

Feb 14, 2019.

Alexander Beilinson. A stacky approach to crystalline cohomology.

I will explain how crystalline cohomology can be seen as cohomology of some natural stack.

Schemes will be affine.

We don't ~~have~~ restrict to p -adically complete schemes.

$$S, \mathcal{Y}_S, \bar{S} = \text{Spec } \mathcal{O}_S / \mathcal{Y}_S$$

$$X/\bar{S} \mapsto X_{\text{crys}} = (X/S)_{\text{crys}}$$

The product in X_{crys} will be denoted by $\begin{matrix} X \\ \text{crys} \end{matrix}$

Let $P \in X_{\text{crys}}$. Have the simplicial object $\{P^{[i]}\}$

in X_{crys} . Let us look at $P^{[i]}$ as at a simplicial scheme.

(retract of a coordinate \mathbb{P}^d -thickening)

Lemma. Suppose P is p -ad smooth.

(i) $\forall T \in X_{\text{crys}}$ the morphism $P \times_{\text{crys}} T \rightarrow T$ is flat. (not smooth!!)

(ii) The maps $P^{[n]} \rightarrow P^{[i]} \times_P P^{[j]} \times_P \dots \times_P P^{[i]}$ are isomorphisms

Reformulation: $P^{[i]}$ is a flat $\neq 0$ scheme-theoretic fiber products groupoid (Strictly speaking, one also needs the involution of $P^{[2]}$).

Fletness of the groupoid is a particular case of (i).

Proof. a) It suffices to consider coordinate thickenings.

A retract of a flat \mathbb{T} -scheme is flat (pass from schemes to rings).

b) ~~1)~~ Coordinate thickenings.

$$X \hookrightarrow A_S^I, \quad P \text{ is the PD-hull.}$$

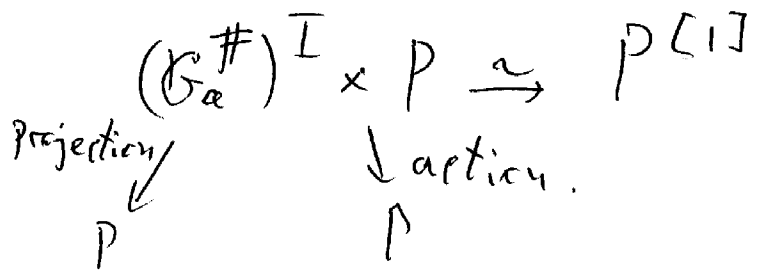
$$A^I = \mathbb{G}_a, \quad \mathbb{G}_a^\# \rightarrow \mathbb{G}_a$$

PD hull of $\{0\}$ in \mathbb{G}_a .

\mathbb{G}_a^I acts on A_S^I .

$(\mathbb{G}_a^\#)^I$ acts on P .

Claim. Our groupoid is the one corresponding to the action of $(\mathbb{G}_a^\#)^I$ on P . Proof: change of variables.



Proving (ii): choose a section $\Gamma \rightarrow P$.

$$X^\square = (X/S)^\square = \mathbb{G}_P/P. \quad (\text{quotient stack}).$$

Def. Crystallization of X/S .

Independence of P : the maps

$$\begin{array}{ccc}
 & P \times_{\text{crys}} P' & \\
 & \swarrow & \searrow \\
 P & & P'
 \end{array}$$

induce isomorphisms of stacks.

Why isomorphism?

$(P \times_{\text{crys}} P')[\cdot]$ is the diagonal part of the bisimplicial object $P[\cdot] \times_{\text{crys}} P'[\cdot]$

$P'_{\text{crys}} \times P$ is a torsor for G_P
 \downarrow
 P

The quotient of a torsor w.r.t. corresponding group is ~~an~~ a point.

Functoriality (and a better explanation of independence of P).

Given

$$f: X \rightarrow Y \quad \text{went} \quad f^\square: X^\square \rightarrow Y^\square$$

$$\downarrow \quad \swarrow$$

$$S \quad \quad \quad S$$

Choose PD-smooth $P_X \in X_{\text{crys}}, P_Y \in Y_{\text{crys}}$

We'll lift $f: X \rightarrow Y$ to a simplicial map

$$P_X[\cdot] \rightarrow P_Y[\cdot]$$

f yields a "pushforward" functor $X_{\text{crys}} \rightarrow Y_{\text{crys}}$

$$\Gamma \rightarrow \Gamma \amalg_X Y$$

In Y_{crys} we have a (noncanonical) morphism $f: P_X \amalg_X Y \rightarrow P_Y$
~~obviously~~
 IT induces $P_X[\cdot] \amalg_X Y \rightarrow P_Y[\cdot]$

For a different choice of f' you get a canonically homotopic map.

Equivalent (and better) construction.

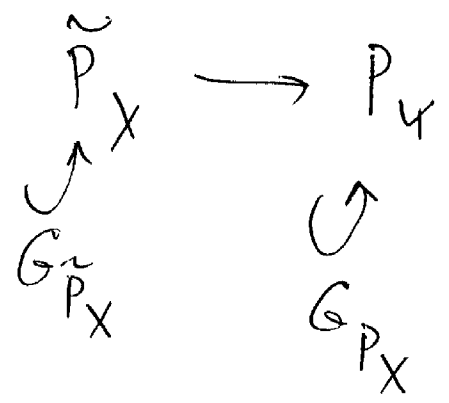
Let $\tilde{P}_X := \text{PD-hull of } X \hookrightarrow P_X \times_S P_Y$
in the relative sense (relative w.r.t. the PD-structures on X and Y).

Or: $P_X \times_S P_Y$ is a PD thickening of $X \times_S Y$.

$X \hookrightarrow X \times_S Y \xrightarrow{\text{PD-thickening}} P_X \times_S P_Y$

\tilde{P}_X is PD-smooth.

We can use \tilde{P}_X to define our stack.



Proposition. ~~Crystallization commutes with limits (!).~~
False!