p-adic schemes.

$$Sch_{\mathbb{Z}_p} = \{\text{p-adic (formal) schemes}\} = \varprojlim_n Sch/(\mathbb{Z}/p^n\mathbb{Z}).$$

A p-adic scheme is a compatible collection of $(\mathbb{Z}/p^n\mathbb{Z})$-schemes, $n \in \mathbb{N}$. One can consider a p-adic scheme as a ringed space (the underlying topol. space depends only on the reduction mod p).

Test rings (maybe postpone this?)

$Rings_p := \{$Rings in which p is nilpotent$\}$. A p-adic scheme is a functor $Rings_p \to Sets$ such that $\forall n \in \mathbb{N}$ its restriction to $\{\mathbb{Z}/p^n\mathbb{Z}\text{-algebras}\}$ is representable by a scheme.

Stacky format of prismatic theory (Bhatt-Lurie, not written).

Prismatization functor $X \longmapsto X^{\Delta}$, where $X \in Sch_{\mathbb{Z}_p}$ and X^{Δ} is a formal stack with a lift of Frobenius $F: X^{\Delta} \to X^{\Delta}$.

A crystal on X is a complex of $\mathcal{O}$-modules on X^{Δ}.
An F-crystal is a pair $(M, F^*M \to M)$ with a certain property: roughly, $Cone(F^*M \to M)$ is supported on the "Hodge-Tate" locus of X^{Δ}. F-crystals are p-adic analogs of $\mathbb{Z}_\ell$-sheaves.

$f: X \to Y$ gives $f^{\Delta}: X^{\Delta} \to Y^{\Delta}$.

$f^{\Delta}_*: \{F\text{-crystals on } X\} \to \{F\text{-crystals on } Y\}$. This is prismatic cohomology.

Last quarter we discussed X^{Δ} for X over $\mathbb{F}_p$. Then we get crystalline cohomology. Recall that $(Spec\ \mathbb{F}_p)^{\Delta} = Spf\ \mathbb{Z}_p$.

Before discussing the general case, I will ~~let me~~ define $\Sigma = (Spf\ \mathbb{Z}_p)^{\Delta}$. This is a stack, & not a (formal) scheme.