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Alexander Beilinson. A stacky approach to crystalline cohomology.

I will explain how crystalline cohomology can be seen as cohomology of some natural stack.

Schemes will be affine.

We don't ~~re~~ restrict to p-adically complete schemes.

$S,\ \mathcal{J}_S,\ \bar{S} = \operatorname{Spec} \mathcal{O}_S/\mathcal{J}_S$.

$X/\bar{S} \longmapsto X_{crys} = (X/S)_{crys}$.

The product in X_{crys} will be denoted by $\underset{crys}{\times}$

Let $P \in X_{crys}$. Have the simplicial object ~~of~~ $P^{[\cdot]}$ in X_{crys}. Let us look at $P^{[\cdot]}$ as at a simplicial scheme.

(retract of a coordinate PD-thickening)

Lemma. Suppose P is $\underbrace{\text{pd smooth}}$.

(i) $\forall\, T \in X_{crys}$ the morphism $P \underset{crys}{\times} T \to T$ is flat. (not smooth!)

(ii) The maps $P^{[n]} \to P^{[1]} \times_P P^{[1]} \times \dots \times_P P^{[1]}$ are isomorphisms

(scheme-theoretic fiber products)

Reformulation: $P^{[1]}$ is a flat ~~gro~~ groupoid (Strictly speaking, one also needs the involution of $P^{[2]}$)

Flatness of the groupoid is a particular case of (i).

Proof. a) It suffices to consider coordinate thickenings.
A retract of a flat T-scheme is flat (pass from schemes to rings).

b) Coordinate thickenings.

$X \hookrightarrow \mathbb{A}^I_S$, P is the PD-hull.

$\mathbb{A}^1 = \mathbb{G}_a$, $\mathbb{G}_a^\# \longrightarrow \mathbb{G}_a$ PD hull of $\{0\}$ in $\mathbb{G}_a$.

$\mathbb{G}_a^I$ acts on $\mathbb{A}^I_S$.
$(\mathbb{G}_a^\#)^I$ acts on P.

Claim. Our groupoid is the one corresponding to the action of $(\mathbb{G}_a^\#)^I$ on P. Proof: change of variables.

$$\begin{tikzcd} & (\mathbb{G}_a^\#)^I \times P \arrow[r, "\sim"] \arrow[dl, "\text{projection}"'] \arrow[d, "\text{action}"] & P[1] \\ P & P & \end{tikzcd}$$

Proving (i): choose a section $T \to P$.

$X^{\square} = (X/S)^{\square} = \mathcal{G}_P \backslash P$. (quotient stack).
~~Cr.~~ Crystallization of X/S.
Independence of P: the maps

$$\begin{tikzcd} & P \times_{\mathrm{crys}} P' \arrow[dl] \arrow[dr] & \\ P & & P' \end{tikzcd}$$

induce isomorphisms of stacks!

Why isomorphism?

$(P \times_{crys} P')^{[\cdot]}$ is the diagonal part of the bisimplicial object $P^{[\cdot]} \times_{crys} P'^{[\cdot]}$

$$\begin{array}{c} P'_{crys} \times P \\ \downarrow \\ P \end{array}$$ is a torsor for G_P'

The quotient of a torsor w.r.t. corresponding group is ~~a~~ an a point.

<u>Functoriality (and a better explanation of independence of P).</u>

Given

$$\begin{tikzcd} f: X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

want $f^{\amalg}: X^{\amalg} \to Y^{\amalg}$

Choose PD-smooth $P_X \in X_{crys}$, $P_Y \in Y_{crys}$

We'll lift $f: X \to Y$ to a simplicial map $P_X^{[\cdot]} \to P_Y^{[\cdot]}$

f yields a "pushforward" functor $X_{crys} \to Y_{crys}$

$T \to T \amalg_X Y$

In Y_{crys} we have a (noncanonical) morphism $\tilde{f}: P_X \amalg_X Y \to P_Y$

It obviously induces $P_X^{[\cdot]} \amalg_X Y \to P_Y^{[\cdot]}$

For a different choice of f' you get a canonically homotopic map.

Equivalent (and better) construction.

Let $\tilde{P}_X :=$ PD-hull of $X \hookrightarrow P_X \times_S P_Y$

in the relative sense (relative w.r.t. the PD-structures on X and Y).

Or: $P_X \times_S P_Y$ is a PD thickening of $X \times_S Y$.

$$X \hookrightarrow X \times_S Y \xhookrightarrow{\text{PD-thickening}} P_X \times_S P_Y$$

$\tilde{P}_X$ is PD-smooth.

We can use $\tilde{P}_X$ to define our stack.

$$\tilde{P}_X \longrightarrow P_Y$$

$G_{\tilde{P}_X}$ $\qquad$ G_{P_X}

~~Proposition: Crystalization commutes with limits (?)~~

False!