Stacks (terminology).

A stack is a functor Rings $\to$ Groupoids which is a sheaf for the fpqc topology. A stack is <u>algebraic</u> if it can be represented as S/Γ, where S is a scheme and Γ is ~~an~~ a flat affine groupoid acting on S ("affine" means that the morphisms $\Gamma \rightrightarrows S$ are affine). Same for "flat". In practice, Γ often comes from a group scheme acting on S. The morphisms $\Gamma \rightrightarrows S$ are not supposed to have finite type, so algebraic stacks in our sense are not necessarily Artin stacks. Reason: we will work with W, which has infinite type.

In addition to algebraic stacks, we'll work with formal ones. A <u>formal stack</u> is a stack of the form $\varinjlim(\mathcal{Y}_0 \hookrightarrow \mathcal{Y}_1 \hookrightarrow \dots)$, where each $\mathcal{Y}_n$ is an algebraic stack and $\mathcal{Y}_n$ is a closed substack of $\mathcal{Y}_{n+1}$ defined by a nilpotent ideal.

<u>Examples</u>: $\mathrm{Spf}\,\mathbb{Z}_p = \varinjlim \mathrm{Spec}\,(\mathbb{Z}/p^n\mathbb{Z})$, $\hat{\mathbb{A}}^1 = \hat{\mathbb{A}}^1_{\mathbb{Z}_p} = \mathrm{Spf}\,\mathbb{Z}_p[[\varepsilon]] = \varinjlim_{m,n} \mathrm{Spec}\,\mathbb{Z}_p[\varepsilon]/(p^m,\varepsilon^n)$, $\hat{\mathbb{A}}^1/\mathbb{G}_m$.

$\mathrm{Spf}\,\mathbb{Z}_p$ as a functor: $R \mapsto \begin{cases} \text{point} & \text{if } p \text{ is nilpotent in } R \\ \varnothing & \text{otherwise} \end{cases}$

Recall: $\mathrm{Rings}_p := \{R \mid p \text{ nilpotent in } R\}$. We will live over $\mathrm{Spf}\,\mathbb{Z}_p$, so instead of functors on Rings we consider functors on Rings_p.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Say that $\mathcal{X}$ is <u>algebraic over $\mathcal{Y}$</u> if for every scheme $S \to \mathcal{Y}$, $\mathcal{X} \times_{\mathcal{Y}} S$ is an algebraic stack. E.g., a p-adic formal scheme is an example of a stack algebraic over $\mathrm{Spf}\,\mathbb{Z}_p$.

Now let us define $\Sigma = (\mathrm{Spf}\,\mathbb{Z}_p)^{\Delta}$. This will be a formal stack algebraic over $\hat{\mathbb{A}}^1/\mathbb{G}_m$. So there will be <u>two</u> "formal directions" (one of them arithmetical).

The stack Σ

$W := \{p\text{-typical Witt vectors}\}$, $W(R) = \{\sum_{i=0}^{\infty} V^i[\xi_i] \mid \xi_i \in R\}$. Recall that $\mathrm{Rings}_p := \{R \in \mathrm{Rings} \mid p \text{ is nilpotent in } R\}$. Let us define $W^{(1)} : \mathrm{Rings}_p \to \mathrm{Sets}$, $W^{(1)} \subset W$. Definition: $W^{(1)}(R) := \{\sum_i V^i[\xi_i] \mid \xi_0 \text{ nilpotent}, \xi_1 \in R^\times\}$.
This functor is (represented by) a formal scheme over $\mathrm{Spf}\,\mathbb{Z}_p$: namely, take $\mathbb{A}^\infty_{\mathbb{Z}}$, formally complete along the locus $\xi_0 = p = 0$ and remove the locus $\xi_1 = 0$. So $W^{(1)} = \mathrm{Spf}\, A$, where A is the completion of $\mathbb{Z}_p[\xi_0, \xi_1, \dots][\xi_1^{-1}]$ w.r.t. the ideal (p, ξ_0). Name: $W^{(1)}$ is the formal scheme of primitive Witt vectors of degree 1.

Definition. $\Sigma := W^{(1)}/W^\times$.
The map $W^{(1)} \xrightarrow{\xi_0} \hat{\mathbb{A}}^1$ induces $\Sigma \to \hat{\mathbb{A}}^1/\mathbb{G}_m$.
Σ is algebraic and flat over $\hat{\mathbb{A}}^1/\mathbb{G}_m$.
The Witt vector Frobenius $F : W \to W$ induces $F : \Sigma \to \Sigma$.
Let $R \in \mathrm{Rings}_p$. By definition, Σ is the fpqc-sheafification of the presheaf $R \mapsto W^{(1)}(R)/W(R)^\times$ (quotient in the sense of groupoids). Sheafification is necessary because $H^1_{fpqc}(\mathrm{Spec}\, R, W^\times) \neq 0$ in general:
Exercise. $H^1_{fpqc}(\mathrm{Spec}\, R, W^\times) = \mathrm{Pic}\, W(R) = \mathrm{Pic}\, R$.
(The exercise implies that Zariski sheafification is enough here.)
Interpreting $W^\times$-torsors on $\mathrm{Spec}\, R$ as invertible $W(R)$-modules, we see that an object of $\Sigma(R)$ is a pair (P, ξ), where P is an invertible $W(R)$-module and $\xi : P \to W(R)$ is a morphism of modules such that $\forall f \in R$ $\forall$ generator e of $P \otimes_{W(R)} W(R_f)$ one has $\xi(e) \in W^{(1)}(R_f)$.
Exercise. The module $P^{\otimes p}$ is (noncanonically) isomorphic to $W(R)$.
Hint: the map f in the diagram $W^{(1)} \to W^{(1)}/\mu_p \xrightarrow{f} W^{(1)}/\mathbb{G}_m \to W^{(1)}/W^\times = \Sigma$ admits a section (to see this, use $\xi_1 : W^{(1)} \to \mathbb{G}_m$). Here I am using the Teichmüller embedding $\mathbb{G}_m \hookrightarrow W^\times$.
Lemma. For any perfect $\mathbb{F}_p$-algebra R one has $\Sigma(R) = \text{point}$.
Proof. Σ is the Zariski sheafification of the presheaf $R \mapsto W^{(1)}(R)/W(R)^\times$, $R \in \mathrm{Rings}_p$. So it remains to show that if $R/\mathbb{F}_p$ is perfect then $W^{(1)}(R)/W(R)^\times = \text{point}$. R is reduced,

so $W^{(1)}(R)=\{Vy \mid y\in W(R)^\times\}$. If $y\in W(R)^\times$ then $\exists!\, u\in W(R)^\times$ such that $u\cdot V(1)=V(y)$, namely $u=F^{-1}(y)$ (the map $F: W(R)^\times\to W(R)^\times$ is bijective by perfectness). ■

Fact. Let $X\in Sch_{\mathbb{Z}_p}$. Let R be a perfect $\mathbb{F}_p$-algebra. Then $X^\Delta(R)=X(R)$. (Note that $X(R)=(X\otimes\mathbb{F}_p)(R)$.) In other words, X^Δ and X have the same image in the category of stacks up to universal homeomorphism.

Thus $\Sigma(\mathbb{F}_p)=\Sigma(\overline{\mathbb{F}}_p)=$ point. What about $\Sigma(\mathbb{Z}_p)$? Here $\mathbb{Z}_p$ is considered as a topological ring (otherwise $\Sigma(\mathbb{Z}_p)=\emptyset$), so

$$\Sigma(\mathbb{Z}_p):=Mor(Spf\,\mathbb{Z}_p,\Sigma)=\varprojlim_n \Sigma(\mathbb{Z}/p^n\mathbb{Z}).$$

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<u>Exercise.</u> (i) The groupoid $\Sigma(\mathbb{Z}_p)$ is a set.

(ii) Give a description of $\Sigma(\mathbb{Z}_p)$ (I don't know a <u>particularly</u> good one). Show that $\Sigma(\mathbb{Z}_p)$ has cardinality continuum.

(iii) Show that $\forall x\in\Sigma(\mathbb{Z}_p)$, $F(x)=p$ (here $p\in\Sigma(\mathbb{Z}_p)$ is the image of $p\in W^{(1)}(\mathbb{Z}_p)$).

<u>Beginning of solution.</u> $\Sigma(\mathbb{Z}_p)=W^{(1)}(\mathbb{Z}_p)/W(\mathbb{Z}_p)^\times$. By Dwork's lemma,
$W(\mathbb{Z}_p)=\{(y_0,y_1,\ldots)\mid y_n\in\mathbb{Z}_p,\ y_n\equiv y_{n-1} \bmod p^n \text{ for } n>0\}$.
(The y_i's are the ghost components, i.e., $y_0=\xi_0$, $y_1=\xi_0^p+p\xi_1$ and so on.)
Let $(y_0,y_1,\ldots)\in W(\mathbb{Z}_p)$. Then

$(y_0,y_1,\ldots)\in W(\mathbb{Z}_p)^\times \Leftrightarrow y_n\in\mathbb{Z}_p^\times$ for all (or some) n,

$(y_0,y_1,\ldots)\in W^{(1)}(\mathbb{Z}_p)\Leftrightarrow y_0\in p\mathbb{Z}_p,\ y_1\notin p^2\mathbb{Z}_p$ (then $y_n\in p\mathbb{Z}_p$ for all n, $y_n\notin p^2\mathbb{Z}_p$ for $n>0$).

The rest is straightforward. ■

We mostly care about <u>two</u> elements of $\Sigma(\mathbb{Z}_p)$, namely about $p, V(1)\in\Sigma(\mathbb{Z}_p)$ (i.e., the images of $p, V(1)\in W^{(1)}(\mathbb{Z}_p)$). These two elements of $\Sigma(\mathbb{Z}_p)$ are different. One has $F(V(1))=p$, $F(p)=p$.

Recall the flat algebraic morphism $\xi_0 : \Sigma \to \hat{\mathbb{A}}^1/\mathbb{G}_m$.

<u>Definition</u>. $\Delta_0 := \xi_0^{-1}(\{0\}/\mathbb{G}_m)$ is called the <u>Hodge-Tate locus</u>.
$\Delta_0 \subset \Sigma$ is an effective Cartier divisor (by flatness of ξ_0).

<u>Remark</u>. $V(1) \in \Delta_0(\mathbb{Z}_p)$, $p \notin \Delta_0(\mathbb{Z}_p)$.

For $n \geq 0$ define $\Delta_n \subset \Sigma$ by $\Delta_n := (F^n)^{-1}(\Delta_0)$. These are effective Cartier divisors on Σ (because $F : \Sigma \to \Sigma$ is flat).

<u>Fact</u>. $\mathrm{Cart\text{-}Div}(\Sigma) = \bigoplus_{n=0}^{\infty} \mathbb{Z} \cdot \Delta_n \oplus \mathbb{Z} \cdot (\underbrace{\text{special fiber}}_{\Sigma \otimes \mathbb{F}_p})$

<u>Exercise</u>. $m < n \Rightarrow \Delta_m \cap \Delta_n = \Delta_m \otimes \mathbb{F}_p$.

<u>Solution</u>. We can assume that $m = 0$. The divisor $\Delta_n \times_\Sigma W^{(1)}$ is defined by the equation $\xi_0^{p^n} + p\xi_1^{p^{n-1}} + \dots + p^n \xi_n = 0$. So $(\Delta_0 \cap \Delta_n) \times_\Sigma W^{(1)}$ is given by the equations $\xi_0 = 0$, $pu = 0$, where $u = \xi_1^{p^{n-1}} + p\xi_2^{p^{n-2}} + \dots + p^{n-1}\xi_{n-1}$. u is invertible because ξ_1 is. ∎

<u>Lemma</u>. $\Delta_0 = (\mathrm{Spf}\, \mathbb{Z}_p)/(W^\times)^{(F)}$ (classifying stack), where $(W^\times)^{(F)} := \mathrm{Ker}(W^\times \xrightarrow{F} W^\times)$.

<u>Proof</u>. We have an isomorphism of ~~form~~ p-adic schemes $W^\times \xrightarrow{\sim} \{\xi \in W^{(1)} \mid \xi_0 = 0\}$, namely $x \mapsto Vx$. If $x, u \in W^\times$ then $u \cdot Vx = V((Fu) \cdot x)$. So $\Delta_0 = \mathrm{Cone}(W^\times \xrightarrow{F} W^\times) = (\mathrm{Spf}\, \mathbb{Z}_p)/(W^\times)^{(F)}$ (because $F : W^\times \to W^\times$ is faithfully flat). ∎

<u>Remark</u>. $(W^\times)^{(F)} = \mathbb{G}_m^\#$, where $\mathbb{G}_m^\#$ is the PD hull of $\{1\} \subset \mathbb{G}_m$. Indeed, in winter we proved that $W^{(F)}$ and $\mathbb{G}_a^{(F)}$ are canonically isomorphic as (non-unital) <u>rings</u> (there is a shorter proof based on Joyal's approach to W). Passing to the multiplicative groups of these rings, we get $(W^\times)^{(F)} = \mathbb{G}_m^\#$.