(a) “Scheme” means affine scheme. The category $\mathcal{A}ff$ of schemes is closed under limits and colimits.

“pd scheme” means a scheme equipped with a pd ideal, $(S, \mathcal{I}_S, \gamma)$ abbreviated to $\tilde{S}$; we set then $\tilde{S} = \text{Spec} O_S/\mathcal{I}_S$ and call $S$ a pd-thickening of $\tilde{S}$. The category of pd schemes $\mathcal{A}ff^{pd}$ is closed under limits and colimits.

Below we fix a base pd scheme $\tilde{S}$ over $\mathbb{Z}/p^n$ and an $S$-scheme $X$ such that the pd structure on $\mathcal{I}_S O_X \subset O_X$ is well defined. (Say, $X$ is a $\tilde{S}$-scheme.) Denote by $X_{\text{crys}} = (X/S)_{\text{crys}}$ the category whose objects are pd thickenings $T$ of $X$ over $S$ (so $T$ is an $S$-scheme such that the pd structure on $\mathcal{I}_T + \mathcal{I}_S O_T$ is well defined). $X_{\text{crys}}$ is closed under nonempty limits (i.e., it has nonempty products and equalizers).

(b) **Alexander-Cech complexes.** We consider presheaves of abelian groups $F$ on $X_{\text{crys}}$, and call them sheaves. They form an abelian category; for any $T \in X_{\text{crys}}$ the functor $F \mapsto F(T)$ is exact. We have a left exact functor $F \mapsto \Gamma(F) = \Gamma(X_{\text{crys}}, F) := \lim_{X_{\text{crys}}} F$ and its derived functor $R\Gamma$.

For $P \in X_{\text{crys}}$ we denote by $F_P$ the sheaf $T \mapsto F(P \times T)$; the functor $F \mapsto F_P$ is exact. There is an evident map $\Gamma(F) \to H^0 F(P[1])$ is an isomorphism, and the exact functor $F \mapsto F_P(1)$ is $R\Gamma$.

(iii) For any $T \in X_{\text{crys}}$ one has $F(T) \xrightarrow{\sim} R\Gamma(F_T)$. Thus $F(P[1]) = \Gamma F_P[1]$.

Proof. (i) means that $\Gamma(F) = H^0 F(P[1])$. Proposition. Suppose $P$ is such that $\text{Hom}(T, P) \neq 0$ for every $T \in X_{\text{crys}}$. Then

(i) $F_P[1]$ is a cosimplicial resolution of $F$.

(ii) The evident map $\Gamma(F) \to H^0 F(P[1])$ is an isomorphism, and the exact functor $F \mapsto F_P[1]$ is $R\Gamma$.

(iii) For any $T \in X_{\text{crys}}$ one has $F(T) \xrightarrow{\sim} R\Gamma(F_T)$. Thus $F(P[1]) = \Gamma F_P[1]$.

(c) **A cosimplicial digression.** Recall that one has the Dold-Puppe equivalence between the category of cosimplicial abelian groups and coconnective complexes of abelian groups. It assigns to a cosimplicial abelian group $A$ the normalized complex $N(A)$ whose components $N(A)^m = \cap \ker(\sigma_1 : A^m \to A^{m-1})$. If $A$ is a cosimplicial ring then $N(A)$ is a dg ring with respect to the Alexander-Whitney $\cup$ product: for $a \in N(A)^m, b \in N(A)^n$ one has $a \cup b := \delta(a) \tau(b)$ where $\delta$ where $\delta : [m] \to [m+n]$, $\tau : [n] \to [m+n]$ are the monotone embeddings with images $[0, m]$ and $[m, m+n]$. If $A$ is an associative ring then so is $N(A)$; a similar fact about commutativity is not necessary true.

Definition. A cosimplicial ring $A$ is said to be of shuffle type if the next property holds: for every monotone embeddings $\delta : [m] \hookrightarrow [\ell], \tau : [n] \hookrightarrow [\ell]$ with $m+n > \ell$ one has $\delta(N(A)^m) \tau(N(A)^n) = 0$. Below $E(m, \ell)$ is the set of increasing embeddings $\delta : [m] \hookrightarrow [\ell]$ with $\delta(0) = 0$.

Lemma. (i) It is enough to check the condition of the definition for $\delta \in E(m, \ell)$, $\tau \in E(n, \ell)$. 

Around the Berthelot theorem
Dold-Puppe decomposition

An abelian group and $\bigcup$-plicial rings is an equivalence of categories depending on $\delta$-monotonous embeddings such that $\delta([m]) \cup \tau([n]) = [m+n]$ then for $a \in N(A)^m$, $b \in N(A)^n$ one has $\delta(a) \tau(b) = \pm a \cup b \in N(A)^{m+n} \subset A^{m+n}$ with the sign $\pm$ depending on $\delta$ and $\tau$.

Proposition. The restriction of $N$ to the subcategory Shuff of shuffle-type cosimplicial rings is an equivalence of categories Shuff $\sim$ (dg rings). It identifies the category of commutative shuffle-type rings with commutative dg rings.

Sketch of a proof. We construct the inverse functor. Let $A$ be a cosimplicial abelian group and $\bigcup$ a product on $N(A)$; we need to construct a shuffle product on $A$ such that the corresponding Alexander-Whitney product equals $\bigcup$. Recall the Dold-Puppe decomposition $A^\ell = \bigoplus_{m, \delta \in E(m, \ell)} \delta N(A)^m$ where $\delta N(A)^m$ is a copy on $N(A)^m$ embedded into $N(A)^\ell$ via $\delta$. The shuffle product $\delta(a) \tau(b)$ for $a \in N(A)^m$, $b \in N(A)^n$ vanishes if $\delta([m]) \cap \tau([n]) \neq \emptyset$; otherwise it equals $\pm \gamma(a \cup b)$ where $\gamma \in E(m+n, \ell)$ is the embedding with image $\delta([m]) \cup \tau([n])$ and $\pm$ is the sign of the permutation $s$ of $[1, m+n]$ such that $\gamma$ equals $\delta$ on $[1, m]$ and is $x \mapsto \tau(x-m)$ on $[m+1, m+n]$.

We call the equivalence of the proposition the Dold-Puppe correspondence.

(ii) Suppose $A$ is of shuffle type. If $\delta : [m] \hookrightarrow [m+n]$, $\tau : [n] \hookrightarrow [m+n]$ are monotonous embeddings such that $\delta([m]) \cup \tau([n]) = [m+n]$ then for $a \in N(A)^m$, $b \in N(A)^n$ one has $\delta(a) \tau(b) = \pm a \cup b \in N(A)^{m+n} \subset A^{m+n}$ with the sign $\pm$ depending on $\delta$ and $\tau$.

Lemma. One has $T_0^0 = T$, $T_1^0$ is the subscheme of $T^{[1]} = T \times_S T$ whose ideal is the square of the ideal $J$ of the diagonal. If $2 \in O_T$ is invertible then $T^{[1]}$ is the largest simplicial subscheme of $T^{[1]}$ with these properties; if not, one adds extra equations $\partial_0(\nu) \partial_2(\nu) = 0$ where $\nu \in J/J^2 \subset O(T^{[1]})$, $\partial_i : [1] \rightarrow [2]$ are the face maps.

Lemma. $\Omega_{pd}(T/S)$ corresponds to $O(T^{[1]}_e)$ by Dold-Puppe.

If $F$ is an $O$-crystal on $X_{crys}$ then the $\Omega_{pd}(T/S)$-module $\Omega(F(T^{[1]}_{pd}))$ equals $\Omega_{pd}(T/S) \otimes_{O(T)} F(T)$ as a plain graded $\Omega_{pd}(T/S)$-module; we call it the $\Omega_{pd}$ de Rham complex of $T$ with coefficients in $F$ and denote by $\Omega_{pd}(T/S, F)$.

(f) An object $P \in X_{crys}$ is said to be pd smooth if every closed embedding $P \hookrightarrow T$ in $X_{crys}$ admits left inverse, i.e., $P$ is a retraction of $T$. For example every coordinate thickening, i.e., the pd-hull of a closed embedding $X \hookrightarrow A^\ell$, is pd smooth. Since every $T$ admits a closed embedding into a coordinate thickening, pd smooth thickenings are the same as retracts of coordinate thickenings. A pd smooth $P$ satisfies the condition of the proposition in (b).
As in (b), for $P \in X_{\text{crys}}$ and a sheaf $\mathcal{F}$ we have the cosimplicial sheaf $\mathcal{F}_{P^{\Omega_{pd}}}$ and the coaugmentation map $\mathcal{F} \to \mathcal{F}_{P^{\Omega_{pd}}}$.

**Theorem.** Suppose $P$ is pd smooth and $\mathcal{F}$ is an $\mathcal{O}$-crystal. Then

(i) $\mathcal{F}_{P^{\Omega_{pd}}}$ is a cosimplicial resolution of $\mathcal{F}$.

(ii) One has $\Omega_{pd}(P/S, \mathcal{F}) \xrightarrow{\sim} N(\Gamma\mathcal{F}_{P^{\Omega_{pd}}}) \xrightarrow{\sim} R\Gamma(\mathcal{F}_{P^{\Omega_{pd}}}) \xrightarrow{\sim} R\Gamma(\mathcal{F})$.

(iii) The resulting quasi-isomorphism $R\Gamma(\mathcal{F}) \to \Omega_{pd}(P)$ can be realized, using the proposition in (b), as the restriction map $\mathcal{F}(P^{\Omega_{pd}}) \to \mathcal{F}(P^{\Omega_{pd}})$.

Proof. Assume (i). The first isomorphism is $\mathcal{F}(P^{\Omega_{pd}}) \xrightarrow{\sim} \Gamma(\mathcal{F}_{P^{\Omega_{pd}}})$ which follows from the first assertion (iii) of the proposition in (b), the second quasi-isomorphism $\Gamma(\mathcal{F}_{P^{\Omega_{pd}}}) \xrightarrow{\sim} R\Gamma(\mathcal{F}_{P^{\Omega_{pd}}})$ follows from loc. cit, the last isomorphism follows from (i). Claim (iii) follows, say, from the last assertion in loc. cit.

It remains to prove (i). We want to check that for $T \in X_{\text{crys}}$ the complex $\mathcal{F}(P^{\Omega_{pd}} \times T)$ is a resolution of $\mathcal{F}(T)$. Since there is a section of $P \times T/T$, the map $\mathcal{F}(T) \to H^0\mathcal{F}(P^{\Omega_{pd}} \times T)$ is injective. Let $P \hookrightarrow P'$ be an embedding of $P$ into coordinate $P'$. Since $P$ is a retract of $P'$, the complex $\mathcal{F}(P^{\Omega_{pd}} \times T)$ is a direct summand of $\mathcal{F}(P'^{\Omega_{pd}} \times T)$, and so our claim for $P$ follows from that about $P'$. Replacing $P$ by $P'$ we can assume that $P$ is a coordinate thickening. Thus $P$ is the relative pd-hull of $\mathcal{A}^I \times T$ at a section $X \to \mathcal{A}^I \times T$ over $X \subset T$. Extending it to a section $s$ over $T$ and applying the substraction of $s$ automorphism of $\mathcal{A}^I \times T/T$, we can assume that our section is zero. Then $P \times T$ is the product of pd schemes $\mathcal{G}^I_a$ and $T$ where $\mathcal{G}^I_a$ is the pd-hull of $\mathcal{G}_a$ at 0. Thus $\mathcal{F}(P^{\Omega_{pd}} \times T) = \mathcal{O}(\mathcal{G}^I_a) \oplus \mathcal{F}(T) = \Omega_{pd}(\mathcal{G}^I_a) \oplus \mathcal{F}(T) \xrightarrow{\sim} \mathcal{F}(T)$, the first equality comes since $\mathcal{F}$ is an $\mathcal{O}$-crystal, the last one since $\Omega_{pd}(\mathcal{G}^I_a) \xrightarrow{\sim} \mathbb{Z}$.

$\square$