We work over a field k. $pt := \operatorname{Spec} k$.

<u>Def.</u> An alg. stack $\mathcal{Y}$ of finite type is <u>pointy</u> if $\mathcal{Y} \overset{\text{open, dense}}{\supset} pt$.
(If you wish, density means "schematically dense" rather than "topologically")

<u>Example.</u> $\mathcal{Y} = X/G$, $X \overset{\text{open dense}}{\supset} G$. E.g., $\mathcal{Y} = \mathbb{A}^1/\mathbb{G}_m$.

Now let C be a smooth curve (e.g., $\mathbb{A}^1$).

Stack $\operatorname{Maps}^\circ(C, \mathcal{Y})$ (stack of nondegenerate maps $C \to \mathcal{Y}$):
an S-point of $\operatorname{Maps}^\circ(C, \mathcal{Y})$ is $f: C \times S \to \mathcal{Y}$ s.t. $f^{-1}(\mathcal{Y} \setminus pt)$ is <u>finite over S</u>.

In the future I will describe k-points of stacks and pretend that this is enough to understand S-points.

<u>Pre-theorem.</u> Assume $\mathcal{Y} \xrightarrow{\text{diag}} \mathcal{Y} \times \mathcal{Y}$ separated (rather than quasi-separated; in practice, it is affine).
Then $\operatorname{Maps}^\circ(C, \mathcal{Y})$ is an alg. space locally of finite type.
(In some important cases this is a theorem, with $\operatorname{Maps}^\circ(C, \mathcal{Y})$ being a scheme.)

<u>Exercise.</u> $\operatorname{Maps}^\circ(C, \mathcal{Y})$ is a "space", i.e., S-points have no (nontrivial) automorphisms. (It is here that the separatedness assumption is used.)

<u>Example.</u> $\operatorname{Maps}^\circ(C, \mathbb{A}^1/\mathbb{G}_m) = \{(\mathcal{L}, s) \mid s \in H^0(\mathcal{L})\}$ ($\mathcal{L}$ line bundle on C; the scheme of zeros of s is finite) $= \operatorname{Div}_+(C) =$
$= \operatorname{Sym} C := \coprod_{n \geq 0} \operatorname{Sym}^n C$.

<u>Exercise.</u> $\operatorname{Maps}^\circ(C, \mathbb{P}^1/\mathbb{G}_a) = ?$ [Answer: $\{(D, \sigma) \mid D \subset C, \sigma \in H^0(\mathcal{O}_C(D)/\mathcal{O}_C)\}$ " tangent bundle of $\operatorname{Sym} C$.]
(Is it true that $\operatorname{Maps}(\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), \mathbb{A}^1/\mathbb{G}_m) = \mathbb{P}^1/\mathbb{G}_a$?)
Note: $\mathbb{P}^1/\mathbb{G}_a = B\backslash PGL(2)/N$. Here is a generalization:

<u>Example.</u> G reductive, $X = G/N$, $\mathcal{Y} = B \backslash X$ (is pointy).
Then $\operatorname{Maps}^\circ(C, \mathcal{Y})$ (known to be a scheme; this is (Mirkovic + Finkelberg)) is Mirkovic's "open zastava" scheme.
In this example the complement of the open orbit is a Cartier divisor.

<u>Generalization.</u> H affine alg. group, $\mathcal{Y} = X/H$ (where X is an H-variety).
Assume $X \supset H$. Assume $\exists$ H-invariant Cartier divisor $\Delta \subset X$ such that $X \setminus \Delta = H$. We'll almost prove the quasi-theorem if X is quasi-affine.
We'll construct a morphism from $\operatorname{Maps}^\circ(C, \mathcal{Y})$ to a certain scheme.

Have $\operatorname{Maps}^\circ(C, \mathcal{Y}) \to \operatorname{Sym} C$. Have something better:

$$f \mapsto D_f := f^{-1}(\Delta/H)$$

We have $\operatorname{Maps}^\circ(C, \mathcal{Y}) \to \operatorname{Maps}^\circ(C, pt/H) = \{H\text{-bundles on } C\}$

$$f \longmapsto E_f$$

E_f is trivialized over $C \setminus D_f$.

$\mathcal{GR}_D := \{H\text{-bundles on } C \text{ trivialized over } C - D\}$ (this is an ind-scheme)

$\mathcal{GR} \to \operatorname{Sym} C$, $\mathcal{GR}_D$ = fiber over D.

Thus we get a map $Maps^\circ(C, \mathcal{Y}) \to \mathcal{GR}$. not merely ind-scheme!

Prove (or maybe disprove) that:

Exercise. 1) X affine $\Rightarrow Maps^\circ(C,\mathcal{Y})$ is a closed subscheme of $\mathcal{GR}$

~~X quasiaffine $\Rightarrow$ locally closed~~

2) $\mathcal{O}_X(\Delta)$ ample $\Rightarrow Maps^\circ(C,\mathcal{Y})$ is a locally closed subscheme of $\mathcal{GR}$

This assumption ~~~~ holds if X is quasi-affine.

Relation to formal arcs.

Grinberg-Kazhdan. $\underset{\text{finite type}}{X} \supset X^{sm}$ - smooth ~~open~~ locus

(γ describes movement of a point on X)

~~$\gamma \in (\mathcal{L}X)(k)$~~ $\gamma: \operatorname{Spec} k[[t]] \to X$, $t =$ "time", $\gamma(t) \in X^{sm}$ for $t \neq 0$.

$\mathcal{L}X := Maps(D, X)$, $D := \operatorname{Spec} k[[t]]$. $\mathcal{L}X$ is a scheme (of infinite type)

$\gamma \in (\mathcal{L}X)(k)$.

Thm. $\widehat{\mathcal{L}X}_\gamma \simeq \widehat{Y}_y \hat{\times} D^\infty$ for some pair (Y, y), where Y is some scheme of finite type ~~~~ and $y \in Y(k)$.

(Before Grinberg-Kazhdan), Finkelberg-Mirković proved for $X = \overline{G/N}$; they showed that one can take $Y = Maps^\circ(\mathbb{A}^1, X/B)$. (Zastava space).

Summary of their argument. (in a slightly more general situation)

Suppose that 1) an affine alg. group H acts on X, $X \supset H$,

2) $\gamma(t) \in H$ for $t \neq 0$,

3) $Maps^\circ(\mathbb{A}^1, X/H)$ is a scheme of locally finite type (i.e., the quasi-theorem holds)

Then Grinberg-Kazhdan holds for $Y = Maps^\circ(\mathbb{A}^1, X/H)$.

(If $X = \overline{G/N}$ take $H = gBg^{-1}$ for a suitable $g \in G$).

What is $y \in Y(k)$?

$$\begin{array}{ccc} y: \mathbb{A}^1 & \longrightarrow & X/H \\ \cup & & \cup \\ \mathbb{G}_m & \longrightarrow & pt \end{array}$$

Have a covering $\mathbb{G}_m \amalg \operatorname{Spec} k[[t]] \to \mathbb{A}^1$ ("morally", an open covering).
Specifying y amounts to specifying $\hat{y}: \operatorname{Spec} k[[t]] \to X/H$ such that $\operatorname{Spec} k((t))$ goes to $pt \subset X/H$ (Believable, can be proved), follows from Beauville-Laszlo

Take $\hat{y} = \bar{\gamma}: \operatorname{Spec} k[[t]] \to X/H$.

How does one get an isomorphism $\widehat{\mathcal{L}X}_\gamma \xrightarrow{\sim} \hat{Y}_y \hat{\times} D^\infty$?
~~There is~~ It is not quite canonical. Canonically, one has $\widehat{\mathcal{L}X}_\gamma \longrightarrow \hat{Y}_y$, torsor over a certain formal group scheme $\mathfrak{g}$.
As a formal scheme, $\mathfrak{g} \simeq D^\infty$. Choosing such an isomorphism and also a section of the torsor, get Grinberg-Kazhdan.

<u>Description of $\mathfrak{g}$:</u> H acts on X, $\mathcal{L}H$ acts on $\mathcal{L}X$, so
$\mathfrak{g} := \widehat{\mathcal{L}H}$ acts on $\widehat{\mathcal{L}X}_\gamma$. As a formal scheme, $\hat{H} \simeq D^n$, so $\widehat{\mathcal{L}H} \simeq D^\infty$.

For ~~a~~ <u>general</u> X, the Finkelberg-Mirković method <u>has to be modified</u>.
(E.g., if X is a curve whose normalization has genus >1 then ~~a~~ connected algebraic group cannot act on X nontrivially.)
<u>Modification:</u> instead of group actions use (smooth) groupoids acting on X. In my article on Grinberg-Kazhdan this idea is used "Behind the scenes" (there is only a hint in the last paragraph).
I will try to explain the hint.

Generalities on groupoids.

(abstract) group — group scheme

(abstract) groupoid — groupoid in $\{\text{Schemes}\}$

Abstract groupoids.

Def. A groupoid is a category in which all morphisms are invertible.

$$\left.\begin{array}{l} X = \{\text{objects}\},\quad \Gamma = \{\text{morphisms}\} \\ \Gamma \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \quad (\text{source, target}). \\ \Gamma \times_X \Gamma \xrightarrow{c} \Gamma \quad \text{composition.} \end{array}\right\} \text{Data}$$

Certain properties should hold. In particular, one should have the $\boxed{\text{"identity" } \underset{x \mapsto \mathrm{id}_x \in \Gamma}{} \; X \xrightarrow{e} \Gamma \text{ and inversion } \Gamma \xrightarrow{i} \Gamma}$

If you wish, you can consider e and i as data; then all the properties become identities (no existence quantifiers).

Groupoid in $\{\text{Schemes}\}$ (or in any category in which fiber products exist).

Two equivalent ways:

① To formulate the notion of composition and the above-mentioned identities one only needs the notion of fiber product, and we have this notion in $\{\text{Schemes}\}$;

② A lazier way is to use the language of S-points:

Data: schemes X, Γ, morphisms $\Gamma \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$, $\Gamma \times_X \Gamma \xrightarrow{c} \Gamma$

Condition: for any scheme S the diagrams

$$\Gamma(S) \overset{p_1}{\underset{p_2}{\rightrightarrows}} X(S), \quad \Gamma(S) \times_{X(S)} \Gamma(S) \xrightarrow{c} \Gamma(S)$$

define a groupoid.

Conventions.

1. I will say "groupoid $\Gamma \rightrightarrows X$" (without mentioning composition) or "groupoid $\Gamma \to X \times X$".
2. One says $\underbrace{\text{"groupoid on } X\text{" or}}$ "groupoid Γ acting on X".

Examples of groupoids (say, in {Schemes}).

(1) An equivalence relation on X is a groupoid $\Gamma \rightrightarrows X$ such that $\Gamma \to X \times X$ is a monomorphism.

Example: $X \times X \rightrightarrows X$ is a groupoid.

(2) Suppose a group G acts on X. Then we have a groupoid Γ acting on X: a morphism $x_1 \to x_2$ is an element $g \in G$ such that $g x_1 = x_2$. So the groupoid is $G \times X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$, $p_1 =$ projection, $p_2 =$ action.

(3) A group scheme over X is the same as a groupoid $\Gamma \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$ with $p_1 = p_2$

(4) $\Gamma \rightrightarrows X$, $U \xrightarrow{f} X$

Pullback groupoid: $\{\text{objects}\} = U$, a morphism $u_1 \to u_2$ is a morphism $f(u_1) \to f(u_2)$.

So we get a groupoid $\Gamma_U \rightrightarrows U$, $\Gamma_U = \Gamma \times_{X \times X} (U \times U)$.

E.g., given a groupoid Γ in {Schemes} acting on X, you can restrict it to an open subscheme of X. Even if Γ comes from a group action, Γ_U usually doesn't.

This example (and many others) shows that groupoids are much more flexible than group actions.

(By the way, groupoids are discussed in Ch. V of SGA3-I).

Smooth groupoids and algebraic stacks.

Let $\Gamma \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$ be a groupoid in {Schemes}.

Def. A groupoid is smooth if p_1 is smooth (equivalently, if p_2 is smooth).

Note: p_1 and p_2 (i.e., source and target) have equal rights because the inversion map $i: \Gamma \to \Gamma$ interchanges them.

Suppose we have a smooth groupoid Γ on X such that the map $\Gamma \to X \times X$ is quasi-compact and separated (or quasi-separated, this depends on the book). To the data one associates the quotient stack $\Gamma \backslash X$ (I prefer to skip the details). (same if Γ is ~~an algebraic space~~) A stack is said to be algebraic if it can be represented in this way (with Γ being an alg. stack and X a scheme if you wish).

Details.

X/Γ is the stack (for the smooth topology) associated to the pre-stack $S \mapsto \{\text{groupoid } \Gamma(S) \rightrightarrows X(S)\}$.

(Here the "substance" is in "sheafification", i.e., in passing from the pre-stack to the associated stack.)

Here is an "explicit" description of $(X/\Gamma)(S)$, which is hard to find in the literature: an S-point of X/Γ is the following data:

(i) a smooth surjective morphism $S' \to S$, where S' is an algebraic space,

(ii) a morphism of groupoids

$$\begin{tikzcd} S' \times_S S' \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ S' \arrow[r] & X \end{tikzcd}$$

such that the squares

$$\begin{tikzcd} S' \times_S S' \arrow[r] \arrow[d, "p_1"'] & \Gamma \arrow[d, "p_1"] \\ S' \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} S' \times_S S' \arrow[r] \arrow[d, "p_2"'] & \Gamma \arrow[d, "p_2"] \\ S' \arrow[r] & X \end{tikzcd}$$

are Cartesian; in fact, it suffices to require <u>one</u> of the squares to be Cartesian.

(Given a morphism $S \to X/\Gamma$, one sets $S' := S \times_{X/\Gamma} X$.)