Let \( f_1, \ldots, f_e \in R[x_1, \ldots, x_n, y_1, \ldots, y_e] \),
\[ f_i = (f_i, \ldots, f_i), \quad X := f^{-1}(0) \subseteq A^{n+e} \]
(i.e., \( X \) is the scheme of common zeros of \( f_1, \ldots, f_e \)). Let \( \Delta \subseteq X \) be the subscheme defined by the equation
\[ \det \frac{\partial f_i}{\partial y} = 0. \]
Fix \( N \) a positive integer \( N \).

Now let \( Z \) be the scheme representing the following functor on the category of all \( k \)-algebras: an \( R \)-point of \( Z \) is a pair consisting of a monic \( \varphi \in R[t] \) of degree \( N \) and an element of \( \lim_{\leftarrow} \operatorname{Spec} R[t]/(q^r) \) such that the scheme-theoretic preimage of \( \Delta \) in \( \operatorname{Spec} R[t]/(q^2) \) equals \( \operatorname{Spec} R[t]/(q) \).

Proposition. \( Z \) is a product of a \( k \)-scheme of finite type and an (infinite-dimensional) affine space.

This proposition is nice and simple, but it is not related directly to formal arcs (e.g., it doesn't involve the ring \( R[[t]] \)). When you pass to formal arcs, things become complicated.

Proof. By definition, \( Z = \lim_{\leftarrow} Z_{\tau} \), where \( Z_{\tau} \) (\( \tau \geq 3 \)) represents the following functor: an \( R \)-point of \( Z_{\tau} \) is a triple \((\varphi, \overline{x}, \overline{y})\), where
\( \varphi \in R[t] \) is monic of degree \( N \),
\( \overline{x} \in (R[t]/(q^\tau))^{n} \), \( \overline{y} \in (R[t]/(q^\tau-1))^{e} \),
\( f(\overline{x}, \overline{y}) \equiv 0 \mod q^\tau \),
\( \det B(\overline{x}, \overline{y}) \equiv 0 \mod q \), where \( B := \frac{\partial f}{\partial y}(\overline{x}, \overline{y}) \)
\( q^{-1} \cdot \det \hat{B} \) is invertible modulo \( q \) (matrix).
\( \hat{B} \cdot f(\overline{x}, \overline{y}) \equiv 0 \mod q^\tau \), where \( \hat{B} \) is the adjugate matrix.
(The last condition makes sense despite the fact that \( \overline{y} \) is defined only modulo \( q^{\tau-1} \). To check this, use that \( \hat{BB} = \det B \) is divisible by \( q \).) Now it remains to prove...
the following lemma.

Lemma: $\mathbb{Z}[r+1]$ is isomorphic to $\mathbb{Z}[r] \times \{\text{affine space}\}$
(isomorphism of schemes over $\mathbb{Z}[r]$).

The lemma is easily proved using "Newton's method" (i.e., calculus).

The goal of my talks is to explain that $\mathbb{Z}[r]$ is, in fact, the space of "nondegenerate" maps $A^1 \to X/\Gamma_r$, where $\Gamma_r$ is a certain smooth groupoid acting on $X$ and $X/\Gamma_r$ is the quotient stack. (You can guess the definition of $\Gamma_r$ if you wish.)

A broader goal is to explain that smooth groupoids acting on $\mathbb{Z}$ (possibly singular) schemes are nice and manageable objects.