To fix the ideas, we will use $\mathbb{Z}$ as the ground ring. So DG categories will be over $\mathbb{Z}$, Lurie's tensor product will be over $\mathbb{Z}$, and dualizability will be over $\mathbb{Z}$. (The latter is, after all, irrelevant because by [GL: DG, Cor. 6.4.2], a DG category over k is dualizable over k if and only if it is dualizable over $\mathbb{Z}$.)

Let X be a locally compact topological space. Let $Sh(X)$ (resp. $\text{Pre-sh}(X)$) denote the DG category of sheaves (resp. pre-sheaves) of $\mathbb{Z}$-modules on X.

<u>Theorem 1.</u> $Sh(X)$ is dualizable. Moreover, the pairing $ev: Sh(X)\otimes Sh(X)\to \mathbb{Z}\text{-mod}$ defined by

$$ev(M,N):=\Gamma_c(X, M\otimes N)$$

induces an equivalence $Sh(X)\xrightarrow{\sim} Sh(X)^\vee$.

It is somewhat suspicious that I do not require X to be a manifold. With a possible exception of this point, the theorem was predicted (and maybe proved) by D. Gaitsgory.

<u>Proof.</u> Let $\pi: \text{Pre-sh}(X)\to Sh(X)$ denote the sheafification functor. Let $\text{Opens}(X)$ denote the category of open subsets of X. The assignment

$$(U,V)\mapsto \Gamma(U\cap V, \mathbb{Z}), \quad U,V\in \text{Opens}(X)$$

defines a presheaf of complexes on $\text{Opens}(X)\times\text{Opens}(X)$, which is the same as a functor

$$\text{pre-coev}: \mathbb{Z}\text{-mod}\to \text{Pre-sh}(X)\otimes\text{Pre-sh}(X).$$

Composing it with $\pi\otimes\pi: \text{Pre-sh}(X)\otimes\text{Pre-sh}(X)\to Sh(X)\otimes Sh(X)$

We get a functor

$$\mathrm{coev}: \mathbb{Z}\text{-mod} \to Sh(X) \otimes Sh(X)$$

It remains to check that ev and coev satisfy the required two conditions. In fact, since ev and coev are symmetric there is only one condition to check. It says that the composition

$$\begin{array}{c} S(X) \xrightarrow{Id \otimes \text{pre-coev}} Sh(X) \otimes Pre\text{-}sh(X) \otimes Pre\text{-}sh(X) \\ \downarrow Id \otimes \pi \otimes Id \\ Sh(X) \otimes Sh(X) \otimes Pre\text{-}sh(X) \\ \downarrow ev \otimes Id \\ Pre\text{-}sh(X) \xrightarrow{\pi} Sh(X) \end{array}$$

is isomorphic to $Id_{Sh(X)}$. Denote this composition by Φ. Explicitly, $\Phi = \pi \circ s$, where $s: Sh(X) \to Pre\text{-}sh(X)$ takes $\mathcal{F} \in Sh(X)$ to the presheaf

$$U \mapsto ev(\mathcal{F}, (j_U)_* \mathbb{Z}_U);$$

here $j_U: U \hookrightarrow X$ is the embedding.

Let us define a morphism $\Phi \to Id_{Sh(X)}$. This amounts to defining a morphism

$$\Gamma_c(X, \mathcal{F} \otimes (j_U)_* \mathbb{Z}_U) \to \Gamma(U, \mathcal{F})$$

functorial in $\mathcal{F}$ and U. We define it to be the morphism

$$\Gamma_c(X, \mathcal{F} \otimes (j_U)_* \mathbb{Z}_U) \to \Gamma(X, (j_U)_* j_U^* \mathcal{F}) = \Gamma(U, \mathcal{F})$$

induced by the canonical morphisms $\Gamma_c \to \Gamma$ and $\mathcal{F} \otimes (j_U)_* \mathbb{Z}_U \to (j_U)_* j_U^* \mathcal{F}$.

Now let us check that our morphism $\phi(\mathcal{F}) \to \mathcal{F}$ is an isomorphism. This can be done stalkwise. Let $x \in X$. Then

$$\phi(\mathcal{F})_x = \operatorname*{colim}_{U \ni x} ev(\mathcal{F}, (j_U)_* \mathbb{Z}_U) = ev(\mathcal{F}, g_x),$$

where $g_x := \operatorname*{colim}_{U \ni x} (j_U)_* \mathbb{Z}_U$. Clearly $g_x = (i_x)_* \mathbb{Z}$, where $i_x : \{x\} \to X$ is the embedding. So $\phi(\mathcal{F})_x = ev(\mathcal{F}, (i_x)_* \mathbb{Z}) = \mathcal{F}_x$, and our morphism $\phi(\mathcal{F})_x \to \mathcal{F}_x$ can be regarded as a morphism $\mathcal{F}_x \to \mathcal{F}_x$. One checks that it is the identity morphism. ■

Remark. The functors $Sh(X) \xrightarrow{S} Pre\text{-}sh(X) \xrightarrow{\pi} Sh(X)$ from the above proof exhibit $Sh(X)$ as a retract of $Pre\text{-}sh(X)$.

Compact objects of $Sh(X)$.

As before, let X be a locally compact topological space.

Proposition 2. An object $\mathcal{F} \in Sh(X)$ is compact if and only if it satisfies the following conditions:

(i) $\mathcal{F}$ is locally constant;

(ii) $\mathcal{F}$ has compact support;

(iii) all stalks of $\mathcal{F}$ are compact objects of $\mathbb{Z}$-mod.

The proposition implies that $Sh(X)$ is not compactly generated unless X is totally disconnected.

Remark. Proposition 2 is a variant of Theorem 0.1 from A. Neeman's work "On the derived category of sheaves on a manifold". (In that work he assumes that X is a noncompact disconnected manifold of positive dimension.)

Proof of Proposition 2. The "if" statement is easy. Indeed, (i) and (iii) imply that

$$Maps(\mathcal{F}, \mathcal{G}) = \Gamma(X, \mathcal{F}' \otimes \mathcal{G}), \quad \mathcal{F}' := \mathcal{H}om(\mathcal{F}, \mathbb{Z}_X),$$

and the functor $\mathcal{G} \mapsto \Gamma(X, \mathcal{F}' \otimes \mathcal{G})$ is continuous by (ii).

Let us prove the "only if" statement. Assume that $\mathcal{F} \in Sh(X)$ is compact.

Let $x \in X$ and $i_x : \{x\} \to X$ be the embedding. The functor i_x^* has a continuous right adjoint, so the object $\mathcal{F}_x = i_x^* \mathcal{F} \in \mathbb{Z}$-mod is compact. This proves (iii).

Set $S_{\mathcal{F}} := \{x \in X \mid \mathcal{F}_x \neq 0\}$. Using an argument from A. Neeman's article we will show that $S_{\mathcal{F}}$ is compact (this is slightly stronger than statement (ii), which says that $\overline{S_{\mathcal{F}}}$ is compact). We have to show that if $\{U_\alpha\}$ is a directed family of open subsets of X that covers $S_{\mathcal{F}}$ then $U_{\alpha_0} \supset S_{\mathcal{F}}$ for some α_0. Note that $\mathcal{F} = \operatorname{colim}_\alpha \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha := (j_\alpha)_! j_\alpha^! \mathcal{F}$ and $j_\alpha : U_\alpha \hookrightarrow X$ is the embedding. So compactness of $\mathcal{F}$ implies that $\mathcal{F}$ is a direct summand of some $\mathcal{F}_{\alpha_0}$. Then $S_{\mathcal{F}} \subset U_{\alpha_0}$.

Now let us prove (i). We have to show that any $x \in X$ has a neighborhood V such that $\mathcal{F}|_V$ is constant. Let $i_x : \{x\} \hookrightarrow X$ be the embedding. For each open $U \subset X$ containing x let $j_U : U \hookrightarrow X$ be the corresponding embedding. Then

$$(i_x)_* \mathcal{F}_x = \operatorname*{colim}_U \pi_U^* \mathcal{F}_x ,$$

where U runs through the set of all open subsets of X containing x and $\pi_U : U \to \{x\}$. Since $\mathcal{F}$ is compact the canonical morphism $\mathcal{F} \to (i_x)_* \mathcal{F}_x$ factors as $\mathcal{F} \to (j_U)_* \pi_U^* \mathcal{F}_x \to (i_x)_* \mathcal{F}_x$ for some U. Set $\mathcal{E} := \operatorname{Cone}(\mathcal{F} \to (j_U)_* \pi_U^* \mathcal{F}_x)$. Since $\mathcal{E}$ is compact so is the set $S_{\mathcal{E}} := \{x' \in X \mid \mathcal{E}_{x'} \neq 0\}$. Thus the set $V := U \setminus S_{\mathcal{E}}$ is a neighborhood of x such that $\mathcal{F}|_V$ is constant. ■