

The q -de Rham prism. ~~8~~ ~~5~~ ~~6~~ ~~1~~

By def, $\Sigma = W^{(1)}/W^*$, where $\dim W^{(1)} = \infty$. Recall that the action of W^* on $\Delta_0 \subset \Sigma$ is transitive. Choosing a transversal to Δ_0 , one can represent Σ as a quotient of a formal scheme of Krull dimension 2 (one "geometric" direction). We'll do something like this.

More precisely, we'll construct a "nice" triple (Q, F, π) , where Q is a formal scheme, $F: Q \rightarrow Q$ a lift of Frobenius, $\pi: (Q, F) \rightarrow (\Sigma, F)$ is faithfully flat. (Q, F, π) is called "q-de Rham prism".

Def. $Q := \hat{G}_m := \text{Spf } \mathbb{Z}_p[[q-1]]$; $F: Q \rightarrow Q, q \mapsto q^p$.

End $\hat{G}_m = \mathbb{Z}_p$, $\text{Aut } \hat{G}_m = \mathbb{Z}_p^\times$ acts on (Q, F) . $n \in \mathbb{Z}_p$ acts by $q \mapsto q^n$.

Explicit formula: $q = 1+y \Rightarrow q^n = \sum_{i=0}^{\infty} \frac{n(n-1)\dots(n-i+1)}{i!} y^i$ (clear if $n \in \mathbb{Z}$. In general, by continuity, the coefficients are in \mathbb{Z}_p by continuity).

We'll define $Q/\mathbb{Z}_p^\times \rightarrow \Sigma$ commuting with F such that Q is flat (and algebraic) over Σ . To tell the truth, \mathbb{Z}_p^\times is considered as a group scheme (any profinite group is an affine group scheme over \mathbb{Z}).

Consider $Q \rightarrow W$ given by $\phi_p([q]) \in W(\mathbb{Z}_p[[q-1]])$, where $\phi_p(y) = \frac{1-y^p}{1-y} = 1+y+\dots+y^{p-1}$ is the cyclotomic polynomial and $[q]$ is the Teichmüller representative.

Claim. In fact, $Q \rightarrow W^{(1)} \subset W$.

Proof. It suffices to look at the unique point of Q . The image of $\phi_p([q])$ in $W(\mathbb{F}_p)$ equals $\phi_p(1) = p = V(1)$. \blacksquare

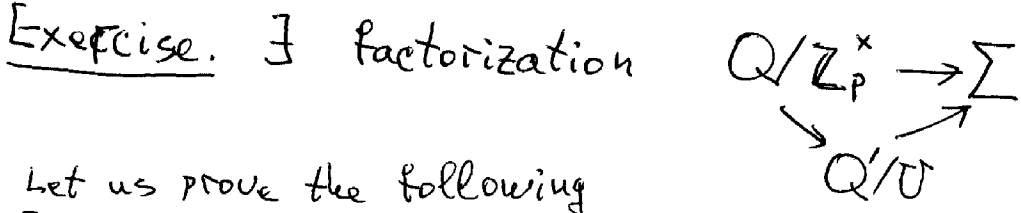
Claim. $Q \rightarrow W^{(1)}$ commutes with F .

Proof. $F(\phi_p([q])) = \phi_p([q^p])$ because $F([q]) = [q^p]$. \blacksquare

Claim. $Q \rightarrow W^{(1)} \rightarrow W^{(1)}/W^* = \Sigma$ factors through Q/\mathbb{Z}_p^\times the stacky quotient

Proof. $n \in \mathbb{Z}_p^\times \Rightarrow \phi_p([q^n]) \in \phi_p([q]) \cdot (\text{unit})$. Indeed, in $\mathbb{Z}_p[[q-1]]$ we have $\phi_p(y^n) = \phi_p(y) \cdot (\text{unit})$ because the subscheme of zeros of $\phi_p(q)$ is \mathbb{Z}_p^\times -stable. \blacksquare

Here is a finer factorization. Have $\mathbb{Z}_p^x = \mathbb{F}_p^x \times U$, $U = 1+p\mathbb{Z}_p \subset \mathbb{Z}_p^x$.
 Consider the geometric quotient $Q' := Q // \mathbb{F}_p^x = \text{Spf } \mathbb{Z}_p[[q-1]]^{\mathbb{F}_p^x}$.
 Note that $\mathbb{Z}_p[[q-1]]^{\mathbb{F}_p^x} \cong \mathbb{Z}_p[[z]]$ because the action of \mathbb{F}_p^x is linearizable. U acts on Q' .



Let us prove the following

Prop. $Q \rightarrow \Sigma$ is faithfully flat.
 "Faithfully" just means that $Q \neq \emptyset$. (Not a big deal.)

Have $\Sigma \rightarrow \hat{A}^1/G_m$, Σ is algebraic and flat over \hat{A}^1/G_m .

Exercise. Q is algebraic and flat over \hat{A}^1/G_m .

(Indeed, the map is $Q \xrightarrow{\phi_p} \hat{A}^1 \rightarrow \hat{A}^1/G_m$).

Lemma. $Y \rightarrow \Sigma$. Suppose Y is algebraic and flat over \hat{A}^1/G_m .

Then Y is (algebraic & and) flat over Σ .

Proof. Have the closed substack $\{0\}/G_m \subset \hat{A}^1/G_m$. Let us

look at its preimages Δ in Y and Σ :



Y flat over $\Sigma \iff Y$ flat over \hat{A}^1/G_m and classifying stack $Y_0 \otimes \mathbb{F}_p$ flat over $\Delta_0 \otimes \mathbb{F}_p$.

But $\Delta_0 \otimes \mathbb{F}_p = (\text{Spec } \mathbb{F}_p)/H_{\mathbb{F}_p}$ ($H := (W^x)^{\mathbb{F}_p} \otimes \mathbb{F}_p = G_m \otimes \mathbb{F}_p$).

Flatness over $\text{Spec } \mathbb{F}_p$ is automatic, so flatness over $(\text{Spec } \mathbb{F}_p)/H$ is automatic. ■

Prop. $H^0(\Sigma, \mathcal{O}_\Sigma) = \mathbb{Z}_p$.

Proof. $H^0(\Sigma, \mathcal{O}_\Sigma) \subset H^0(Q, \mathcal{O}_Q) = \mathbb{Z}_p[[q-1]]^{\mathbb{Z}_p^x} = \mathbb{Z}_p$ (because \mathbb{Z}_p^x is infinite). ■

Prop. Cart-Div (Σ) = $\bigoplus_{n=0}^{\infty} \mathbb{Z} \Delta_n \oplus \mathbb{Z} \cdot$ (special fiber)

Cart-Div (Σ) \subset Div (Q) \mathbb{Z}_p^*

$D_n \subset Q$ the divisor $\phi_{p^n}(q) = 0$ (i.e., the subscheme of primitive roots of 1 of degree p^n).

E.g., $D_0 = \text{Spt } \mathbb{Z}_p^* \subset \mathbb{A}^1_{\mathbb{Z}_p} \cong \mathbb{Z}[q]/(q-1)$

Exercise. Div (Q) $\mathbb{Z}_p^* = \bigoplus_{n=0}^{\infty} \mathbb{Z} \cdot D_n \oplus \mathbb{Z} \cdot$ (special fiber).

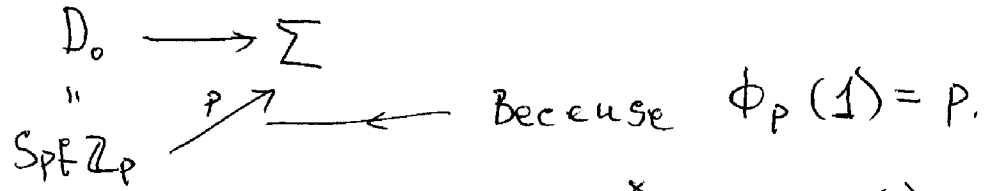
Lemma. $\Delta_n \times_Q \Sigma = D_{n+1}$.

Proof. $\Delta_n := (F^n)^* \Delta_0$, $D_{n+1} = (F^n)^* D_1$, so we can assume $n=0$.

By definition, $\Delta_0 \times_Q \Sigma \subset \Sigma$ is defined by the equation $\phi_p(q) = 0$.

It remains to show that the preimage in Q of a divisor in Σ cannot contain D_0 .

Claim. D_0 has ^{strongly} dense image in Σ not contained in a closed substack of Σ



Reformulation. The map $W^* \rightarrow W^{(1)}$ has ^{strongly} dense image $u \mapsto pu$

~~(i.e., the tangent map at $u=1$ is $W \xrightarrow{p} \text{Lie } W \rightarrow \text{Lie } W$.~~

Moral. The groupoid $Q \times_{\Sigma} Q$ is quite different from $\mathbb{Z}_p^* \times Q$. This follows from the next exercise.

Exercise. If $a \in W(R)$ maps to $p \in W(R/p^2R)$ then $a = pu$, $u \in W(R)^*$.
 Hint: $[p^2] \in pW(\mathbb{Z}_p)$ by Dwork's lemma, or by a formula for $p^2 [p^2]$.

Solution of Exercise. $a - p = \sum_i V^i [p^2 x_i] = p \sum_i V^i \left(\frac{[p^2]}{p} [x_i] \right)$.

Moral. The groupoid $Q \times_{\Sigma} Q$ is "bigger" than $\mathbb{Z}_p^* \times Q$: the map $\mathbb{Z}_p^* \times Q \rightarrow Q \times_{\Sigma} Q$ is far from being an isomorphism.