category of cochain complexes of p-complete pro-free abelian groups.*

Proof. If U^* is a cochain complex of free abelian groups representing an object of $\widehat{D(\mathbf{Z})_p}$, then the (levelwise) p-completion $\widehat{U}^*_p$ is quasi-isomorphic to U^*. It follows that any object of $\widehat{D(\mathbf{Z})_p}$ can be represented by a cochain complex of pro-free p-complete abelian groups.

Let X^*, Y^* be cochain complexes of p-complete pro-free abelian groups. Choose a cochain complex $\widetilde{X}^*$ of free abelian groups equipped with a quasi-isomorphism $\widetilde{X}^* \to X^*$. We have that $\mathrm{Hom}_{D(\mathbf{Z})}(X, Y)$ is the group of chain homotopy classes of maps $\widetilde{X}^* \to Y^*$. Replacing $\widetilde{X}^*$ by its (levelwise) p-completion Z^* does not change the quasi-isomorphism type, and we have a factorization $\widetilde{X}^* \to Z^* \to X^*$, and maps $\widetilde{X}^* \to Y^*$ are equivalent to maps $Z^* \to Y^*$. Combining these observations, it suffices to show now that $Z^* \to X^*$ is a chain homotopy equivalence.

For this, we observe that X^* is K-projective [25] as a cochain complex in the category of derived p-complete abelian groups. Indeed, any complex of pro-free p-complete abelian groups T^* is K-projective in this sense: T^* is a filtered colimit of its truncations $\tau^{\le n} T^*$, which are bounded-above cochain complexes of pro-free p-complete abelian groups. Applying [25, Cor. 2.8], we conclude that X^* is K-projective, and similarly so is Z^*. Therefore, any quasi-isomorphism $Z^* \to X^*$ admits a chain homotopy inverse. □

8.2. **The Functor $L\eta_p$.** Next, we review some of the basic properties of the operator $L\eta_p$. For a more extensive treatment, we refer the reader to [4, Sec. 6].

Recall from Construction 2.1.3 the functor η_p on the category $\mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z})$ of p-torsion free cochain complexes, which takes an object M^* to the subcomplex $\eta_p M^* \subset M^*[p^{-1}]$ with $(\eta_p M)^n = \{x \in p^n M^n : dx \in p^{n+1} M^{n+1}\}$. One has the following basic property of η_p.

Proposition 8.2.1. *If (M^*, d) is a cochain complex of p-torsion-free abelian groups, then there is a functorial isomorphism of graded abelian groups $H^*(\eta_p M) \simeq H^*(M)/H^*(M)[p]$, where $H^*(M)[p] \subset H^*(M)$ denotes the p-torsion.*

Proof. Given a cycle $z \in M^n$, the element $p^n z \in (\eta_p M)^n$ is a cycle, and if z is a boundary, then so is $p^n z \in (\eta_p M)^n$. This defines a map $H^n(M) \to H^n(\eta_p M)$. Unwinding the definitions, we see that the class of a cycle $z \in M^n$ belongs to the kernel of this map precisely if $pz \in M^n$ is a boundary, as claimed. □

We now study the derived version of this functor.

Definition 8.2.2. The functor $\eta_p : \mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z}) \to \mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z})$ defined above preserves quasi-isomorphisms by Proposition 8.2.1. Therefore, η_p descends to a functor $L\eta_p : D(\mathbf{Z}) \to D(\mathbf{Z})$.

Given an object $X \in D(\mathbf{Z})$, we have $H^*(L\eta_p X) \simeq H^*(X)/H^*(X)[p]$ by Proposition 8.2.1. In particular, $L\eta_p$ is an operator on $D(\mathbf{Z})$ which reduces the p-torsion by a small amount.

Proposition 8.2.3. *$L\eta_p$ naturally has the structure of a symmetric monoidal functor $D(\mathbf{Z}) \to D(\mathbf{Z})$.*

Proof. In fact, η_p has the structure of a *lax* symmetric monoidal functor: given p-torsion-free cochain complex K^*, L^*, there is a natural transformation $\eta_p K^* \otimes \eta_p L^* \to \eta_p(K^* \otimes L^*)$. The derived functor $L\eta_p$ is strictly symmetric monoidal. Since both sides are quasi-isomorphism invariants, it suffices to show that the natural map $L\eta_p X \otimes_{\mathbf{Z}} L\eta_p Y \to L\eta_p(X \otimes_{\mathbf{Z}} Y)$ is a quasi-isomorphism for $X, Y \in D(\mathbf{Z})$. As in [4, Prop. 6.8], one reduces to the case where X, Y are both cyclic, and then checks directly. □

Proposition 8.2.4 (Compare [4, Lem. 6.19])**.** *The functor $L\eta_p$ preserves $\widehat{D(\mathbf{Z})}_p$ and descends to a symmetric monoidal functor $\widehat{D(\mathbf{Z})}_p \to \widehat{D(\mathbf{Z})}_p$.*

We recall also Proposition 2.4.4, which shows that the composite functor

$$D(\mathbf{Z}) \overset{L\eta_p}{\to} D(\mathbf{Z}) \overset{\otimes \mathbf{Z}/p}{\to} D(\mathbf{Z}/p)$$

has a very simple description, as the cohomology modulo p equipped with the Bockstein differential. In particular, $L\eta_p/p$ has a canonical cochain representative. This suggests that $L\eta_p$ should have a strictification function, and we explore this in the rest of the section.

8.3. **Fixed Points of $L\eta_p$: 1–categorical version.** In this subsection, we prove the basic result (Theorem 8.3.11 below) that the category of strict Dieudonné complexes can be identified with the fixed point category of $L\eta_p$ on the p-complete derived category. In particular, any fixed point will admit a canonical representative as a cochain complex. We use the following definition of fixed points.

Definition 8.3.1. Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. The *fixed points* $\mathcal{C}^F$ is the category of pairs $(X, \varphi : X \simeq F(X))$ where $X \in \mathcal{C}$ and φ is an isomorphism between X and $F(X)$. Morphisms in $\mathcal{C}^F$ are defined in the evident manner.

Construction 8.3.2. We clearly have a functor $\mathbf{KD}_{\mathrm{sat}} \to \mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z})$ which sends a saturated Dieudonné complex X^* to its underlying cochain complex. Furthermore, the operator F enables us to lift this to a functor

$$\mathbf{KD}_{\mathrm{sat}} \to (\mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z}))^{\eta_p}, \tag{13}$$

to the fixed point category $(\mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z}))^{\eta_p}$. Essentially by definition, this induces an equivalence of categories $\mathbf{KD}_{\mathrm{sat}} \simeq (\mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z}))^{\eta_p}$.

Our main result is an analog of Construction 8.3.2 for strict Dieudonné complexes and for the derived category. To begin with, we will need a number of homological preliminaries.

Notation 8.3.3. Let X^* be a torsion-free cochain complex. For each n, we have a canonical reduction map $X^*/p^{n+1} \to X^*/p^n$ and an inclusion map $X^*/p^n \to X^*/p^{n+1}$ obtained from tensoring the inclusion $\mathbf{Z}/p^n \subset \mathbf{Z}/p^{n+1}$ with X^*. We denote by

$$F : H^*(X/p^{n+1}) \to H^*(X/p^n), \quad V : H^*(X/p^n) \to H^*(X/p^{n+1})$$

the induced maps in cohomology. This choice of notation is informed by its compatibility with similar maps for Dieudonné complexes under the strictification result in Theorem 8.3.11.

Construction 8.3.4 (The Bockstein homomorphism)**.** Let X^* be a cochain complex of torsion-free abelian groups and let $n \geq 1$ be an integer. Then we define a differential β_{p^n} on the cohomology groups $H^*(X/p^n)$ in one of two equivalent ways:

(1) β_{p^n} is the connecting homomorphism arising from the short exact sequence of cochain complexes $0 \to X^*/p^n \to X^*/p^{2n} \to X^*/p^n \to 0$.
(2) Explicitly, given $x \in X^i$ such that dx is divisible by p^n (so that x represents a class $[x]$ in $H^i(X/p^n)$), $\beta_{p^n}([x])$ is the class $[dx/p^n]$.

In particular, this procedure manufactures a cochain complex of $\mathbf{Z}/p^n$-modules from a torsion-free cochain complex of abelian groups.

Remark 8.3.5. The Bockstein homomorphism has the following basic property. Consider the reduction map $X^*/p^{n+1} \to X^*/p^n$. Then a class $a \in H^i(X/p^n)$ admits a lift to $H^i(X/p^{n+1})$ (i.e., belongs to the image of the map F) if and only if $\beta_{p^n}(a)$ is divisible by p.

We will need the following general construction.

Construction 8.3.6. Let X^* be a torsion-free cochain complex. Then, for each n, we have a natural map

$$\mu_n : H^*(X/p^n) \to H^*((\eta_p X)/p^{n-1}),$$

such that:

(1) μ_n commutes with the Bockstein differentials on both sides.
(2) μ_n annihilates any $x \in H^*(X/p^n)$ with $px = 0$.
(3) μ_n is surjective, and its kernel is the chain complex generated by the image of the map $V^{n-1} : H^*(X/p) \to H^*(X/p^n)$ induced by the natural map $X/p \to X/p^n$ arising from the embedding $\mathbf{Z}/p \subset \mathbf{Z}/p^n$.

(4) For any $k \le n$, we have a commutative square

$$\begin{array}{ccc} H^*(X/p^n) & \xrightarrow{\mu_n} & H^*(L\eta_p X/p^{n-1}) \\ \downarrow{\scriptstyle F^{n-k}} & & \downarrow{\scriptstyle F^{n-k}} \\ H^*(X/p^k) & \xrightarrow{\mu_k} & H^*(L\eta_p X/p^{k-1}) \end{array}.$$

(5) For any $k \le n$, we have a commutative square

$$\begin{array}{ccc} H^*(X/p^n) & \xrightarrow{\mu_n} & H^*(L\eta_p X/p^{n-1}) \\ {\scriptstyle V^{n-k}}\uparrow & & {\scriptstyle V^{n-k}}\uparrow \\ H^*(X/p^k) & \xrightarrow{\mu_k} & H^*(L\eta_p X/p^{n-1}) \end{array}.$$

To construct μ_n, we proceed as follows. We represent a class in $H^k(X/p^n)$ by $x \in X^k$ such that $p^n \mid dx$ and map $x \mapsto p^k x \in (\eta_p X)^k$. Then $d(p^k x) \in (\eta_p X)^{k+1}$ is divisible by p^{n-1}, so that $p^k x$ defines a cycle in $(\eta_p X/p^{n-1})^k$. Passing to cohomology classes gives the map desired.

Proof. We prove that the above construction is well-defined. Suppose first that $x \in X^k$ has the property that p^n divides dx and that px represents the zero class in $H^k(X/p^n)$. Then $px = p^n x' + dy$, where dy is divisible by p. Furthermore, dx' is divisible by p since dx is divisible by p^n. It follows that $p^k x \in (\eta_p X)^k$ can be written as $p^{n-1}(p^k x') + d(p^{k-1}y)$, which yields the zero class in $H^k(\eta_p X/p^{n-1})$. This implies that the construction μ_n is well-defined and implies further that μ_n annihilates the p-torsion in $H^*(X/p^n)$.

Consider a class $x \in X^k$ such that p^n divides dx. Then $\mu_n([x]) = [p^k x]$. We observe that the Bockstein β_{p^n} carries $[x]$ to $[dx/p^n] \in H^{k+1}(X/p^n)$, while the Bockstein $\beta_{p^{n-1}}$ carries $[p^k x] \in H^k(\eta_p X^*/p^{n-1})$ to $[d(p^k x)/p^{n-1}]$. Furthermore, we have $\mu_n([dx/p^n]) = [p^{k+1} dx/p^n]$. Comparing shows that μ_n commutes with the Bockstein.

Next, we argue that μ_n is surjective. Indeed, fix a chain $z \in (\eta_p X)^k$ such that $dz \in (\eta_p X)^{k+1}$ is divisible by p^{n-1}. We have $z = p^k z'$ for some $z \in X^k$, and it suffices to observe that dz is divisible by p^n in X^{k+1}, because $p^k dz$ is divisible by p^{n-1} in a subgroup of $p^{k+1} X^{k+1}$. Thus $\mu_n([z']) = [z']$.

Suppose now that $x \in X^k$ has dx divisible by p^n and that $\mu_n([x]) = 0$. In other words, in $(\eta_p X)^k$, we have $p^k x = p^{n-1}(p^k x') + d(p^{k-1}y')$ for some $p^k x' \in (\eta_p X)^k, p^{k-1}y' \in (\eta_p X)^{k-1}$. Then we have that dx' is divisible by p, and same for dy'. We have $x = p^{n-1}x' + d(p^{n-1}y')/p^n$. We can then write $[x] = V^{n-1}([x']) + \beta_{p^n} V^{n-1}([y'])$ in $H^k(X/p^n)$. □

Remark 8.3.7. We now observe that the above constructions pass to the *derived* category. Given $X \in D(\mathbf{Z})$, we let $X/p^n = X \overset{\mathbf{L}}{\otimes} \mathbf{Z}/p^n$ be the derived tensor product of X with $\mathbf{Z}/p^n$; explicitly, if X is represented by a *torsion-free* cochain complex X^*, then X/p^n is represented by the quotient X^*/p^n.

We will now need to observe that Notation 8.3.3, Construction 8.3.4, and Construction 8.3.6 are clearly quasi-isomorphism invariant in the torsion-free cochain complex X^*, so that they are well-defined and functorial for an object $X \in D(\mathbf{Z})$. That is, we find:

(1) Bockstein homomorphisms and operators F, V on the cohomology $\{H^*(X/p^n)\}_{n \geq 1}$.
(2) Natural maps $\mu_n : H^*(X/p^n) \to H^*(L\eta_p X/p^{n-1})$ for $n \geq 1$.

In the remainder of the section, we will pass to the derived category $D(\mathbf{Z})$, and carry out all of our constructions there. In particular, our goal is to construct a strict Dieudonné tower. The main new construction we need is that of the restriction map.

Construction 8.3.8. Let $M \in D(\mathbf{Z})$ be an object equipped with a map $\varphi : L\eta_p M \to M$ in $D(\mathbf{Z})$. Then we obtain, for each n, a map of graded abelian groups

$$\mathcal{R} : H^*(M/p^n) \xrightarrow{\mu_n} H^*(L\eta_p M/p^{n-1}) \xrightarrow{\varphi} H^*(M/p^{n-1}).$$

If we consider the source and the target as cochain complexes, equipped with the Bockstein differential, then the map $\mathcal{R}$ is a map of cochain complexes.

Let M^* be a saturated Dieudonné complex. By Proposition 2.5.2, we have an isomorphism

$$F^r : \mathcal{W}_r(M)^* \xrightarrow{\simeq} H^*(M/p^r M).$$

In addition, we have an isomorphism $\eta_p M^* \simeq M^*$ via F. We observe now that the restriction maps in cohomology from Construction 8.3.8 are compatible with the restriction (i.e., quotient) maps on the tower $\{\mathcal{W}_r(M)^*\}$. In fact, we can obtain the entire structure of a strict Dieudonné tower in this way.

Proposition 8.3.9. *Let M^* be a saturated Dieudonné complex, so that we obtain an isomorphism $\eta_p M^* \simeq M^*$ and an induced isomorphism $\varphi : L\eta_p M \simeq M$ in $D(\mathbf{Z})$. We have commutative diagrams*

$$\begin{array}{ccc} \mathcal{W}_r(M)^* & \xrightarrow{\simeq} & H^*(M/p^r) \\ \downarrow R & & \downarrow \mathcal{R} \\ \mathcal{W}_{r-1}(M)^* & \xrightarrow{\simeq} & H^*(M/p^{r-1}) \end{array},$$

$$\begin{array}{ccc} \mathcal{W}_r(M)^* & \xrightarrow{\simeq} & H^*(M/p^r) \\ \downarrow F & & \downarrow F \\ \mathcal{W}_{r-1}(M)^* & \xrightarrow{\simeq} & H^*(M/p^{r-1}) \end{array}, \qquad \begin{array}{ccc} \mathcal{W}_{r-1}(M)^* & \xrightarrow{\simeq} & H^*(M/p^{r-1}) \\ \downarrow V & & \downarrow V \\ \mathcal{W}_r(M)^* & \xrightarrow{\simeq} & H^*(M/p^r) \end{array}$$

Proof. Let $x \in M^i$ have the property that dx is divisible by p^r, so that x defines a class in $H^i(M/p^r)$. Unwinding the definitions, we find that $\mu_r x \in H^i(L\eta_p M/p^{r-1})$ is represented by the class $p^i x \in (\eta_p M)^i$. The map $\eta_p M^i \to M^i$ is given by $p^i z \mapsto F^{-1}(z)$, so we find that the map $\mathcal{R}$ carries the class of x to the class of $F^{-1}(x)$. Since the map $\mathcal{W}_r(M)^* \to H^*(M/p^r)$ sends an element $y \in M^i$ representing a class in $\mathcal{W}_r(M)^i$ to the class of $F^r y$ in $H^i(M/p^r)$, the commutativity of the first diagram follows. The commutativity of the second and third diagrams is easier, and is left to the reader.

□

Let M^* be a saturated Dieudonné complex, so that we can construct a strict Dieudonné tower $\{\mathcal{W}_r(M)^*\}$. The above discussion shows that the strict Dieudonné tower (which in particular determines the strictification of M^*) depends only on the quasi-isomorphism class $M \in D(\mathbf{Z})$ of M^* and the equivalence $M \simeq L\eta_p M$ in $D(\mathbf{Z})$ (rather than the isomorphism $M^* \simeq \eta_p M^*$ of cochain complexes). We formulate this more explicitly.

Construction 8.3.10. Suppose given a pair (X, φ_X) where $X \in \widehat{D(\mathbf{Z})_p}$ and $\varphi_X : X \simeq L\eta_p X$. For each n, we consider the cochain complex

$$(X_n^*, d) = (H^*(X/p^n), \beta_{p^n}),$$

where β_{p^n} is the Bockstein differential on $H^*(X/p^n)$. We assemble the $\{X_n^*\}$ into a tower of cochain complexes via the maps

$$\mathcal{R}_n : X_n^* \to X_{n-1}^*$$

from Construction 8.3.8. This is a tower of cochain complexes, and since the map μ_n is surjective, it follows that each $\mathcal{R}_n$ is surjective. In addition, we consider the maps $F : H^*(X/p^{n+1}) \to H^*(X/p^n)$ and $V : H^*(X/p^n) \to H^*(X/p^{n+1})$ as in Construction 8.3.3. The list of properties in Construction 8.3.6 (and Remark 8.3.5) shows that the tower $\{X_n^*\}$ together with the F, V operators is a strict Dieudonné tower.

The above defines a functor

$$\Psi : \widehat{D(\mathbf{Z})_p}^{L\eta_p} \to \mathbf{KD}_{\mathrm{str}}$$

as the composite of the above construction with the equivalence $\mathbf{TD} \simeq \mathbf{KD}_{\mathrm{str}}$ between strict Dieudonné towers and strict Dieudonné complexes.

With this in mind, the following is the main homological result of this section.

Theorem 8.3.11. *Construction 8.3.2 together with projection to the derived category and Construction 8.3.10. induce an inverse equivalence of categories* $\mathbf{KD}_{\mathrm{str}} \simeq \widehat{D(\mathbf{Z})_p}^{L\eta_p}$.

The proof of Theorem 8.3.11 will require some additional preliminaries.

Definition 8.3.12. Let X^*, Y^* be p-torsion free Dieudonné complexes. Let $f : X^* \to Y^*$ be a map of the underlying cochain complexes. We say that f is *weakly F-compatible* if the following diagram of cochain complexes

$$\begin{array}{ccc} X^* & \xrightarrow{F} & \eta_p X^* \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \eta_p f} \\ Y^* & \xrightarrow{F} & \eta_p Y^* \end{array}$$

commutes up to chain homotopy. Equivalently, we require that the map $F^{-1} \circ f \circ F$ of cochain complexes $X^* \to Y^*$ is chain homotopic to f.

Proposition 8.3.13. *Let $f : X^* \to Y^*$ be a weakly F-compatible map between the underlying cochain complexes of strict Dieudonné complexes X^*, Y^*. Then f is chain homotopic to a unique map of strict Dieudonné complexes $\overline{f} : X^* \to Y^*$.*

Proof. Let $u : X^* \to Y^{*-1}$ be an arbitrary map of graded abelian groups. Then we have

$$F^{-1} \circ (du + ud) \circ F = d(Vu) + (Vu)d.$$

Write $F^{-1} \circ f \circ F = f + dh + hd$ for some map $h : X^* \to Y^{*-1}$. We now set $u = \sum_{n=0}^{\infty} V^n h$ and $\overline{f} = f + du + ud$. It follows that

$$\begin{aligned} F^{-1} \circ \overline{f} \circ F - \overline{f} &= dh + hd + d(Vu) + (Vu)d - (du + ud) \\ &= dh + hd + \sum_{n=0}^{\infty} (dV^{n+1}h + V^{n+1}hd) - (\sum_{n=0}^{\infty} dV^n h + V^n hd) \\ &= 0. \end{aligned}$$

It follows that $\overline{f} : X^* \to Y^*$ is a map of strict Dieudonné complexes and, on underlying cochain complexes, $\overline{f}$ is chain homotopic to f.

Finally, let $g : X^* \to Y^*$ be a map of strict Dieudonné complexes which is chain homotopic to zero. Then g induces the zero map $\mathcal{W}_r(X)^* \to \mathcal{W}_r(Y)^*$ thanks to the identifications $\mathcal{W}_r(X)^* \simeq H^*(X/p^r), \mathcal{W}_r(Y)^* \simeq H^*(Y/p^r)$. It follows that g is zero, since both X^*, Y^* are strict. This proves uniqueness. □

Proof of Theorem 8.3.11. Let $\Phi : \mathbf{KD}_{\mathrm{str}} \to \widehat{D(\mathbf{Z})_p}$ be the evident functor which carries a strict Dieudonné complex X^* to the class in $D(\mathbf{Z})$ of the isomorphism $X^* \to \eta_p X^*$. In Construction 8.3.10, we obtained a functor Ψ in the other direction. Thanks to Proposition 8.3.9, we have seen that the above construction, when one inputs a strict Dieudonné complex M^*, produces the strict Dieudonné tower $\{\mathcal{W}_r(M)^*\}$. In particular, we have an equivalence $\Psi \circ \Phi \simeq \mathrm{id}$.

To complete the proof, we need to show that Φ is fully faithful and essentially surjective. First, Φ is fully faithful thanks to Proposition 8.3.13 and the description of the p-complete derived category in Proposition 8.1.10. We now check essential surjectivity. Given an object $X \in \widehat{D(\mathbf{Z})}_p$ together with an equivalence $\varphi : X \simeq L\eta_p X$ in $D(\mathbf{Z})$, choose a complex Y^* of free abelian groups representing X. Then the equivalence φ is represented by a map (well-defined up to chain homotopy) $Y^* \to \eta_p Y^*$ of cochain complexes, i.e., Y^* can be given the structure of a Dieudonné complex. Now the map $Y^* \to \mathcal{W}(Y^*)$ is a quasi-isomorphism since it is a quasi-isomorphism modulo p and both sides are p-complete. It follows that the pair (X, φ) is represented by a strict Dieudonné complex. In particular, Φ is essentially surjective. $\square$

8.4. **Fixed Points of $L\eta_p$: ∞–categorical version.** From a homotopy-theoretic point of view, the analysis of $D(\mathbf{Z})$ and fixed points of functors on it is somewhat unnatural: the more natural object is not $D(\mathbf{Z})$ itself, but its ∞-categorical enhancement which remembers the *space* of maps between objects. In particular, one expects to perform operations such as fixed points at the ∞-categorical level to obtain a well-behaved object.

In this subsection, we observe that the above analysis of the fixed points of $L\eta_p$ works at the ∞-categorical level. That is, it does not matter whether one considers $D(\mathbf{Z})$ or its ∞-categorical enhancement. This subsection, in which we will freely use the theory of ∞-categories following [22], will not be used in the sequel.

Definition 8.4.1. Given an ∞-category $\mathcal{D}$ and objects $X, Y \in \mathcal{D}$ and parallel arrows $X \rightrightarrows Y$, an *equalizer* of the parallel arrows is a limit of the diagram $\Delta^1 \cup_{\partial \Delta^1} \Delta^1 \to \mathcal{D}$.

Example 8.4.2. Suppose given a functor $F : B\mathbf{Z}_{\geq 0} \to \mathcal{D}$, classifying an object $X \in \mathcal{D}$ together with an endomorphism $f : X \to X$. Then $\varprojlim F$ can be identified with an equalizer of the pair $(f, \mathrm{id}_X) : X \rightrightarrows X$. In fact, the map (of simplicial sets) $\Delta^1 \cup_{\partial \Delta^1} \Delta^1 \to B\mathbf{Z}_{\geq 0}$ is both left and right cofinal, as one sees easily using the ∞-categorical version of Quillen's Theorem A [22, Th. 4.1.3.1].

Example 8.4.3. Given spectra X, Y and maps $f, g : X \rightrightarrows Y$, the equalizer $\mathrm{eq}(f, g)$ can be identified with the fiber $\mathrm{fib}(f - g)$.

Definition 8.4.4. Let $\mathcal{C}$ be an ∞-category and let $F : \mathcal{C} \to \mathcal{C}$ be a functor. We define the *fixed points* $\mathcal{C}^F$ as the homotopy fixed points of the $\mathbf{Z}_{\geq 0}$-action on $\mathcal{C}$ thus induced. We can describe it in any of the following manners:

(1) $\mathcal{C}^F$ is the equalizer of the pair $(F, \mathrm{id}_{\mathcal{C}}) : \mathcal{C} \rightrightarrows \mathcal{C})$ in the ∞-category of ∞-categories.
(2) Using the Grothendieck construction [22, Ch. 2], we can identify the pair $(\mathcal{C}, \mathcal{F})$ with a cocartesian fibration $\widetilde{\mathcal{C}} \to \Delta^1/\partial\Delta^0$, and we consider the ∞-category of cocartesian sections of this fibration.

By construction, an object of $\mathcal{C}^F$ is represented by a pair (X, φ_X) where $X \in \mathcal{C}$ and $\varphi_X : X \simeq FX$. In addition, we can describe the space of maps between two objects $(X, \varphi_X), (Y, \varphi_Y)$ via the homotopy equalizer

$$\mathrm{Hom}_{\mathcal{C}^F}((X, \varphi_X), (Y, \varphi_Y)) = \mathrm{eq}(\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X, FY))$$

where the two maps of spaces $\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X, FY)$ are given by $f \mapsto \varphi_Y \circ f$ and $f \mapsto F(f) \circ \varphi_X$.

Definition 8.4.5. We let $\mathcal{D}(\mathbf{Z})$ be the natural ∞-categorical enhancement of $D(\mathbf{Z})$. Namely, $\mathcal{D}(\mathbf{Z})$ is defined in one of two ways:

(1) $\mathcal{D}(\mathbf{Z})$ is the ∞-categorical localization of the category $\mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z})$ of torsion-free cochain complexes at the quasi-isomorphisms.
(2) $\mathcal{D}(\mathbf{Z})$ is defined as an appropriate homotopy coherent nerve [?] of the *differential graded category* of cochain complexes of free abelian groups.

Similarly, we let $\widehat{\mathcal{D}(\mathbf{Z})}_p \subset \mathcal{D}(\mathbf{Z})$ be the full subcategory spanned by the p-complete objects.

Definition 8.4.6. The functor $\eta_p : \mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z}) \to \mathrm{CoCh}^{\mathrm{t.fr}}(\mathbf{Z})$ respects quasi-isomorphisms, so it induces a functor $L\eta_p : \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$ (which lifts the functor $D(\mathbf{Z}) \to D(\mathbf{Z})$).

Our goal is to calculate the ∞-categorical fixed points $\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$. We first study the action of $L\eta_p$ on $\mathcal{D}(\mathbf{Z})$ further. The functor $L\eta_p$ is an additive functor on $\mathcal{D}(\mathbf{Z})$, but it is not exact. In fact, its failure to commute with suspension is exactly multiplication by p, which leads precisely to the aforementioned discreteness. We unwind this next. In order to do this, it will be helpful and concrete to view $\mathcal{D}(\mathbf{Z})$ as arising from a differential graded category, rather than as a localization.

Definition 8.4.7. We let $\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})$ be the *differential graded category* whose objects are the cochain complexes of free abelian groups. Given X^*, Y^*, we define the cochain complex $\mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(X^*, Y^*)$ of maps $X^* \to Y^*$ such that

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(X^*, Y^*)^r = \begin{cases} 0 & r > 0 \\ \mathrm{Hom}_{\mathrm{CoCh}(\mathbf{Z})^{\mathrm{free}}}(X^*, Y^*) & r = 0 \\ \prod_{n \in \mathbf{Z}} \mathrm{Hom}(X^n, Y^{n+r}) & r < 0 \end{cases}.$$

The differential graded nerve of the differential graded category $\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})$ is equivalent to $\mathcal{D}(\mathbf{Z})$. This is slightly different from the usual differential graded category structure on cochain complexes, since we have truncated the cochain complexes of maps. However, the underlying ∞-categories only depend on the connective covers of the mapping complexes.

Construction 8.4.8. We have a functor $\eta_p : \mathcal{D}_{\mathrm{diff}}(\mathbf{Z}) \to \mathcal{D}_{\mathrm{diff}}(\mathbf{Z})$ of differential graded categories defined as follows:

(1) On objects, it sends $X^* \mapsto \eta_p X^*$.
(2) On morphisms, we have a map of cochain complexes $\mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$ which is zero in degrees > 0. In degrees $r > 0$, given a system of maps $f_i : X^i \to Y^{i+r}$, we obtain maps $f_i[1/p] : X^i[1/p] \to Y^{i+r}[1/p]$ and which carry $\eta_p X^i$ into $p^i Y^i \subset \eta_p Y^{i+r}$, thereby obtain the system of maps $\eta_p f_i : X^i \to Y^{i+r}$. In degree $r = 0$, the same construction works for a map of complexes.

On underlying ∞-categories, this produces the functor $L\eta_p : \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$, thanks to the universal property to $\mathcal{D}(\mathbf{Z})$ as a localization.

The crucial feature of η_p for the present section is the following divisibility property.

Proposition 8.4.9. *Let X^*, Y^* be cochain complexes of free abelian groups. Then, for $r \leq 0$, the map between r-cochains*

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(X^*, Y^*)^r \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)^r$$

is divisible by $p^{-(r+1)}$. Furthermore, the map between r-cocycles

$$Z^r \mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(X^*, Y^*) \to Z^r \mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$$

is divisible by p^{-r}. Therefore, the map on rth cohomology

$$H^r \mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(X^*, Y^*) \to H^r \mathrm{Hom}_{\mathcal{D}_{\mathrm{diff}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$$

is divisible by p^{-r}.

Proof. An r-cocycle is given by a system of maps $f_n : X^n \to Y^{n+r}$ such that $df_n = (-1)^r f_n d$. It follows that $f_n[1/p] : X^n[1/p] \to Y^{n+r}[1/p]$ carries $\eta_p X^n$ into $p^{-r}\eta_p Y^{n+r}$. In particular, we have a system of maps $\{p^r f_n : \eta_p X^n \to \eta_p Y^{n+r}\}$. This proves the desired divisibility of the map between r-cocycles. The analogous (but slightly weaker) statement for cochains is similar. Together, the first two statements imply the third. □

Proposition 8.4.10. *Let $X, Y \in \mathcal{D}(\mathbf{Z})$. Then, for $i \geq 0$, the map*

$$\pi_i \mathrm{Hom}_{\mathcal{D}(\mathbf{Z})}(X, Y) \to \pi_i \mathrm{Hom}_{\mathcal{D}(\mathbf{Z})}(L\eta_p X, L\eta_p Y)$$

is divisible by p^i.

Theorem 8.4.11. *The fixed points $\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$ are equivalent to a 1-category. That is, the mapping spaces in $\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$ are homotopy discrete. Furthermore, the natural functor induces an equivalence*

$$(\widehat{\mathcal{D}(\mathbf{Z})}_p)^{L\eta_p} \simeq (\widehat{D(\mathbf{Z})_p})^{L\eta_p} \simeq \mathbf{KD}_{\mathrm{str}}.$$

Proof. Fix (X, φ_X) and $(Y, \varphi_Y) \in \widehat{D(\mathbf{Z})_p}^{L\eta_p}$, where $\varphi_X : X \simeq L\eta_p X$ and $\varphi_Y : Y \simeq L\eta_p Y$. We then see that the space of maps $\mathrm{Hom}_{\widehat{D(\mathbf{Z})_p}^{L\eta_p}}((X, \varphi_X), (Y, \varphi_Y))$ can be computed as the homotopy fiber

$$\mathrm{fib}(f_1 - f_2 : \mathrm{Hom}_{\mathcal{D}(\mathbf{Z})}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(\mathbf{Z})}(X, L\eta_p Y)),$$

where f_1 is given by composition with φ_Y and f_2 is the composite of $L\eta_p$ and precomposition with φ_X. We see that f_1 is an isomorphism. Combining Lemma 8.4.12 below and Lemma 8.4.10, we find that $f_1 - f_2$ induces an isomorphism on positive-dimensional homotopy groups. This implies that the homotopy fiber is homotopy discrete, and on π_0 recovers the kernel of $f_1 - f_2$ on π_0. This implies that $\widehat{\mathcal{D}(\mathbf{Z})_p}^{L\eta_p}$ is equivalent to a 1-category and the first displayed equivalence; meanwhile, the second is Theorem 8.3.11. □

Lemma 8.4.12. *Let A be a derived p-complete abelian group and let $f, g : A \to A$ be maps. Suppose f is an automorphism of A. Then $f + pg$ is an automorphism of A.*

Proof. In fact, since A is derived p-complete, it suffices to show that $f + pg$ induces an automorphism of $A \otimes^L \mathbf{Z}/p\mathbf{Z} \in D(\mathbf{Z})$. However, $f + pg$ induces the same self-map of $A \otimes^L \mathbf{Z}/p\mathbf{Z} \in D(\mathbf{Z})$ as does f, and this is an automorphism. □

Remark 8.4.13. Another formulation of the above argument is that since the functor $L\eta_p : \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$ preserves the zero object, we obtain a natural transformation $\Sigma L\eta_p \to L\eta_p \Sigma X$ for $X \in \mathcal{D}(\mathbf{Z})$. This map is p times a natural equivalence $\Sigma L\eta_p \simeq L\eta_p \Sigma X$, which leads to the above divisibility.

9. Application: Crystalline Comparison for $A\Omega$

The purpose of this section is to give a proof of the crystalline comparison theorem for the $A\Omega$-complexes of [4] using the description of strict Dieudonné complexes in terms of fixed points of $L\eta_p$ on the derived category, as explained in Theorem 8.3.11. Let us first recall the relevant notation from [4, Example 3.16 and Proposition 3.17].

Notation 9.0.14. Let $C/\mathbf{Q}_p$ is a complete nonarchimedean extension with $\mu_{p^\infty} \subset C$. Let $\mathscr{O}_C$ be the valuation ring, $\mathfrak{m} \subset \mathscr{O}_C$ the maximal ideal, and $k = \mathscr{O}_C/\mathfrak{m}$ the residue field. Write $A_{\mathrm{inf}} = W(\mathscr{O}_C^\flat)$ equipped with its Frobenius automorphism φ. Let $\theta : A_{\mathrm{inf}} \to \mathscr{O}_C$ be Fontaine's θ-map, and set $\tilde{\theta} := \theta \circ \varphi^{-1}$. Let $W = W(k) \simeq A_{\mathrm{inf}}(k)$; there is a unique map $A_{\mathrm{inf}} \to W$ lifting the map $A_{\mathrm{inf}} \xrightarrow{\tilde{\theta}} \mathscr{O}_C \to k$, and we refer to this as the canonical map[3]. In particular, any base change functor of the form $- \otimes^L_{A_{\mathrm{inf}}} W$ is along the canonical map.

We also need to specific elements of A_{inf}. Choose a compatible system $\epsilon_{p^n} \in \mu_{p^n}(C)$ of primitive p^n-th roots of unity. This choice defines an element $\underline{\epsilon} \in \mathscr{O}_C^\flat$, and thus an element $\mu = [\underline{\epsilon}] - 1 \in A_{\mathrm{inf}}$, normalized as in [4], satisfying $\varphi^{-1}(\mu) \mid \mu$. Write $\xi = \frac{\mu}{\varphi^{-1}(\mu)}$ and $\tilde{\xi} = \varphi(\xi)$; our normalizations are chosen to ensure that $\xi = \frac{[\underline{\epsilon}]-1}{[\underline{\epsilon}^{\frac{1}{p}}]-1}$ is a generator of $\ker(\theta)$, and thus $\tilde{\xi} = \frac{[\underline{\epsilon}^p]-1}{[\underline{\epsilon}]-1}$ is a generator of $\ker(\tilde{\theta})$. The following two facts shall be useful in the sequel:

- The canonical map $A_{\mathrm{inf}} \to W$ carries $\tilde{\xi}$ to p.
- For $r \geq 1$, the map $\tilde{\theta} : A_{\mathrm{inf}} \to \mathscr{O}_C$ carries $\varphi^{-r}(\mu)$ maps to $\epsilon_{p^r} - 1 \in \mathscr{O}_C$. In particular, this element is nonzero as $r \geq 1$.

Finally, let $\mathfrak{X}$ be a smooth formal $\mathscr{O}_C$-scheme, and write $\mathfrak{X}_k$ for its special fibre. We shall use the complex $A\Omega_{\mathfrak{X}} \in D(\mathfrak{X}, A_{\mathrm{inf}})$ from [4]. Recall that this complex is defined as

$$A\Omega_{\mathfrak{X}} := L\eta_\mu R\nu_* A_{\mathrm{inf},X}$$

where $\nu : X_{proet} \to \mathfrak{X}$ is the nearby cycles map from the generic fibre X of $\mathfrak{X}$. This is a commutative algebra object of $D(\mathfrak{X}, A_{\mathrm{inf}})$, and it comes equipped with a multiplicative isomorphism

$$\widetilde{\varphi}_{\mathfrak{X}} : A\Omega_{\mathfrak{X}} \xrightarrow{\simeq} \varphi_* L\eta_{\tilde{\xi}} A\Omega_{\mathfrak{X}} \tag{14}$$

[3]This choice differs from the usual map $A_{\mathrm{inf}} \to W$ coming from θ by a Frobenius. Our perhaps nonstandard choice is dictated by the prismatic picture: the complex $\mathcal{W}\Omega^*_R$ from Notation 4.1.2 is most naturally identified with $\varphi_* W\Omega^*_R$, i.e., there is an implicit Frobenius twist in identifying the de Rham-Witt complex introduced in this paper with the usual one (though this is invisible in our results as we have restricted ourselves to the absolute theory of $\mathbf{F}_p$). This twist is reflected in a similar twist present in the de Rham comparison theorem for $A\Omega$. Indeed, even though the formulation in [4, Theorem 14.1 (ii)] does not involve a Frobenius twist, this is an error in [4] (pointed out to the first author by T. Koshikawa): there is a missing φ_* on the second displayed isomorphism in the "alternative" proof of Theorem 14.1 (ii) in [4, page 112].

that factors the commutative $A_{\inf}$-algebra map

$$\varphi_{\mathfrak{X}} : A\Omega_{\mathfrak{X}} \to \varphi_* A\Omega_{\mathfrak{X}}$$

induced by $L\eta_\mu$ functoriality from the Frobenius automorphism φ_X of the sheaf $A_{\inf,X}$ on X_{proet}.

The goal of this section is to give a simple proof of the following result from [4, Theorem 1.10 (i)], relating $A\Omega_{\mathfrak{X}}$ to the de Rham-Witt complex $\mathcal{W}\Omega^*_{\mathfrak{X}_k}$.

Theorem 9.0.15. *There is a natural quasi-isomorphism $\mathcal{W}\Omega^*_{\mathfrak{X}_k} \simeq A\Omega_{\mathfrak{X}} \widehat{\otimes}^L_{A_{\inf}} W$ of commutative algebras in $D(\mathfrak{X}, W)$ compatibly with Frobenius.*

There are two constructions of the comparison map given in [4]. Both these constructions entail contemplating deformations of $\mathcal{W}\Omega^*_{\mathfrak{X}_k}$ over much larger subsets of $\mathrm{Spec}(A_{\inf})$ than the locus $\mathrm{Spec}(W) \subset \mathrm{Spec}(A_{\inf})$ where $\mathcal{W}\Omega^*_{\mathfrak{X}_k}$ is supported: the first construction uses mixed characteristic de Rham-Witt complexes that live over $\mathrm{Spec}(W(\mathscr{O}_C)) \to \mathrm{Spec}(A_{\inf})$, while the second entails working over the even larger $\mathrm{Spec}(A_{\mathrm{crys}}) \to \mathrm{Spec}(A_{\inf})$ and using embeddings into infinite dimensional affine spaces. In contrast, the construction of the comparison map we give here uses only intrinsic properties of the complex $A\Omega_{\mathfrak{X}} \widehat{\otimes}^L_{A_{\inf}} W$ to produce the comparison map from $\mathcal{W}\Omega^*_{\mathfrak{X}_k}$, and does not involve deforming outside $\mathrm{Spec}(W) \subset \mathrm{Spec}(A_{\inf})$. Moreover, using the language introduced in this paper, we shall prove a more precise statement matching up the additional structures on either side: one can promote $A\Omega_{\mathfrak{X}} \widehat{\otimes}^L_{A_{\inf}} W$ to a strict Dieudonné algebra that matches up with $\mathcal{W}\Omega^*_{\mathfrak{X}_k}$.

Our construction will be local on $\mathfrak{X}$, so it suffices to give a functorial identification of the (derived) global sections of the two sheaves appearing in Theorem 9.0.15 on a basis of the topology of $\mathfrak{X}$; this reduction enables us to avoid discussing sheaves of strict Dieudonné algebras. A technically convenient basis for the topology is provided by the "small" affine opens, so we assume from now that $\mathfrak{X} = \mathrm{Spf}(R)$ where R is a small formally smooth $\mathscr{O}_C$-algebra R in the sense of [4, Definition 8.5]. In this case, write

$$A\Omega_R := R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}}).$$

This is a commutative algebra in $D(A_{\inf})$ equipped with a Frobenius map

$$\varphi_R : A\Omega_R \to \varphi_* A\Omega_R$$

by functoriality of global sections. Moreover, this map factors over a multiplicative isomorphism

$$\widetilde{\varphi}_R : A\Omega_R \xrightarrow{\simeq} \varphi_* L\eta_{\widetilde{\xi}} A\Omega_R \tag{15}$$

as in (14); this last property is not formal from (14), and a consequence of the smallness of R, see [4, Proposition 9.15] (and the related [2, Remark 7.11]). With this notation, our more precise comparison theorem is the following:

Theorem 9.0.16. *Fix notation as above.*

(1) *After base change along $A_{\inf} \to W$, the map $\widetilde{\varphi}_R$ induces a commutative W-algebra isomorphism*

$$A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W \simeq \varphi_* L\eta_p(A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W).$$

In particular, (ignoring the W-linear structure and thus the φ_* on the right hand side and) applying the equivalence in Theorem 8.3.11, this datum yields a strict Dieudonne complex $A\Omega^*_{R,W} \in \mathbf{KD}_{\mathrm{str}}$ equipped with a commutative algebra structure.

(2) *The strict Dieudonne complex $A\Omega^*_{R,W}$ from (1) is a Dieudonné algebra, and is naturally identified with the de Rham-Witt complex $\mathcal{W}\Omega^*_{R_k}$.*

In particular, one obtains quasi-isomorphisms

$$A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W \simeq A\Omega^*_{R,W} \simeq \mathcal{W}\Omega^*_{R_k}$$

of commutative algebras in $D(W)$ that are compatible with Frobenius.

The rest of this section will be dedicated to the proof of this theorem. We shall first check that base change along $A_{\inf} \to W$ carries $A\Omega_R$ to a strict Dieudonné complex via a base change of the isomorphism in (15). The key assertion here is the following result, which expresses a commutativity of the $L\eta_{\tilde{\xi}}$ operator with base change along $A_{\inf} \to W$.

Lemma 9.0.17. *The natural map $A_{\inf} \to W$ carries $\tilde{\xi}$ to p and induces a quasi-isomorphism*

$$(L\eta_{\tilde{\xi}} A\Omega_R) \widehat{\otimes}^L_{A_{\inf}} W \simeq L\eta_p(A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W)$$

compatibly with Frobenius.

Proof. Note that we can write W as p-adic completion of $\varinjlim_r A_{\inf}/\varphi^{-r}(\mu)$. We shall show that the desired compatibility holds true for each finite r and thus also for the filtered colimit, and then pass it to the completion.

Let us first check that for $r \geq 1$, the natural map $A_{\inf} \to A_{\inf}/\varphi^{-r}(\mu)$ induces a quasiisomorphism

$$(L\eta_{\tilde{\xi}} A\Omega_R) \otimes^L_{A_{\inf}} A_{\inf}/\varphi^{-r}(\mu) \simeq L\eta_{\tilde{\xi}}\big(A\Omega_R \otimes^L_{A_{\inf}} A_{\inf}/\varphi^{-r}(\mu)\big).$$

Via the criterion in [2, Lemma 5.16], it is enough to show that the $\mathscr{O}_C$-modules $H^i(A\Omega_R \otimes^L_{A_{\inf},\tilde{\theta}} \mathscr{O}_C)$ have no $\varphi^{-r}(\mu)$-torsion for all i. As $\varphi^{-r}(\mu)$ maps to the nonzerodivisor $\epsilon_{p^r} - 1 \in \mathscr{O}_C$ under $\tilde{\theta}$, it enough to show that these modules are $\mathscr{O}_C$-flat. Now [4, Theorem 9.2 (i)] (see also [2, Proposition 7.5 and Remark 7.11] for a shorter argument) identifies $A\Omega_R \otimes^L_{A_{\inf},\tilde{\theta}} \mathscr{O}_C$ with $\widetilde{\Omega}_R$. Then [4, Theorem 8.7] identifies $H^i(\widetilde{\Omega}_R)$ with (a Breuil-Kisin twist of) the sheaf $\Omega^i_{R/\mathscr{O}_C}$ of continuous i-forms on R. In particular, each $H^i(\widetilde{\Omega}_R)$ is a finite projective R-module. As R

is flat over $\mathscr{O}_C$, the same holds true for any finite projective R-module, so our claim follows.

Next, as both $L\eta_{\tilde{\xi}}$ and $\otimes^L$ commute with filtered colimits, we get a natural identification

$$(L\eta_{\tilde{\xi}} A\Omega_R) \otimes^L_{A_{\inf}} \varinjlim_r A_{\inf}/\varphi^{-r}(\mu) \simeq L\eta_{\tilde{\xi}}\big(A\Omega_R \otimes^L_{A_{\inf}} \varinjlim_r A_{\inf}/\varphi^{-r}(\mu)\big).$$

The lemma now follows by applying derived p-adic completions to both sides, observing that W is the derived p-adic completion of $\varinjlim_r A_{\inf}/\varphi^{-r}(\mu)$, and that the $L\eta_{\tilde{\xi}}$ functor commutes with derived p-adic completions (see [4, Lemma 6.20]). □

Applying the preceding lemma to the isomorphism $\widetilde{\varphi}_R$ from (15) thus yields a commutative algebra isomorphism

$$A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W \simeq \varphi_* L\eta_p (A\Omega_R \widehat{\otimes}^L_{A_{\inf}} W),$$

which proves first assertion in Theorem 9.0.16 (1). Ignoring the W-linear structure (and thus also the φ_* on the right hand side), this defines an object of the category $\widehat{D(\mathbf{Z}_p)}^{L\eta_p}$ from §8.3. Via Theorem 8.3.11, it strictifies to a strict Dieudonné complex $A\Omega^*_{R,W} \in \mathbf{KD}_{\mathrm{str}}$. As all our constructions are compatible with the multiplicative structure, the complex underlying $A\Omega^*_{R,W}$ is a naturally a cdga. This gives the second assertion in Theorem 8.3.11 (1).

To check that $A\Omega^*_{R,W}$ is a Dieudonné algebra, we use the criterion in Proposition 3.4.2. Unwinding the equivalence in Theorem 8.3.11, and observing that A^0/VA^0 is F-equivariantly identified with $H^0(A^*/p)$ for any $A \in \mathbf{KD}_{\mathrm{str}}$, we are reduced to the following lemma:

Lemma 9.0.18. *The endomorphism of $H^0(A\Omega_R \otimes^L_{A_{\inf}} k)$ induced by φ_R is the Frobenius map.*

Proof. As R is small, we can fix a framing of R which gives rise to a map $R \to R_\infty$ as in [4, Definition 8.5]. In particular, R_∞ is perfectoid, and $R \to R_\infty$ is faithfully flat modulo p. By functoriality, we obtain a φ-equivariant map

$$\eta : A\Omega_R \to A_{\inf}(R_\infty).$$

Now recall that if S is a perfectoid $\mathscr{O}_C$-algebra, then $A_{\inf}(S) \widehat{\otimes}^L_{A_{\inf}} W \simeq W(S_k)$ compatibly with Frobenius. Applying this to $S = R_\infty$, we obtain a φ-equivariant map

$$H^0(\eta \otimes^L_{A_{\inf}} k) : H^0(A\Omega_R \otimes^L_{A_{\inf}} k) \to R_{\infty,k}.$$

As φ acts as the Frobenius on $R_{\infty,k}$, it also acts as the Frobenius on any φ-stable subring. In particular, the lemma would follow if $H^0(\eta \otimes^L_{A_{\inf}} k)$ were injective. To check this injectivity, we use the Hodge-Tate comparison theorem [4, Theorem

9.2 (i), Theorem 9.4(iii)] (see [2, Proposition 7.5 and Remark 7.11] for a simpler argument), which identifies

$$A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathscr{O}_C \simeq \widetilde{\Omega}_R.$$

Via this identification, one can identify $H^0(\eta \otimes^L_{A_{\mathrm{inf}}} k)$ as H^0 of the map

$$\widetilde{\Omega}_R \otimes^L_{\mathscr{O}_C} k \to R_{\infty,k}.$$

But on passage to H^0, this simply gives the natural map $R_k \to R_{\infty,k}$ (using [4, Theorem 8.7] to identify the left hand side). As $R \to R_\infty$ is faithfully flat modulo p, it is also faithfully flat after base change to k, so the injectivity follows. □

Thus, we have strictified $A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W$ to a strict Dieudonné algebra $A\Omega^*_{R,W} \in \mathbf{AD}_{\mathrm{str}}$. We shall complete the proof of Theorem 9.0.16 by identifying this strict Dieudonné algebra with the de Rham-Witt complex $\mathcal{W}\Omega^*_{R_k}$. As explained in the proof above, the Hodge-Tate comparison gives a natural quasi-isomorphism between $A\Omega^*_{R,W}/p$ and $\widetilde{\Omega}_R \otimes^L_{\mathscr{O}_C} k$. In particular, there is a natural map

$$\gamma : R_k \to H^0(A\Omega^*_{R,W}/p)$$

which is an isomorphism. By the universal property of the de Rham-Witt complex resulting from Theorem 4.4.12 and Definition 4.1.1, we obtain a unique map

$$\widetilde{\gamma} : \mathcal{W}\Omega^*_{R_k} \to A\Omega^*_{R,W}$$

of strict Dieudonné algebras such that the composition

$$R_k \xrightarrow{f} H^0(\mathcal{W}\Omega^*_{R_k}/p) \xrightarrow{H^0(\widetilde{\gamma}\otimes^L_W k)} H^0(A\Omega^*_{\mathfrak{X},W}/p)$$

agrees with γ. To finish proving Theorem 9.0.16, it is now enough to check the following:

Lemma 9.0.19. *The map $\widetilde{\gamma}$ is an isomorphism of strict Dieudonné algebras.*

Proof. By Corollary 2.5.5 and strictness of both $\mathcal{W}\Omega^*_{R_k}$ and $A\Omega^*_{R,W}$, it is enough to prove that $\widetilde{\gamma}\otimes^L_{\mathbf{Z}_p}\mathbf{F}_p$ is a quasiisomorphism, i.e., that $H^*(\widetilde{\gamma}\otimes^L_{\mathbf{Z}_p}\mathbf{F}_p)$ is an isomorphism. Via Proposition 4.3.3 for $\mathcal{W}\Omega^*_{R_k} \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$ and the Hodge-Tate isomorphism for $\widetilde{\Omega}_R$ [4, Theorem 8.7] for $A\Omega^*_{R,W} \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$, we can view $H^*(\widetilde{\gamma} \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p)$ as an endomorphism $\overline{\gamma}$ of the graded ring $\Omega^*_{R_k}$ of differential forms on R_k. This endomorphism intertwines the Bockstein differentials coming from the natural $\mathbf{Z}/p^2$-lift $\widetilde{\gamma} \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p^2$ of the map $\widetilde{\gamma} \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$. Now the Bockstein differential on the source of $\overline{\gamma}$ is the de Rham differential by the Cartier isomorphism. By the universal property of the de Rham complex, it is thus enough to prove that the Bockstein differential on $H^*((A\Omega_R\widehat{\otimes}^L_{A_{\mathrm{inf}}}W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p)$ attached to the element $p \in \mathbf{Z}_p$ coincides with the de Rham differential under the Hodge-Tate isomorphism. We shall deduce this compatibility from the de Rham comparison theorem for $A\Omega_R$. As

$\tilde{\xi} \in A_{\mathrm{inf}}$ maps to $p \in W$, there is a natural diagram in $D(A_{\mathrm{inf}})$ comparing the Bockstein constructions for $\tilde{\xi}$ and p:

$$\begin{array}{ccccc} A\Omega_R/\tilde{\xi} \simeq A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathscr{O}_C & \xrightarrow{\tilde{\xi}} & A\Omega/\tilde{\xi}^2 & \longrightarrow & A\Omega_R/\tilde{\xi} \simeq A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathscr{O}_C \\ \downarrow & & \downarrow & & \downarrow \\ A\Omega_R/\tilde{\xi} \otimes^L_{A_{\mathrm{inf}}} W & \xrightarrow{\tilde{\xi}} & A\Omega/\tilde{\xi}^2 \otimes^L_{A_{\mathrm{inf}}} W & \longrightarrow & A\Omega_R/\tilde{\xi} \otimes^L_{A_{\mathrm{inf}}} W \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ (A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p & \xrightarrow{p} & (A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p^2 & \longrightarrow & (A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p \end{array}$$

here all rows are exact triangles, and the second row is obtained via base change from the first one. Comparing boundary maps on cohomology for the first and last row above gives a commutative diagram

$$\begin{array}{ccc} H^i(A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathscr{O}_C) & \xrightarrow{\beta_{\tilde{\xi}}} & H^{i+1}(A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathscr{O}_C) \\ \downarrow & & \downarrow \\ H^i((A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p) & \xrightarrow{\beta_p} & H^{i+1}((A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p). \end{array}$$

Our goal was to identify the bottom horizontal map with the de Rham differential. Via the Hodge-Tate comparison (and ignoring Breuil-Kisin twists), this square identifies with

$$\begin{array}{ccc} \Omega^i_{R/\mathscr{O}_C} & \xrightarrow{\beta_{\tilde{\xi}}} & \Omega^{i+1}_{R/\mathscr{O}_C} \\ \downarrow & & \downarrow \\ \Omega^i_{R_k} \simeq \Omega^i_{R_k/k} & \xrightarrow{\beta_p} & \Omega^{i+1}_{R_k} \simeq \Omega^{i+1}_{R_k/k}. \end{array}$$

As the vertical maps are the natural ones, it is enough to prove that the top horizontal map is the de Rham differential. But this is precisely the calculation that deduces the the de Rham comparison theorem for $A\Omega$ from the Hodge-Tate one (see [2, Proposition 7.9]), so we are done. □

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