The q-de Rham prism.

By def, $\Sigma = W^{(1)}/W^\times$, where $\dim W^{(1)} = \infty$. Recall that the action of $W^\times$ on $\Delta_0 \subset \Sigma$ is transitive. Choosing a transversal to Δ_0, one can represent Σ as a quotient of a formal scheme of Krull dimension 2 (one "geometric" direction). We'll do something like this.

More precisely, we'll construct a "nice" triple (Q, F, π), where Q is a formal scheme, $F: Q \to Q$ a lift of Frobenius, $\pi : (Q, F) \to (\Sigma, F)$ is faithfully flat. (Q, F, π) is called "q-de Rham prism".

Def. $Q := \widehat{\mathbb{G}}_m := \mathrm{Spf}\, \mathbb{Z}_p[[q-1]]$; $F: Q \to Q$, $q \mapsto q^p$.

End $\widehat{\mathbb{G}}_m = \mathbb{Z}_p$, Aut $\widehat{\mathbb{G}}_m = \mathbb{Z}_p^\times$ acts on (Q, F).

$n \in \mathbb{Z}_p$ acts by $q \mapsto q^n$ ← clear if $n \in \mathbb{Z}$. In general, by continuity.

Explicit formula: $q = 1 + y \Rightarrow q^n = \sum_{i=0}^{\infty} \frac{n(n-1)\dots(n-i+1)}{i!} y^i$

(the coefficients are in $\mathbb{Z}_p$ by continuity).

We'll define $\underset{\text{stacky quotient}}{Q/\mathbb{Z}_p^\times} \to \Sigma$ commuting with F, such that Q is flat (and algebraic) over Σ. To tell the truth, $\mathbb{Z}_p^\times$ is considered as a group scheme (any pro-finite group is an affine group scheme over $\mathbb{Z}$).

Consider $Q \to W$ given by $\Phi_p([q]) \in W(\mathbb{Z}_p[[q-1]])$, where $\Phi_p(y) = \frac{1-y^p}{1-y} = 1 + y + \dots + y^{p-1}$ is the cyclotomic polynomial and $[q]$ is the Teichmüller representative.

Claim. In fact, $Q \to W^{(1)} \subset W$.

Proof. It suffices to look at the unique point of Q. The image of $\Phi_p([q])$ in $W(\mathbb{F}_p)$ equals $\Phi_p(1) = p = V(1)$. ∎

Claim. $Q \to W^{(1)}$ commutes with F.

Proof. $F(\Phi_p([q])) = \Phi_p([q^p])$ because $F([q]) = [q^p]$. ∎

Claim. $Q \to W^{(1)} \to W^{(1)}/W^\times = \Sigma$ factors $\overbrace{\text{through } Q/\mathbb{Z}_p^\times}^{\text{the stacky quotient}}$

Proof. $n \in \mathbb{Z}_p^\times \Rightarrow \Phi_p([q^n]) \overset{?}{\in} \Phi_p([q]) \cdot (\text{unit})$. Indeed, in $\mathbb{Z}_p[[y-1]]$ we have $\Phi_p(y^n) = \Phi_p(y) \cdot (\text{unit})$ because the subscheme of zeros of $\Phi_p(y)$ is $\mathbb{Z}_p^\times$-stable. ∎

Here is a finer factorization. Have $\mathbb{Z}_p^\times = \mathbb{F}_p^\times \times U$, $U = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$.
Consider the <u>geometric</u> quotient $Q' := Q /\!/ \mathbb{F}_p^\times = \mathrm{Spf}\, \mathbb{Z}_p[[q-1]]^{\mathbb{F}_p^\times}$.
Note that $\mathbb{Z}_p[[q-1]]^{\mathbb{F}_p^\times} \simeq \mathbb{Z}_p[[z]]$ because the action of $\mathbb{F}_p^\times$ is linearizable. U acts on Q'.

<u>Exercise.</u> $\exists$ factorization

$$\begin{tikzcd} Q/\mathbb{Z}_p^\times \arrow[r] \arrow[dr] & \Sigma \\ & Q'/U \arrow[u] \end{tikzcd}$$

Let us prove the following

<u>Prop.</u> $Q \to \Sigma$ is faithfully flat.

"Faithfully" just means that $Q \neq \emptyset$. (Not a big deal.)

Have $\Sigma \to \hat{\mathbb{A}}^1/\mathbb{G}_m$, Σ is algebraic and flat over $\hat{\mathbb{A}}^1/\mathbb{G}_m$.

<u>Exercise.</u> Q is algebraic and flat over $\hat{\mathbb{A}}^1/\mathbb{G}_m$.

(Indeed, the map is $Q \xrightarrow{\phi_p} \hat{\mathbb{A}}^1 \to \hat{\mathbb{A}}^1/\mathbb{G}_m$).

<u>Lemma.</u> stack $\mathcal{Y} \to \Sigma$. Suppose $\mathcal{Y}$ is algebraic and flat over $\hat{\mathbb{A}}^1/\mathbb{G}_m$. Then $\mathcal{Y}$ is (algebraic and) flat over Σ.

<u>Proof.</u> Have the closed substack $\{0\}/\mathbb{G}_m \subset \hat{\mathbb{A}}^1/\mathbb{G}_m$. Let us look at its preimages in $\mathcal{Y}$ and Σ:

$$\begin{tikzcd} \mathcal{Y} \arrow[d] & \supset & \mathcal{Y}_0 \arrow[d] \\ \hat{\mathbb{A}}^1/\mathbb{G}_m & \supset \arrow[u, phantom, "\square"] & \{0\}/\mathbb{G}_m \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[d] & \supset & \Delta_0 \arrow[d] \\ \hat{\mathbb{A}}^1/\mathbb{G}_m & \supset \arrow[u, phantom, "\square"] & \{0\}/\mathbb{G}_m \end{tikzcd}$$

$\mathcal{Y}$ flat over $\Sigma \Leftrightarrow \mathcal{Y}$ flat over $\hat{\mathbb{A}}^1/\mathbb{G}_m$ and $\mathcal{Y}_0 \otimes \mathbb{F}_p$ flat over $\Delta_0 \otimes \mathbb{F}_p$.

classifying stack

But $\Delta_0 \otimes \mathbb{F}_p = (\mathrm{Spec}\, \mathbb{F}_p)/H$ $(H := (W^\times)^{(F)} \otimes \mathbb{F}_p = \mathbb{G}_m^\sharp \otimes \mathbb{F}_p)$.

Flatness over $\mathrm{Spec}\, \mathbb{F}_p$ is automatic, so flatness over $(\mathrm{Spec}\, \mathbb{F}_p)/H$ is automatic. ∎

<u>Prop.</u> $H^0(\Sigma, \mathcal{O}_\Sigma) = \mathbb{Z}_p$.

<u>Proof.</u> $H^0(\Sigma, \mathcal{O}_\Sigma) \subset H^0(Q, \mathcal{O}_Q)^{\mathbb{Z}_p^\times} = \mathbb{Z}_p[[q-1]]^{\mathbb{Z}_p^\times} = \mathbb{Z}_p$ (because $\mathbb{Z}_p^\times$ is infinite). ∎

$\Sigma\otimes\mathbb{F}_p$

<u>Prop.</u> $\mathrm{Cart\text{-}Div}(\Sigma)=\bigoplus_{n=0}^{\infty}\mathbb{Z}\Delta_n\oplus\mathbb{Z}\cdot$ (special fiber)

$\mathrm{Cart\text{-}Div}(\Sigma)\subset \mathrm{Div}(Q)^{\mathbb{Z}_p^\times}$

$D_n\subset Q$ the divisor $\Phi_{p^n}(q)=0$ (i.e., the subscheme of primitive roots of 1 of degree p^n).

E.g., $D_0=\mathrm{Spf}\,\mathbb{Z}_p[q]/(q-1)$

<u>Exercise.</u> $\mathrm{Div}(Q)^{\mathbb{Z}_p^\times}=\bigoplus_{n=0}^{\infty}\mathbb{Z}\cdot D_n\oplus\mathbb{Z}\cdot$ (special fiber).

<u>Lemma.</u> $\Delta_n\times_Q\Sigma=D_{n+1}$.

<u>Proof.</u> $\Delta_n:=(F^n)^*\Delta_0$, $D_{n+1}=(F^n)^*D_1$, so we can assume that $n=0$. By definition, $\Delta_0\times_Q\Sigma\subset\Sigma$ is defined by the equation $\Phi_p(q)=0$. ∎

It remains to show that the preimage in Q of a divisor in Σ cannot contain D_0.

<u>Claim.</u> D_0 has strongly dense image in Σ (not contained in a closed substack of Σ)

$$\begin{tikzcd} D_0 \arrow[r] \arrow[d, equal] & \Sigma \\ \mathrm{Spf}\,\mathbb{Z}_p \arrow[ur, "p"] & \end{tikzcd}$$

Because $\Phi_p(1)=p$.

<u>Reformulation.</u> The map $W^\times\longrightarrow W^{(1)}$, $u\mapsto pu$ has strongly dense image

~~(i.e., Exe Indeed, the tangent map at $u=1$ is~~ ~~$\mathrm{Lie}\,W\xrightarrow{p}\mathrm{Lie}\,W$.~~

<u>Moral.</u> ~~The groupoid~~ $Q\times_\Sigma Q$ is quite different from $\mathbb{Z}_p^\times\times Q$. This follows from the next exercise.

<u>Exercise.</u> If $a\in W(R)$ maps to $p\in W(R/p^2R)$ then $a=pu$, $u\in W(R)^\times$.

<u>Hint:</u> $[p^2]\in pW(\mathbb{Z}_p)$ by Dwork's Lemma, or by a formula for $p^{-1}[p^2]$. (See file "Formula for [p^2]/p.pdf")

<u>Solution of Exercise.</u> $a-p=\sum_i V^i[p^2x_i]=p\sum_i V^i\left(\frac{[p^2]}{p}[x_i]\right)$. ∎

<u>Moral.</u> The groupoid $Q\times_\Sigma Q$ is "bigger" than $\mathbb{Z}_p^\times\times Q$: the map $\mathbb{Z}_p^\times\times Q\to Q\times_\Sigma Q$ is far from being an isomorphism.