Derived setting (it makes formulations cleaner!)

$\text{Rings} := \{\infty\text{-}\underbrace{\text{category of simplicial}}_{\text{"non-derived"}} \text{ commutative rings}\}$ ← Let's pretend we know the definition of this ∞-category!

$\cup$

$\text{Rings}^{cl} := \{\text{classical rings}\}$

$$\text{Rings}^{cl} \hookrightarrow \text{Rings} \xrightarrow{\text{left adjoint}} \text{Rings}^{cl}$$

$A \mapsto A^{cl} := H^0(A)$ ("classical truncation").

$\text{Sch}_{\mathbb{Z}_p} = \{\text{Derived } p\text{-adic (formal) schemes}\}$

$\cup$

$\text{Aff}_{\mathbb{Z}_p} = \{\text{affine ones}\}$.

$\text{Aff}^{op}_{\mathbb{Z}_p} \xleftarrow{\sim} \text{Rings}_{p\text{-complete}}$ (derived) $= \{A \in \text{Rings} \mid A \xrightarrow{\sim} \varprojlim_n A \otimes \mathbb{Z}/p^n\mathbb{Z}\}$.

This depends only on A as an object of the derived category of $\mathcal{A}b$.

$\text{Spf} A \longleftarrow A$

("classical truncation")

$$\text{Sch}^{cl}_{\mathbb{Z}_p} \hookrightarrow \text{Sch}_{\mathbb{Z}_p} \xrightarrow{\text{right adjoint}} \text{Sch}^{cl}_{\mathbb{Z}_p}, \quad X \mapsto X^{cl}$$

Sublety: there are <u>two</u> versions of $\text{Sch}^{cl}_{\mathbb{Z}_p}$, but what I said (and will say in the future) holds for each of them. For ~~a class~~ $A \in \text{Rings}^{cl}$ (or $A \in \mathcal{A}b$) one can require derived p-completeness, i.e., $A \xrightarrow{\sim} R\varprojlim_n (A \otimes \mathbb{Z}/p^n\mathbb{Z})$ or ~~naive~~ ~~p-~~ a traditional p-adic completeness $A \xrightarrow{\sim} \varprojlim_n A/p^nA$, which is a $\boxed{\text{stronger}}$ condition. The less traditional notion defines an abelian subcategory in $\mathcal{A}b$ (unlike the traditional notion). If $f: A_1 \to A_2$ is a morphism of p-adically complete abelian groups then $\text{Coker} f$ is derived p-complete but not necessarily p-adically complete. More details: see Bhatt's Lecture 3 at Columbia.

The functor $X \mapsto X^{cl}$ commutes with $\varprojlim$'s (it has a left adjoint) but the functor $\text{Sch}^{cl}_{\mathbb{Z}_p} \hookrightarrow \text{Sch}_{\mathbb{Z}_p}$ does <u>not</u> commute with $\varprojlim$'s (the intersection of classical subschemes needn't be classical!)

<u>Similarly</u>: $\text{Stacks}_\Sigma = \{\text{derived stacks algebraic over } \Sigma\} \supset \text{Stacks}^{cl}_\Sigma$, have $\text{Stacks}_\Sigma \to \text{Stacks}^{cl}_\Sigma$ ("classical truncation").

We'll define the prismatization functor $\text{Sch}_{\mathbb{Z}_p} \xrightarrow{X \mapsto X^{\Delta}} \text{Stacks}_\Sigma$, which commutes with $\varprojlim$'s (this is conceptually nice and also convenient). If you wish, you can define a "classical version" of this functor as follows:

$$\begin{tikzcd} \mathrm{Sch}^{cl}_{\mathbb{Z}_p} \arrow[r, "X\mapsto X^{\Delta,cl}"] \arrow[d, hook] & \mathrm{Stacks}^{cl}_{\Sigma} \\ \mathrm{Sch}_{\mathbb{Z}_p} \arrow[r, "X\mapsto X^{\Delta}"] & \mathrm{Stacks}_{\Sigma} \arrow[u] \end{tikzcd}$$

But the functor $X\mapsto X^{\Delta,cl}$, $X\in \mathrm{Sch}^{cl}_{\mathbb{Z}_p}$, doesn't commute with $\varprojlim$'s. It doesn't have to (because the functor $\mathrm{Sch}^{cl}_{\mathbb{Z}_p} \hookrightarrow \mathrm{Sch}_{\mathbb{Z}_p}$ doesn't commute with limits). Moreover, it has no chance to commute with $\varprojlim$'s. This is clear a priori: in $\mathrm{Sch}^{cl}_{\mathbb{Z}_p}$ we have $X \times_{X\times X} X \xleftarrow{\sim} X$, so commutation with $\varprojlim$'s would imply that $\underbrace{X^{\Delta,cl} \times_{X^{\Delta,cl}\times_{\Sigma} X^{\Delta,cl}} X^{\Delta,cl}}_{\text{Inertia stack of } X^{\Delta,cl} \text{ over } \Sigma} \xleftarrow{\sim} X^{\Delta,cl}$; this would ~~imply th~~ mean that for any map $\mathrm{Spec}\, R\to\Sigma$, $X^{\Delta,cl}\times_{\Sigma} \mathrm{Spec}\, R$ is a space (i.e., ~~poi~~ all points have no nontrivial automorphisms). ~~But $(\mathrm{Spf}\,\mathbb{Z}_p)^{\Delta}=\Sigma$ is not a space!~~ ~~(In fact,~~ We will see that this is false for $X=\mathbb{A}^1$.

Test rings. $\mathrm{Rings}^{cl}_p := \{R\in \mathrm{Rings}^{cl} \mid p \text{ nilpotent in } R\}$.

$\mathrm{Rings}_p := \{R\in\mathrm{Rings} \mid R^{cl}\in \mathrm{Rings}^{cl}\}$.

Fact. ~~Keep it on the blackboard!~~ $\varinjlim_n \{\text{(derived)}\ \mathbb{Z}/p^n\mathbb{Z}\text{-algebras}\} \xrightarrow{\sim} \mathrm{Rings}_p$.

Proof: see file "My version of Akhil's explanation.pdf".

The functor $X\mapsto X^{\Delta}$ is first defined for $X\in \mathrm{Aff}_{\mathbb{Z}_p}$. The construction turns out to be etale-local, so it extends to $\mathrm{Sch}_{\mathbb{Z}_p}$.

Let $X\in\mathrm{Aff}_{\mathbb{Z}_p}$, $R\in\mathrm{Rings}_p$. Want an ∞-groupoid $X^{\Delta}(R)$ equipped with a map $X^{\Delta}(R)\longrightarrow \Sigma(R)$.

Recall $\Sigma(R)$. An object of $\Sigma(R)$ is (P,ξ), where $P\in W(R)$-mod and $\xi: P\to W(R)$ satisfy some conditions:

$R\in\mathrm{Rings}^{cl}_p \Rightarrow$ P should be invertible, in cohomological degree 0.
Two conditions for ξ: something is nilpotent, something else is invertible.

$R\in \mathrm{Rings}_p \Rightarrow$ the conditions should hold after base change $R\to R^{cl}$.

Def. Let $X = \mathrm{Spf}\, A$. An object of $X^{\Delta}(R)$ is (P, ξ, α), where $(P,\xi) \in \Sigma(R)$, $\alpha \in \mathrm{Hom}(A, \mathrm{Cone}(P \xrightarrow{\xi} W(R)))$.

Here Hom is computed in the ∞-category of derived rings, so Hom is a "space" or an ∞-groupoid; accordingly, α is a point of the space or an object of the ∞-groupoid.

Remark. The functor $X \mapsto X^{\Delta}$ commutes with $\varprojlim$'s. Indeed, $\mathrm{Hom}(\varinjlim_i A_i, B) = \varprojlim_i \mathrm{Hom}(A_i, B)$. We use this for $B = \mathrm{Cone}(\xi)$.

What is $X^{\Delta}(R)$ if X, R are classical? Then $\mathrm{Cone}(P \xrightarrow{\xi} W(R))$ is slightly derived, so $\mathrm{Hom}(A, \mathrm{Cone}(P \xrightarrow{\xi} W(R)))$ is a 1-groupoid, which was described (or defined) in winter: an object of this groupoid is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & P & \xrightarrow{v} & \tilde{A} & \to & A & \to & 0 \\ & & & \underset{\xi}{\searrow} & \downarrow f & & & & \\ & & & & W(R) & & & & \end{array}$$

The upper row is a ring extension.
f is a ring morphism
v is $\tilde{A}$-linear ($\tilde{A}$ acts on P via f).
If you wish, v is f-linear.

Lemma. Let $R/\mathbb{F}_p$ be (classical and) perfect. Then $X^{\Delta}(R) = X(R)$ $(= X^{\mathrm{perf}}(R)$, where $X^{\mathrm{perf}} := ((X \otimes \mathbb{F}_p)^{cl})^{\mathrm{perf}})$.

Proof. R perfect $\Rightarrow \Sigma(R) = \mathrm{point} = \{p\}$, $p \in \Sigma(R)$ (this was given as an exercise). So $\mathrm{Cone}(P \xrightarrow{\xi} W(R)) = \mathrm{Cone}(W(R) \xrightarrow{p} W(R)) = R$. (We have used perfectness again.) ∎

Let us return to the derived setting.

Prop. $\varinjlim_n (X \otimes \mathbb{Z}/p^n\mathbb{Z})^{\Delta} \xrightarrow{\sim} X^{\Delta}$.

Lemma. Let $R \in \mathrm{Rings}_p$, $(P, \xi) \in \Sigma(R)$. Then $\mathrm{Cone}(P \xrightarrow{\xi} W(R)) \in \mathrm{Rings}_p$.

The lemma reduces to the following exercise (see file "Nilpotence lemma.pdf").

Exercise. Let $R \in \mathrm{Rings}_p^{cl}$, $\xi \in W^{(1)}(R)$. Then $W(R)/(\xi) \in \mathrm{Rings}_p^{cl}$ (which means that $\xi \mid p^n$ for some n).

Proof of the Proposition. Let $X = \mathrm{Spf}\, A$, $(P, \xi) \in \Sigma(R)$. Let $B := \mathrm{Cone}(P \xrightarrow{\xi} W(R))$.

Want: $\mathrm{Hom}(A, B) = \varinjlim_n \mathrm{Hom}(A \otimes \mathbb{Z}/p^n\mathbb{Z}, B)$. This is true for any $B \in \mathrm{Rings}_p$ and any derived ring A, see file "My version of Akhil's explanations.pdf".

Prop. $\forall X \in \mathrm{Aff}_{\mathbb{Z}_p}$ the stack X^Δ is algebraic over Σ.

① Reduction to the case $X = \mathbb{A}^1 (= \mathbb{A}^1_{\mathbb{Z}_p})$ using commutation with $\varprojlim$'s and the following

Remark. Any $X \in \mathrm{Aff}_{\mathbb{Z}_p}$ can be represented as $\varprojlim (X^0 \rightrightarrows X^1 \rightrightarrows X^2 \rightrightarrows \dots)$, where $X^\bullet$ is a cosimplicial scheme and each X^i is an affine space (over $\mathbb{Z}_p$).

② Let $X = \mathbb{A}^1$. An object of $(\mathbb{A}^1)^\Delta(R)$ is (P, ξ, α), where $(P,\xi) \in \Sigma(R)$, $\alpha \in \mathrm{Cone}(P \xrightarrow{\xi} W(R))$. So $(\mathbb{A}^1)^\Delta \times_\Sigma W^{(1)} = \mathrm{Cone}(W \times W^{(1)} \to W \times W^{(1)})$, $(x,\xi) \mapsto (x\xi, \xi)$, which is a stack algebraic over $W^{(1)}$. [Here $W \times W^{(1)}$ is viewed as ring scheme over $W^{(1)}$]

Explicitly: $(\mathbb{A}^1)^\Delta = (W \times W^{(1)})/G$, $G = \begin{pmatrix} 1 & W \\ 0 & W^\times \end{pmatrix} \subset GL(2, W)$.

(G acts on W^2 preserving $W \times W^{(1)}$). Note: $\begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix}\begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} y + x\xi \\ u\xi \end{pmatrix}$.

Fact. If $X \hookrightarrow Y$ is a closed immersion the map $X^\Delta \to Y^\Delta$ is schematic and affine. (This should follow formally from commutation with limits.) In the classical setting, see file "Prismatization of closed embeddings.pdf".

Fact. The functor $X \mapsto X^\Delta$ (where X is affine) is uniquely determined by two properties: commutation with $\varprojlim$'s and the above description of $(\mathbb{A}^1)^\Delta$ as a ring stack over Σ (This is how X^Δ was defined in winter.) Idea: commutation implies that for fixed R the functor $A \mapsto (\mathrm{Spf}\, A)^\Delta(R)$ is corepresentable by the derived ring $\mathcal{Y}(R) = (\mathbb{A}^1)^\Delta(R)$).

A recipe for computing $(X^\Delta)^{cl}$, $X = \mathrm{Spf}\, A$.

Let $A_\bullet \to A$ be a polynomial resolution. Let $X^i = \mathrm{Spf}\, A_i$. We know the cosimplicial stack $(X^\bullet)^\Delta$ (including the morphisms between $(X^i)^\Delta$).

Claim. $(X^\Delta)^{cl} = [\varprojlim ((X^0)^\Delta \rightrightarrows (X^1)^\Delta \rightrightarrows (X^2)^\Delta)]^{cl}$ Good news: $X^3, X^4, \dots$ irrelevant

$$\cap \leftarrow \text{closed embedding}$$

$$[\varprojlim ((X^0)^\Delta \rightrightarrows (X^1)^\Delta)]^{cl} = \text{"naive guess"}$$

Proof. To recover $(X^\Delta)^{cl}$, it's enough to know $X^\Delta(R)$ for R classical.

Set $\Gamma^\bullet = (X^\bullet)^\Delta(R)$; this is a cosimplicial $\underline{\underline{1}}$-groupoid:

$$\Gamma^0 \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} \Gamma^1 \underset{\partial^{12},\, \partial^{02}}{\overset{\partial^{01}}{\rightrightarrows}} \Gamma^2 \rightrightarrows \dots$$

Fact: $\varprojlim \Gamma^\bullet$ is the groupoid of pairs (γ, f), where $\gamma \in \Gamma^0$ and $f : \partial^0(\gamma) \xrightarrow{\sim} \partial^1(\gamma)$ satisfies the "cocycle relation" $\partial^{02}(f) = \partial^{12}(f)\partial^{01}(f)$ (familiar from descent theory!). (If you forget about the "cocycle relation", you only get an upper bound.)

$\Sigma_n := (\operatorname{Spec} \mathbb{Z}/p^n\mathbb{Z})^\Delta$. $\Sigma_n = ?$

Promised: $\Sigma_1 = \operatorname{Spf} \mathbb{Z}_p$. Know: $\varinjlim_n \Sigma_n = \Sigma$. So Σ_n's "interpolate" between the formal scheme $\operatorname{Spf} \mathbb{Z}_p$ and the "very stacky" Σ.

<u>Description of Σ_n.</u> Let $\mathrm{Fact}_n \subset \underset{\xi}{W^{(1)}} \times \underset{x}{W}$ be the derived subscheme $\xi x = p^n$ ("Fact" stands for "factorizations" of p^n).

<u>Exercise.</u> Fact_n is classical.

(Lemma: the multiplication map $W_r \times W_r \to W_r$ is flat, $\forall r$. This can be checked by showing that the fibers have the "expected" dimension r.)

<u>Exercise.</u> $\Sigma_n = \mathrm{Fact}_n / W^\times$, where $u \in W^\times$ takes (ξ, x) to $(\xi u, x u^{-1})$. <u>Hint</u>: $\operatorname{Spec} \mathbb{Z}/p^n\mathbb{Z} = \varprojlim (\operatorname{Spf} \mathbb{Z}_p \underset{0}{\overset{p^n}{\rightrightarrows}} \mathbb{A}^1)$.

<u>Conjecture.</u> X^Δ is classical if X is an l.c.i.

<u>The case $n=1$.</u> If $p = \xi x$ and $\xi \in W^{(1)}(R)$ then $x \in W^\times(R)$.

So $\mathrm{Fact}_1 = W^\times$, $\Sigma_1 = \mathrm{Fact}_1 / W^\times = \operatorname{Spf} \mathbb{Z}_p$.

<u>Remark.</u> The map $\underbrace{\Sigma_1}_{= \operatorname{Spf} \mathbb{Z}_p} \to \Sigma$ corresponds to $p \in \Sigma(\mathbb{Z}_p)$. This is clear from ~~the~~ above (and also from the fact that $\Sigma^F(\mathbb{Z}_p) = \{p\}$).

<u>Exercise.</u> If $X = \operatorname{Spf} A$ is over $\mathbb{F}_p$ then $X^\Delta(R) = \operatorname{Hom}_{\mathbb{F}_p}(A, \operatorname{Cone}(W(R) \xrightarrow{p} W(R)))$ (the cone is an $\mathbb{F}_p$-algebra, so $\operatorname{Hom}_{\mathbb{F}_p}$ makes sense).

(This is how X^Δ was defined in winter.)

~~Particular case. If $R/\mathbb{F}_p$ is perfect then $X^\Delta(R) = \operatorname{Hom}_{\mathbb{F}_p}(A, R) = X(R)$.~~

<u>Solution of Exercise.</u> An object of $X^\Delta(R)$ is (P, ξ, α), where $(P, \xi) \in \Sigma(R)$, $\alpha \in \operatorname{Hom}(A, \operatorname{Cone}(P \xrightarrow{\xi} W(R)))$. Note: $\operatorname{Hom}(\mathbb{F}_p, \operatorname{Cone}(P \xrightarrow{\xi} W(R))) = \operatorname{Isom}((P, \xi), (W(R), p))$. So $X^\Delta(R)$ is the fiber of the map $\operatorname{Hom}(A, \operatorname{Cone}(W(R) \xrightarrow{p} W(R))) \to \operatorname{Hom}(\mathbb{F}_p, \operatorname{Cone}(W(R) \xrightarrow{p} W(R)))$ over the canonical point of $\operatorname{Hom}(\mathbb{F}_p, \operatorname{Cone}(W(R) \xrightarrow{p} W(R)))$. ∎

Prismatization table (commented below). In this table (n') is a particular case of (n)

In these examples X^Δ is (almost) scheme

	X	X^Δ
(1)	$\operatorname{Spec} B$, $B/\mathbb{F}_p$ perfect	$\operatorname{Spf} W(B)$
(2)	$\operatorname{Spf} W(B)$, $B/\mathbb{F}_p$ perfect	$\operatorname{Spf} W(B) \times \Sigma$
(3)	$\operatorname{Spec} B$, $B/\mathbb{F}_p$ semiperfect	$\operatorname{Spf} A_{cris}(B)$
(3′)	$\operatorname{Spec} \mathcal{O}_C/(p)$	$\operatorname{Spf} A_{cris}$
(4′)	$\operatorname{Spf} \mathcal{O}_C$	$\operatorname{Spf} A_{inf} = \operatorname{Spf} W(B)$, where $B = (\mathcal{O}_C/p)^\flat$ (Fontainization). ("projective limit perfection")
(4)	$\operatorname{Spf} W(B)/(\eta)$, where $B/\mathbb{F}_p$ perfect, $\eta = \sum_{i=0}^{\infty} V^i[b_i]$, $b_1 \in B^\times$, $B \xrightarrow{\sim} \varprojlim_n B/(b_0^n)$ (b_0 "kind of nilpotent")	$\operatorname{Spf} W(B)$ (the topology on $W(B)$ and the morphism $\operatorname{Spf} W(B) \to \Sigma$ depends on η).

(1) is a particular case of (3) and also of (4) (take $\eta = p = V(1)$).

Rings of the form $W(B)/(\eta)$, where B, η are as in (4), are called (integral) perfectoid. See Lecture IV of Bhatt's Columbia course on prismatic cohomology.

Comments on the prismatization table.

1 a) $B/\mathbb{F}_p$ is perfect if $Fr: B \xrightarrow{\sim} B$ is an isomorphism. Then B is automatically classical. ~~Eo~~ This is a corollary of the following fact (see proof of Prop. 21 on p. 5 of the file "Akhil's ~~sou~~ perfectoid spaces course. pdf"): if A is a derived $\mathbb{F}_p$-algebra then the map $Fr: H^i(A) \to H^i(A)$ is zero for $i \neq 0$.

b) If A is over $\mathbb{F}_p$ then the map $(\operatorname{Spec} A)^\Delta \to \Sigma$ is the composed map $(\operatorname{Spec} A)^\Delta \to (\operatorname{Spec} \mathbb{F}_p)^\Delta = \operatorname{Spf} \mathbb{Z}_p \xrightarrow{p} \Sigma$.

3 a) $B/\mathbb{F}_p$ is semi-perfect if $Fr: B \to B$ is an epimorphism (i.e., induces a epimorphism $H^0(B) \to H^0(B)$). Then $\exists$ epimorphism $\tilde{B} \twoheadrightarrow B$ with $\tilde{B}$ perfect (e.g., $\tilde{B} = B^\flat := \varprojlim (B \xleftarrow{Fr} B \leftarrow \ldots)$, this is the "inverse limit perfection" or "Fonteinization").

b) The fact that $(\operatorname{Spec} B)^\Delta$ is a ~~formal~~ p-adic scheme is clear from above: indeed, $\operatorname{Spec} B$ is a closed subscheme of $\operatorname{Spec} \tilde{B}$, so the map $(\operatorname{Spec} B)^\Delta \to (\operatorname{Spec} \tilde{B})^\Delta = \operatorname{Spf} W(\tilde{B})$ is schematic.

c) The ~~fact~~ equality $(\operatorname{Spec} B)^\Delta = \operatorname{Spf} A_{cris}(B)$ can be viewed as a definition of $A_{cris}(B)$. The ring defined by Fontaine is $A_{cris}(B)^{cl}$ (because he used the non-derived PD hull). ~~The~~ It is known (Scholze-Weinstein!) that $A_{cris}(B)$ is classical if $B = \tilde{B}/(f_1, \ldots, f_m)$, where $\tilde{B}$ is a perfect $\mathbb{F}_p$-algebra and $f_1, \ldots, f_m$ form a regular sequence. E.g., $O_C/(p)$ is in this class (for $m=1$).

4 a) The derived quotient by η is classical because η is not a zero-divisor, see Lemma 2.33 of file "Bhatt-Scholze prismatic article draft2.pdf").

Proof. Suppose $\sum V^i [b_i] \cdot \sum\limits_j V^j [x_j] = 0$. Then $b_0 x_0 = 0$, $b_0^p x_1 + x_0^p b_1 = 0$, so $x_0^{2p} = -\frac{x_1}{b_1} (b_0 x_0)^p = 0$, so $x_0 = 0$. And so on. ∎

(b) To define $\operatorname{Spf} W(B)$ I have to specify a topology on $W(B)$. (It depends on b_0!) Here are two equivalent definitions. First, the topology of the limit $\varprojlim\limits_{m,n} W_m(B/(b_0^n))$. Second, the I-adic topology, where $I = (p, \eta) = (p, [b_0])$.

c) Let me describe the map $\mathrm{Spf}\, W(B) \to \Sigma = W^{(1)}/W^\times$. We'll even construct $\xi \in \mathrm{Mor}(\mathrm{Spf}\, W(B), W^{(1)}) \subset \mathrm{Mor}(\mathrm{Spf}\, W(B), W) = W(W(B))$. Namely, $\xi = f(\eta)$, where $f: W(B) \to W(W(B))$ comes from the comonad structure on ~~B~~ the functor W (so f is defined for <u>any</u> ring B). The comonad structure comes from ~~the adjuction $\mathrm{Hom}(R, W(R')) = \mathrm{Hom}(\Phi(R), R')$~~, ~~where~~ Joyal's theory. We have $\{\delta\text{-rings}\} \underset{\mathrm{Forget}^*}{\overset{\mathrm{Forget}}{\rightleftarrows}} \{\text{rings}\}$, and $W = \mathrm{Forget} \circ \mathrm{Forget}^*$. This gives $W \to W \circ W$.

<u>Exercise.</u> $f([B]) = [[B]]$.

~~For~~ This formula holds for <u>any</u> ring B. ~~If~~ Since our B is a perfect $\mathbb{F}_p$-algebra, it uniquely determines f because any element of $W(B)$ has the form $\sum_{i=0}^{\infty} [b_i] p^i$, $b_i \in B$.

<u>Remarks for myself.</u> ① The map $f: W(B) \to W(W(B))$ can be easily ~~de~~ described ~~via~~ in Joyal's coordinates: ~~x~~ if $y \in W(B)$ has Joyal's coordinates $y_0, y_1, \ldots$ then $f(y)$ has Joyal's coordinates $z_{m,n} = y_{m+n}$. E.g., if $y_i = 0$ for $i > 0$ then $z_{m,n} = 0$ if $(m,n) \neq (0,0)$; this means that $f([B]) = [[B]]$.

② The above element $\xi \in \mathrm{Mor}(\mathrm{Spf}\, W(B), W^{(1)})$ is in $\mathrm{Mor}(\mathrm{Spf}\, W(B), W^{(1)})$. ~~Because~~ Indeed, let $R_{m,n} := W_m(B/(b_0^n))$, then the image of $\xi \in W(W(B))$ in $W(R_{m,n})$ belongs to $W^{(1)}(R_{m,n})$.