[On dualizable DG categories. II]

<u>Proposition II-1.</u> Let $\mathcal{D}$ be a cocomplete DG category and C a compactly generated one. Suppose we have continuous functors $\mathcal{D} \xrightarrow{s} C \xrightarrow{\pi} \mathcal{D}$ and an isomorphism $\pi \circ s \simeq \mathrm{Id}_{\mathcal{D}}$. In addition, assume that the restriction of π to C^c is fully faithful. Then π is a localization functor.

The proof will be based on the following lemma, which seems to be interesting in its own right.

<u>Lemma II-2.</u> In the situation of Proposition II-1 the map

$$\mathrm{Maps}(s\pi, \mathrm{Id}_C) \xrightarrow{\pi} \mathrm{Maps}(\pi s \pi, \pi) = \mathrm{Maps}(\pi, \pi)$$

is an equivalence.

<u>Proof.</u> A continuous functor $C \to C$ (resp. $C \to \mathcal{D}$) is the same as a functor $C^c \to C$ (resp. $C^c \to \mathcal{D}$). So $\mathrm{Maps}(s\pi, \mathrm{Id}_C)$ is the "end" of the bifunctor

$$(c, c') \mapsto \mathrm{Maps}(s \circ \pi(c), c'), \quad c, c' \in C^c.$$

Similarly, $\mathrm{Maps}(\pi s \pi, \pi)$ is the "end" of the bifunctor

$$(c, c') \mapsto \mathrm{Maps}(\pi \circ s \circ \pi(c), \pi(c')), \quad c, c' \in C^c.$$

So it suffices to show that for any $c, c' \in C^c$ the map

$$\mathrm{Maps}(s \circ \pi(c), c') \xrightarrow{\pi} \mathrm{Maps}(\pi \circ s \circ \pi(c), \pi(c'))$$

is an equivalence. In fact, for any $c'' \in C$, $c' \in C^c$ the map

$$\mathrm{Maps}(c'', c') \xrightarrow{\pi} \mathrm{Maps}(\pi(c''), \pi(c'))$$

is an equivalence. Indeed, if $c'' \in C^c$ this is just fully faithfulness of $\pi|_{C^c}$, and the general case follows. ■

Now let $\varepsilon \in \mathrm{Maps}(s\pi, \mathrm{Id}_C)$ be the preimage of $\mathrm{id}_\pi \in \mathrm{Maps}(\pi,\pi)$ with respect to the map $\mathrm{Maps}(s\pi, \mathrm{Id}_C) \to \mathrm{Maps}(\pi,\pi)$ from Lemma II-2. In other words, the composition

$$\pi \xrightarrow{\sim} \pi\circ s\circ\pi \xrightarrow{\mathrm{id}\circ\varepsilon} \pi$$

equals the identity.

<u>Remark.</u> I do <u>not</u> claim that the composition

$$s \to s\circ\pi\circ s \to s$$

is isomorphic to the identity. If this were true then s would be left adjoint to π. For a given $\pi: C \to \mathcal{D}$ there is at most one such s; on the other hand, there is no reason to believe that the condition $\pi\circ s \simeq \mathrm{Id}_{\mathcal{D}}$ from Proposition II-1 determines s uniquely.

<u>Remark.</u> Since $\pi\circ s \simeq \mathrm{Id}_{\mathcal{D}}$ the functor $s\circ\pi$ is an idempotent coalgebra in the monoidal category $\mathrm{Funct}_{cont}(C,C)$. Our $\varepsilon \in \mathrm{Maps}(s\pi, \mathrm{Id})$ is a <u>right</u> counit of this coalgebra: indeed, the composition

$$s\circ\pi \xrightarrow{\sim} s\circ\pi\circ s\circ\pi \xrightarrow{\mathrm{id}\circ\varepsilon} s\circ\pi$$

equals $\mathrm{id}_{s\circ\pi}$ because the composition $\pi \xrightarrow{\mathrm{id}\circ\varepsilon} \pi\circ s\circ\pi \xrightarrow{\sim} \pi$ equals id_π. Note that the coalgebra $s\circ\pi$ is counital (i.e., a comonad) if and only if s is left adjoint to π.

Proof of Proposition II-1. Let $\pi^*: \mathcal{D} \to \mathcal{C}$ and $s^*: \mathcal{C} \to \mathcal{D}$ be the (discontinuous) right adjoints of $\pi: \mathcal{C} \to \mathcal{D}$ and $s: \mathcal{D} \to \mathcal{C}$. We have an isomorphism $s^* \circ \pi^* \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$ and a morphism $\varepsilon^*: \mathrm{Id}_{\mathcal{C}} \to \pi^* \circ s^*$ such that the composition

$$\pi^* \xrightarrow{\varepsilon^* \circ \pi^*} \pi^* \circ s^* \circ \pi^* \xrightarrow{\sim} \pi^*$$

equals Id_{π^*}. This formally implies that π^* induces an equivalence $\mathcal{D} \xrightarrow{\sim} \mathcal{C}' \subset \mathcal{C}$, where $\mathcal{C}'$ is the full subcategory of those $c \in \mathcal{C}$ for which the morphism $(\varepsilon^*)_c : c \to \pi^* s^*(c)$ is an isomorphism (the inverse functor $\mathcal{C}' \to \mathcal{D}$ is the restriction of $s^*: \mathcal{C} \to \mathcal{D}$). Thus π^* is fully faithful, Q.E.D. ∎

Remark. Define $\Phi: \mathcal{C} \to \mathcal{C}$ by $\Phi := \mathrm{Cone}(s\pi \xrightarrow{\varepsilon} \mathrm{Id}_{\mathcal{C}})$. It is easy to see that $\mathrm{Ker}\,\pi$ is equal to $\Phi(\mathcal{C})$, i.e., to the essential image of Φ (so the subcategory $\Phi(\mathcal{C}) \subset \mathcal{C}$ is cocomplete). Thus

$$\mathcal{D} = \mathcal{C}/\mathrm{Ker}\,\pi = \mathcal{C}/\Phi(\mathcal{C}) = \mathrm{Coker}\,\Phi.$$

Corollary II-3. Suppose that a cocomplete DG category $\mathcal{D}$ is a retract of a compactly generated one. Then $\mathcal{D}$ is presentable (and therefore dualizable by Theorem 1 from part I of these notes).

Proof. We have

$$\mathcal{D} \xrightarrow{s} \mathcal{C} \xrightarrow{\pi} \mathcal{D}, \quad \pi \circ s \simeq \mathrm{Id}_{\mathcal{D}},$$

where $\mathcal{C}$ is compactly generated and π, s are continuous. Without loss of generality, we can assume that the restriction of π to C^c is fully faithful (otherwise replace C by $\mathrm{Ind}(\mathcal{A})$, where $\mathcal{A}\subset\mathcal{D}$ is a small triangulated full subcategory such that $\pi(C^c)\subset\mathcal{A}$). Then the previous Remark yields a "presentation" $\mathcal{D}=\mathrm{Coker}(C\xrightarrow{\Phi}C)$, which shows that $\mathcal{D}$ is presentable. ∎

Remark. Presentability of $\mathcal{D}$ (without a concrete "presentation" as a cokernel of a continuous functor between compactly generated DG categories) follows from the very general fact that a retract of a presentable category $\mathcal{C}$ is presentable. Indeed, such a retract is the limit (a priori, in $\mathcal{DG}Cat$) of a certain functor $\mathrm{Idem}\to\mathcal{DG}\,Cat_{cont}$ (here Idem is the ∞-category from §4.4.5 of "Higher Topos Theory" and the functor $\mathrm{Idem}\to\mathcal{DG}Cat_{cont}$ takes the unique object of Idem to $\mathcal{C}\in\mathcal{DG}Cat_{cont}$). Such a limit belongs to $\mathcal{DG}\,Cat_{cont}$. (The above argument also works for not necessarily stable ∞-categories instead of DG categories.)

Question 1. Do Proposition II-1 and Lemma II-2 hold for not necessarily stable ∞-categories (instead of DG categories)?

Another proof of Proposition 1 (which uses more familiar "stable" arguments). Set $\bar{C} := C/\mathrm{Ker}\,\pi$. Let $\bar{\pi}: \bar{C} \to \mathcal{D}$ be induced by $\pi: C \to \mathcal{D}$. Let $\bar{s}: \mathcal{D} \to \bar{C}$ be the composition $\mathcal{D} \xrightarrow{s} C \to \bar{C}$. We have $\bar{\pi} \circ \bar{s} \simeq Id_{\mathcal{D}}$. It remains to show that $\bar{s} \circ \bar{\pi} \simeq Id_{\bar{C}}$. The cone of the morphism $\varepsilon: s\pi \to Id_C$ maps C to $\mathrm{Ker}\,\pi$, so ε induces an isomorphism $\bar{s} \circ \bar{\pi} \simeq Id_{\bar{C}}$. ∎

Question 2. Let $\mathcal{D}$ be a dualizable DG category. Does there exist a diagram

$$C \underset{s}{\overset{\pi}{\rightleftarrows}} \mathcal{D}, \quad \pi \circ s \simeq Id_{\mathcal{D}}$$

with C compactly generated and π, s continuous such that <u>s is left adjoint to π</u> with the isomorphism $Id_{\mathcal{D}} \xrightarrow{\sim} \pi \circ s$ being the adjunction?
(In this case $\mathcal{D}$ is a colocalization of C.)

Example. The answer to Question 2 is "yes" for $\mathcal{D} = QCoh(\mathcal{Y})$, where $\mathcal{Y}$ is an eventually coconnected QCA stack locally almost of finite type over a field of characteristic 0; moreover, one can take $C = IndCoh(\mathcal{Y})$. This is explained in §4.3 of the article "On some finiteness questions..." by D. Gaitsgory and me.

Remark. If π has a continuous <u>right</u> adjoint then π preserves compactness, so $\mathcal{D}$ has to be compactly generated. This is why Question 2 is about <u>left</u> adjoints.

Reformulation of Question 2. Let $\mathcal{D}$ be a dualizable DG category. Can one represent $\mathcal{D}$ as a retract $C \underset{s}{\overset{\pi}{\rightleftarrows}} \mathcal{D}$ of a compactly generated DG category C so that the maps

(1) $\mathrm{Maps}(s\pi, \mathrm{Id}_C) \xrightarrow{f \mapsto \mathrm{id}_\pi \circ f} \mathrm{Maps}(\pi s \pi, \pi) = \mathrm{Maps}(\pi, \pi)$

(2) $\mathrm{Maps}(s\pi, \mathrm{Id}_C) \xrightarrow{f \mapsto f \circ \mathrm{id}_s} \mathrm{Maps}(s\pi s, s) = \mathrm{Maps}(s, s)$

are equivalences?

Proposition II-4. Let $\mathcal{D}$ be a dualizable DG category. Then $\mathcal{D}$ admits a representation as a retract of a compactly generated DG category C such that the map (1) is an equivalence. It also admits (another!) representation as a retract of a compactly generated DG category such that the map (2) is an equivalence.

Proof. (a) As mentioned in the proof of Corollary II-3, $\mathcal{D}$ can be represented as a retract of a compactly generated DG category C so that the restriction of $\pi : C \to \mathcal{D}$ to C^c is fully faithful. Then the map (1) is an equivalence by Lemma II-2.

(b) To ensure that the map (2) is an equivalence, apply the previous construction to the Lurie dual $\mathcal{D}^\vee$, i.e., represent it as a retract $\tilde{C} \underset{\tilde{s}}{\overset{\tilde{\pi}}{\rightleftarrows}} \mathcal{D}^\vee$ of a compactly generated DG category $\tilde{C}$ so that the corresponding map (1) is an equivalence. Then the diagram $\tilde{C}^\vee \underset{\tilde{\pi}^\vee}{\overset{\tilde{s}^\vee}{\rightleftarrows}} \mathcal{D}$ represents $\mathcal{D}$ as a retract so that the corresponding map (2) is an equivalence. ∎