The functor $\mathrm{Lie}: \{\text{smooth groupoids on } X\} \to \left\{\begin{matrix}\text{locally free Lie}\\ \text{algebroids on } X\end{matrix}\right\}$

X/k, ~~of~~ locally of finite type (for safety).

~~Def. A Lie algebroid on X~~ This functor is mentioned in ~~Book~~ some books by diff. geometers (e.g., the Crainic-Fernandes e-print), but not in the books on algebraic stacks.

Def. A Lie algebroid on X is a sheaf $\mathfrak{a}$ on X equipped with compatible structures of $\mathcal{O}_X$-module, Lie k-algebra and anchor map $\mathfrak{a} \xrightarrow{\tau} \Theta_X$ (typical example: $\mathfrak{a} = \Theta_X$, $\tau = \mathrm{id}$), where compatibility means ~~that identity~~ τ is a morphism of $\mathcal{O}_X$-modules and Lie k-algebras, and for $v_1, v_2 \in \mathfrak{a}$, $f \in \mathcal{O}_X$ one has $[v_1, f v_2] = f[v_1, v_2] + \tau_{v_1}(f) v_2$.

E.g., a Lie algebroid with $\tau = 0$ is the same as a Lie $\mathcal{O}_X$-algebra.

Def. $\mathfrak{a}$ is locally free if $\mathfrak{a}$ is a locally free coherent $\mathcal{O}_X$-module.

Before defining the functor Lie, let me give two examples:

Example 1. Let $\Gamma = X \times X$; this is a groupoid acting on X; it is smooth if X is smooth. Then $\mathrm{Lie}\,\Gamma = \Theta_X$.

Example 2. If $p_1 = p_2$ then Γ is a smooth group scheme over X and $\mathrm{Lie}\,\Gamma$ is its Lie algebra (with $\tau = 0$).

Now let $\Gamma \rightrightarrows X$ be any smooth groupoid on X (we don't require X to be smooth). Have $e: X \hookrightarrow \Gamma$ (the identity).

Def. $\mathrm{Lie}\,\Gamma$ is the normal bundle of $X \overset{e}{\hookrightarrow} \Gamma$.

Γ smooth $\Rightarrow$ $\mathrm{Lie}\,\Gamma$ is a locally free coherent $\mathcal{O}_X$-module.

The anchor map $\mathrm{Lie}\,\Gamma \xrightarrow{\tau} \Theta_X = \{\text{normal bundle of } X \overset{\text{diag}}{\hookrightarrow} X \times X\}$ comes from the diagram

$$\begin{tikzcd} X \arrow[r, hook, "e"] \arrow[dr, hook] & \Gamma \arrow[d] \\ & X \times X \end{tikzcd}$$

How to define the commutator in $\mathrm{Lie}\,\Gamma$?

The commutator in Θ_X comes from the fact that $H^0(X, \Theta_X) = \mathrm{Lie}\,\mathcal{D}iff(X)$, where $\mathcal{D}iff(X)$ is the formal group of diffeomorphisms, i.e., the functor $A \mapsto \{\varphi \in \mathrm{Aut}(X \otimes A) \mid \varphi \bmod m \text{ is trivial}\}$.

local Artinian k-algebra.

For $A = k[\varepsilon]/(\varepsilon^2)$ get Θ_X.

Now replace the formal group of diffeomorphisms by the functor F_Γ, where $F_\Gamma(A) = \left\{ \begin{array}{l} \text{subschemes } \Sigma \subset \Gamma \otimes A \text{ flat over } A \text{ such that} \\ \Sigma \text{ mod } m_A \text{ equals } e(X) \subset \Gamma \end{array} \right\}$.

In other words, $F(A) = \{(\operatorname{Spec} A)\text{-deformations of the subscheme } e(X) \subset \Gamma\}$.

Note: $p_1, p_2 : \Sigma \to X \otimes A$ are isomorphisms (flatness plus the fact that is true modulo m_A).

The group structure on $F(A)$ comes from $\Gamma \times_X \Gamma \xrightarrow{c} \Gamma$.

(This is a "set-theoretical" construction. Let $\Gamma \rightrightarrows X$ be a groupoid in Sets. A subset $\Sigma \subset \Gamma$ such that both projections $\Sigma \rightrightarrows X$ are bijective is the same as a bijection $\sigma : X \xrightarrow{\sim} X$ plus an isomorphism $x \xrightarrow{\sim} \sigma(x)$ for every $x \in X$. Now the group structure on the set of all such subsets $\Sigma \subset \Gamma$ is clear.)

(Differential geometers prefer to define Lie Γ using "right-invariant" vector fields, see Crainic-Fernandes, arXiv:0611259)

Recovering Γ from Lie Γ is not automatic (even in characteristic 0). But one can use Lie algebroids as a <u>heuristic tool</u> to construct some groupoids in {Schemes}. I will give some examples.

<u>Exercise.</u> Let $\mathfrak{A}$ be a locally free Lie algebroid on X. Let $\mathcal{D} \subset X$ be an (effective) Cartier divisor. Then $\mathfrak{A}(-\mathcal{D}) \subset \mathfrak{A}$ is a subalgebroid (obviously, locally free).

E.g., if (X is smooth and) $\mathfrak{A} = \Theta_X$ then $\Theta_X(-\mathcal{D}) = \left\{ \begin{array}{c} \text{vector fields} \\ \text{vanishing on } \mathcal{D} \end{array} \right\}$ is the Lie algebra of the (formal) group of those diffeomorphisms of X whose restriction to $\mathcal{D}$ equals $\mathrm{id}_{\mathcal{D}}$.

<u>Question.</u> Is there a similar construction for smooth groupoids? Given $\Gamma \rightrightarrows X$ and a Cartier divisor $\mathcal{D} \subset X$ want $\begin{array}{ccc} \Gamma' & \to & \Gamma \\ \downarrow\downarrow & \swarrow\swarrow & \\ X & & \end{array}$ with Lie $\Gamma' = (\text{Lie } \Gamma)(-\mathcal{D})$? Yes, there is!!

Before defining Γ', let me discuss an example.

<u>Example.</u> $X = \mathbb{A}^1$, $\mathcal{D} = \{0\} \subset \mathbb{A}^1$, $\Gamma = X \times X$. Then Lie $\Gamma = \Theta_{\mathbb{A}^1}$, $(\text{Lie } \Gamma)(-\mathcal{D})$ is generated by $x \frac{d}{dx}$. $x \frac{d}{dx}$ is the generator of the $\mathbb{G}_m$-action on $\mathbb{A}^1$. Guess: $\Gamma' = \mathbb{G}_m \times \mathbb{A}^1 \underset{\text{mult}}{\overset{\text{pr}}{\rightrightarrows}} \mathbb{A}^1$ (groupoid associated to the action of $\mathbb{G}_m$ on $\mathbb{A}^1$). This approach woudn't work for $3\mathcal{D}$ instead of $\mathcal{D}$ (because $x^3 \frac{d}{dx}$ is not a generator of an algebraic group action). But there is a way to look at the above Γ' that works in the general case (e.g., $3\mathcal{D}$ instead of $\mathcal{D}$). Namely, look at the map

$\Gamma' \to X \times X = \mathbb{A}^2$. It is birational (a kind of affine blow-up).
$(\lambda, x) \mapsto (x, \underset{y}{\underset{\shortparallel}{\lambda x}})$ Draw the picture. $\Gamma' = \operatorname{Spec} k[x, \lambda, \lambda^{-1}] = \operatorname{Spec} k[x, \frac{y}{x}, \frac{x}{y}]$.
$\widehat{\mathbb{A}}^2 :=$ blow-up of $\mathbb{A}^2$ at $(0,0)$. Then $\Gamma' = \widehat{\mathbb{A}}^2 \setminus \left\{ \begin{matrix} \text{strict transforms of} \\ \text{the coordinate axes} \end{matrix} \right\}$.
Fiber of Γ' over $(0,0)$: $\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m$ (as it should be).

<u>Definition of $\Gamma' \to \Gamma$ for any smooth groupoid $\Gamma \rightrightarrows X$ and any $\mathcal{D}$.</u>

Have
$$\begin{matrix} \Gamma & \supset & p_1^{-1}(\mathcal{D}) \\ \cup & & \cup \\ p_2^{-1}(\mathcal{D}) & \supset & e(\mathcal{D}) \end{matrix}$$

By the way, each of the 3 subschemes of Γ is regularly embedded (i.e., locally it is the subscheme of zeros of a regular sequence of functions); they are not smooth in general.

For simplicity, assume that $e(\mathcal{D})$ is closed (automatic if $p_i : \Gamma \to X$ is separated).
Then $\Gamma' := \{\text{blow-up of } \Gamma \text{ at } e(\mathcal{D})\} \setminus \{\text{strict trasforms of } p_1^{-1}(\mathcal{D}) \text{ and } p_2^{-1}(\mathcal{D})\}$
$$\shortparallel$$
$$\{\text{blow-up}\} \setminus \overline{Y}, \quad Y := (p_1^{-1}(\mathcal{D}) \cup p_2^{-1}(\mathcal{D})) \setminus e(\mathcal{D})$$
(note that $Y \subset \{\text{blow-up}\}$)

It is eesy to show that $\Gamma' \to \Gamma$ is affine, so $\Gamma' = \operatorname{Spec} \mathcal{A}$ for some ~~$\mathcal{O}_\Gamma$-alge~~ quasicoherent algebra $\mathcal{A}$ on Γ. Let me describe $\mathcal{A}$.
Set $\Delta := p_1^{-1}(\mathcal{D})$, $\tilde{\Delta} = p_2^{-1}(\mathcal{D})$, $U := \Gamma \setminus (\Delta \cup \tilde{\Delta}) \xhookrightarrow{j} \Gamma$
Let $I \subset \mathcal{O}_\Gamma$ be the ideal of $e(X) \subset \Gamma$.
(You can also use the ideal of $e(\mathcal{D})$; the result is the same.)
$\mathcal{A} \subset j_* \mathcal{O}_U$ is the $\mathcal{O}_\Gamma$-subalgebra generated by $\underset{I \otimes \mathcal{O}_\Gamma(\Delta)}{\overset{\shortparallel}{I(\Delta)}}$ and $I(\tilde{\Delta})$.

<u>Example</u>. $X = \mathbb{A}^n$, $\Gamma = X \times X$, $\mathcal{D} = (p)$, $p \in k[x_1, \ldots, x_n]$.
$$\Gamma' = \operatorname{Spec} k[x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n, \frac{\tilde{x}_i - x_i}{p(x)}, \frac{\tilde{x}_i - x_i}{p(\tilde{x})}].$$
Idea: $\tilde{x}_i - x_i$ should be small if $p(x)$ or $p(\tilde{x})$ is small.
If you wish, $\tilde{x} = x + p(x) \cdot \Delta x$.
(If $e(\mathcal{D})$ is not closed then first remove $\overline{e(\mathcal{D})} \setminus e(\mathcal{D})$ from Γ, after which apply the previous construction.)

<u>Exercise</u>. Prove (or disprove; I am more confident in the situation of the Example):
(i) The morphisms $\Gamma' \overset{p_1'}{\underset{p_2'}{\rightrightarrows}} X$ are smooth;
(ii) Γ' has a (unique) structure of groupoid on X such that $\Gamma' \to \Gamma$ is a morphism of groupoids;
(iii) $\operatorname{Lie} \Gamma' = \mathfrak{a}(-\mathcal{D})$, where $\mathfrak{a} := \operatorname{Lie} \Gamma$;
(iii') ~~Formulate~~ Formulate and prove a version of ~~(ii)~~ (iii) for the formal groups associated to Γ' and Γ.

Here is the formulation. Recall that for a local Artinian A we defined $F_\Gamma(A) := \{(\operatorname{Spec} A)\text{-deformations of } e(X) \subset \Gamma\} = \{(\operatorname{Spec} A)\text{-deformations of } e: X \to \Gamma \text{ as a section of } p_1: \Gamma \to X\}$.
Let us use the (less symmetric) interpretation of F_Γ in terms of sections.
Let $s \in F_\Gamma(A)$, $s: X \otimes A \to \Gamma \otimes A$, $p_1 \circ s = id$.

<u>Claim.</u> $s \in \operatorname{Im}(F_{\Gamma'}(A) \hookrightarrow F_\Gamma(A))$ iff $s|_{\mathcal{D} \otimes A} = e|_{\mathcal{D} \otimes A}$

~~(iv) The groupoid $\Gamma' \times_{X^2} \mathcal{D}^2 \rightrightarrows \mathcal{D}$ (i.e., the restric~~

(iv) The action of Γ' on X equals the identity when restricted to $\mathcal{D} \subset X$; this just means that $p_2'|_{(p_1')^{-1}(\mathcal{D})} = p_1'|_{(p_1')^{-1}(\mathcal{D})}$
So the restriction of Γ' to $\mathcal{D}$ (i.e., $\Gamma' \times_{X^2} \mathcal{D}^2$) is a smooth <u>group scheme</u> over $\mathcal{D}$.
(iv′) Describe this group scheme in terms of $\mathcal{D}$ and $\mathfrak{a} := \operatorname{Lie} \Gamma$.
I think I know such a description (although I don't see an "a priori" reason for its existence in nonzero characteristic): see page 9a.

<u>Possible notation:</u> $\Gamma' = \Gamma(-\mathcal{D})$.

<u>Pre-theorem.</u> Suppose that X is affine (then X is reduced), equidimensional, has a (schematically) dense smooth open. Then $\exists$ smooth groupoid $\Gamma \rightrightarrows X$ such that the map $\Gamma \to X \times X$ is birational.

If X is singular one cannot take $\Gamma = X \times X$ (and usually there is no way to construct a suitable group action). If you have some Γ then you can "~~shrink~~" ~~it by~~ replacing it by $\Gamma(-\mathcal{D})$ (and possibly there are other ways of shrinking Γ).

<u>Key construction: Newton groupoid</u> ("kind of pullback" w.r.t. a ramified map)
Let $\varphi: X' \to X$, X smooth, X' a locally complete intersection (l.c.i.) not necessarily a relative one (e.g., X' could be a blow-up of X at a smooth submanifold). Let $U := \{x \in X' \mid \varphi \text{ etale at } x\}$.
Assume U is dense in X'. Let $\mathcal{D} \subset X'$ be <u>the "different" of φ</u> (name used by number theorists) (a certain Cartier divisor canonically associated to φ such that $X' \setminus \mathcal{D} = U$).
I will recall the definition of $\mathcal{D}$ later. It implies immediately that $\varphi^* \Theta_X \subset \Theta_{X'}(\mathcal{D})$ (Recall: $\varphi^* \Omega^1_X \to \Omega^1_{X'}$, no map $\varphi^* \Theta_X \to \Theta_{X'}$). Moreover, $\varphi^* \Theta_X(-\mathcal{D}) \subset \Theta_{X'}$ is a Lie subalgebroid.
So $\varphi^* \Theta_X(-r\mathcal{D})$ is a Lie algebroid for all $r \geq 1$.

The Newton groupoid is a certain ~~su~~ groupoid $\Gamma \overset{p_1}{\underset{p_2}{\rightrightarrows}} X'$ ~~Prop~~ canonically associated to φ. Properties (I don't know if they characterize φ):

(i) Γ is smooth,

(ii) Lie $\Gamma = \varphi^* \Theta_X(-2\mathcal{D})$,

(iii) $\Gamma \to X' \times X'$ is an isomorphism over $U \times U$,

(iv) The action of Γ becomes the identity when restricted to $\mathcal{D}$; that is, $p_2|_{p_1^{-1}(\mathcal{D})} = p_1|_{p_1^{-1}(\mathcal{D})}$

"Explanation" of $2\mathcal{D}$ assuming we want the last property: $\overbrace{\text{Lie}\,\Gamma \subset \Theta_{X'}(-\mathcal{D})}^{\text{we want}}$.
$\varphi^* \Theta_X(-\mathcal{D}) \not\subset \Theta_{X'}(-\mathcal{D})$, but $\varphi^* \Theta_X(-2\mathcal{D}) \subset \Theta_{X'}(-\mathcal{D})$.

Definition of $\mathcal{D}$: In our situation $\Omega^1_{X'/X}$ has homological dimension 1, and it vanishes on U. $\mathcal{D} = \mathrm{div}(\Omega^1_{X'/X})$ (in the sense of Knudsen-Mumford, i.e. write $\Omega^1_{X'/X}$ as Coker g, $g: \mathcal{O}^n \to \mathcal{O}^n$, and then ~~take~~ $\mathcal{D} = \mathrm{div} \det g$). Locally we can assume that ~~$X' \subset X$~~ $X' \subset X \times \mathbb{A}^\ell$, $X' = f^{-1}(0)$, $f: X \times \mathbb{A}^\ell \to \mathbb{A}^\ell$; then $\mathcal{D} = \det\left(\frac{\partial f}{\partial y}\right)$, where $y_1, \dots, y_\ell$ are the coordinates on $\mathbb{A}^\ell$.

Construction of Γ. For simplicity, assume that X, X' are separated.

$$X' \times X' \supset X' \times_X X' \supset X'_{diag} \quad \text{closed (by separatedness).}$$

$$\mathcal{O}_{X' \times X'} \supset \underset{\text{ideal of } X'_{diag}}{I_1} \supset \underset{\text{ideal of } X' \times_X X'}{I_2}$$

$j: U \times U \hookrightarrow X' \times X'$, $(X' \times X') \setminus (U \times U) = \Delta \cup \tilde{\Delta}$, where
$\Delta := \mathcal{D} \times X'$, $\tilde{\Delta} := X' \times \mathcal{D}$, $\Delta, \tilde{\Delta} \subset X' \times X'$ divisors

Then $\Gamma := \operatorname{Spec} \mathcal{A}$, where $\mathcal{A} \subset j_* \mathcal{O}_{U \times U}$ is the $\mathcal{O}_{X' \times X'}$-subalgebra generated by $I_1(\Delta)$, $I_2(2\Delta)$, $I_1(\tilde{\Delta})$, $I_2(2\tilde{\Delta})$.

The required properties of Γ are checked by a local computation in the spirit of calculus, see my write-up "Newton groupoid". The Newton groupoid is used behind the scenes in my article on formal arcs and also in Chau's talks and notes, and in the Bouthier-Kazhdan article.

Answer to Part (iv′) of the exercise on p. 9.

The group scheme over $\mathcal{D}$ is canonically isomorphic to the fiber product $\mathcal{G}$ in the following Cartesian diagram of group schemes over $\mathcal{D}$:

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{G}_m \arrow[d, hook, "\text{conjugation by } 1\in\mathbb{G}_a"] \\
\mathbb{G}_m \ltimes A \arrow[r, "f"'] & \mathbb{G}_m \ltimes \mathbb{G}_a
\end{tikzcd}$$

where A is the additive group of the locally free $\mathcal{O}_{\mathcal{D}}$-module $\mathfrak{a}(-\mathcal{D})/\mathfrak{a}(-2\mathcal{D})$, and f is induced by the composition

$$\begin{tikzcd}
A = \mathfrak{a}(-\mathcal{D})/\mathfrak{a}(-2\mathcal{D}) \arrow[r, "\lambda"] \arrow[d] & \mathcal{O}_{\mathcal{D}} \arrow[d, phantom, "\shortparallel"] \\
\Theta_X(-\mathcal{D})/\Theta_X(-2\mathcal{D}) \arrow[r] & \mathcal{N}_{\mathcal{D}} \otimes \mathcal{O}_X(-\mathcal{D})
\end{tikzcd}$$

Here $\mathcal{N}_{\mathcal{D}} := \mathcal{O}_X(-\mathcal{D})/\mathcal{O}_X$ is the normal bundle of $\mathcal{D}$.

Another description of $\mathcal{G}$: as a scheme, $\mathcal{G} \subset A$ is the open subscheme $\lambda \neq -1$, and the group operation is

$$u \circ v = u + (1+\lambda(u))\,v.$$