Bhatt-Scholze computed the prismatic cohomology of a p-adic scheme etale over A^n. Let me formulate their answer in the stacky language in the particular case of $\mathbb{A}^1\setminus\{0\}$.

Crystalline cohomology.

$$\widehat{\mathbb{Z}_p[x,x^{-1}]} \xrightarrow{d} \widehat{\mathbb{Z}_p[x,x^{-1}]}\cdot\frac{dx}{x}, \quad \varphi(x)=x^p$$

Basis: $x^n \qquad \frac{1}{n}\frac{d(x^n)}{x^n}$

$$H^i_{cris}(\mathbb{A}^1_{\mathbb{F}_p}\setminus\{0\}) = \begin{cases} 0, & i\neq 0,1 \\ \mathbb{Z}_p, & i=0 \\ \bigoplus_n M_n, & i=1 \end{cases} \qquad \varphi: M_n\to M_{pn}$$

$M_0=\mathbb{Z}_p$, $\varphi|_{M_0}=p$, so $(M_0,\varphi)=\mathbb{Z}_p(-1)$

$n\neq 0$, $n=p^k m$, $p\nmid m$ $\Rightarrow$

$$\begin{array}{ccc} M_n & = & p^{-k}\mathbb{Z}/\mathbb{Z} \\ \varphi\downarrow & & \downarrow \\ M_{pn} & = & p^{-k-1}\mathbb{Z}/\mathbb{Z} \end{array} \quad \leftarrow \text{natural inclusion.}$$

Prismatic cohomology.

$$\begin{tikzcd} \mathbb{A}^1\setminus\{0\} \arrow[r, "\pi"] & \mathrm{Spf}\,\mathbb{Z}_p & (\mathbb{A}^1\setminus\{0\})^\Delta \arrow[r, "\pi^\Delta"] & \Sigma \\ \mathbb{A}^1_{\mathbb{F}_p}\setminus\{0\} \arrow[r, "\bar\pi"] & \mathrm{Spec}\,\mathbb{F}_p \arrow[u, hook] & (\mathbb{A}^1_{\mathbb{F}_p}\setminus\{0\})^\Delta \arrow[r, "\bar\pi^\Delta"] & \mathrm{Spf}\,\mathbb{Z}_p \arrow[u, "p"] \end{tikzcd}$$

$R^i\pi^\Delta_*\mathcal{O}=?$

$p^*R\pi^\Delta_*\mathcal{O}$ $=$ crystalline cohomology.

$$R^i\pi^\Delta_*\mathcal{O} = \begin{cases} 0, & i\neq 0,1 \\ \mathcal{O}_\Sigma, & i=0 \\ \bigoplus_n \mathcal{M}_n, & i=1 \end{cases} \qquad \varphi: F^*\mathcal{M}_n\to\mathcal{M}_{pn}$$

$(\mathcal{M}_0, \varphi: F^*\mathcal{M}_0\to\mathcal{M}_0) = \mathcal{O}_\Sigma\{-1\}$ Recall: $\Delta_i=(F^i)^*\Delta_0$

$n\neq 0$, $n=p^k m$, $p\nmid m$ $\Rightarrow$ $\mathcal{M}_n=\mathcal{O}_\Sigma(\Delta_0+\dots\Delta_{k-1})/\mathcal{O}_\Sigma$

$$\begin{array}{ccc} F^*\mathcal{M}_n & = & \mathcal{O}_\Sigma(\Delta_1+\dots+\Delta_k)/\mathcal{O}_\Sigma \\ \varphi\downarrow & & \downarrow \\ \mathcal{M}_{pn} & = & \mathcal{O}_\Sigma(\Delta_0+\dots+\Delta_k)/\mathcal{O}_\Sigma \end{array}$$

$p: \mathrm{Spf}\,\mathbb{Z}_p\to\Sigma$

$p^*\mathcal{O}_\Sigma\{-1\}=\mathbb{Z}_p(-1)$

$p^*\Delta_i = \mathrm{Spec}\,\mathbb{F}_p\subset\mathrm{Spf}\,\mathbb{Z}_p$.

Notice that $\forall n$ $\mathrm{Supp}\,\mathrm{Coker}(F^*\mathcal{M}_n\xrightarrow{\varphi}\mathcal{M}_{n+1})\subset\Delta_0$.

Bhatt-Scholze: Let $\pi: X\to S$ be smooth, $X,S\in Sch_{\mathbb{Z}_p}$, then $\mathrm{Cone}(F^*(\pi_\Delta)_*\mathcal{O}_{X^\Delta}\xrightarrow{\varphi}\mathcal{O}_{X^\Delta})$ is supported on $n\cdot(S^\Delta\times_\Sigma\Delta_0)$, $n\in\mathbb{N}$. (Probably $n=\dim X$. Probably something along these lines can be formulated without S assuming smoothness and even without assuming finite type.

Prismatic cohomology is a cohomology theory for $Sch_{\mathbb{Z}_p}$, which knows about "all other" cohomology theories. Most important, it knows the etale cohomology of the generic fiber, but I don't understand this. Let's discuss something easier:

de Rham and Hodge-Tate cohomology.

$$\begin{array}{ccccccc} X^{dR} & \longrightarrow & X^{\Delta} & \longleftarrow & X^{\underline{HT}} & \longleftarrow & X^{HT} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spf}\mathbb{Z}_p & \xrightarrow{p} & \Sigma & \hookleftarrow & \Delta_0 & \xleftarrow{V(1)} & \operatorname{Spf}\mathbb{Z}_p \\ \| & & \| & & & & \\ (\operatorname{Spec}\mathbb{F}_p)^{\Delta} & \to & (\operatorname{Spf}\mathbb{Z}_p)^{\Delta} & & & & \end{array}$$

Roughly, each point of Σ gives a cohomology theory. Roughly, Σ is the stack parametrizing cohomology theories.

by commutation with $\varinjlim$'s

So $X^{dR} = (X \otimes \mathbb{F}_p)^{\Delta}$ depends only on $X \otimes \mathbb{F}_p$

$X^{dR} \otimes \mathbb{F}_p = X^{HT} \otimes \mathbb{F}_p$ because $V(1), p \in \Sigma(\mathbb{Z}_p)$ have equal images in $\Sigma(\mathbb{F}_p)$.

Here $\Gamma = R\Gamma$

<u>Fact</u> $\cancel{R}\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) = \cancel{R}\Gamma_{dR}(X)$ (<u>derived</u> de Rham if $X/\mathbb{Z}_p$ is not smooth)

<u>"Proof"</u>. $X^{dR} = (X \otimes \mathbb{F}_p)^{\Delta}$, so $\cancel{R}\Gamma(X^{dR}, \mathcal{O}_{X^{dR}})$ = crystalline cohomology of $X \otimes \mathbb{F}_p$ " $\cancel{R}\Gamma_{dR}(X)$ ∎

$\cancel{R}\Gamma(X^{HT}, \mathcal{O}_{X^{HT}})$ = "Hodge-Tate cohomology" of X. The name is due to Bhatt-Scholze; they have a good reason for it, but I don't understand ~~it~~. More important, I don't understand this cohomology theory.

~~$X^{\underline{HT}} \otimes \mathbb{F}_p = X^{dR} \otimes$~~

" $\Gamma_{dR/\mathbb{F}_p}(X \otimes \mathbb{F}_p, \mathcal{O})$ ".

Good news: $\Gamma(X^{HT}, \mathcal{O}_{X^{HT}}) \otimes \mathbb{F}_p = \Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) \otimes \mathbb{F}_p$ because $X^{dR} \otimes \mathbb{F}_p = X^{HT} \otimes \mathbb{F}_p$.

<u>Fact</u>. Let $X/\mathbb{Z}_p$ be smooth. Then $\exists$ canonical spectral sequence $E_2^{p,q} = H^p(X, \Omega^q_X) \Rightarrow H^{p+q}(X^{HT}, \mathcal{O}_{X^{HT}})$ whose reduction mod p is the <u>conjugate</u> spectral sequence for the $dR/\mathbb{F}_p$ cohomology of $\mathbb{F}_p$ (the one that comes from Cartier isomorphism). ~~So the good news is~~ The statement (about reduction mod p) is ~~not clean~~ ~~even~~ somewhat sloppy; to correct it, one has to talk about a filtration on $(R)\Gamma(X^{HT}, \mathcal{O})$ rather than a spectral sequence.

Good news: the conjugate spectral sequence exists not only mod p (but converges to something mysterious).

Geometry behind cohomological statements.

$p,\ V(1) \in \Sigma(\mathbb{Z}_p)$, $F(p) = p$, $F(V(1)) = p$.

So ~~we the~~ $F: X^{\Delta} \to X^{\Delta}$ induces $F: X^{dR} \to X^{dR}$, $F: X^{HT} \to X^{dR}$

(Equivalently, $F: X^{dR} \to X^{dR}$ comes from $X^{dR} = (X \otimes \mathbb{F}_p)^{\Delta}$ and $Fr: X \otimes \mathbb{F}_p \to X \otimes \mathbb{F}_p$).

We'll construct a canonical factorization
$$\begin{tikzcd} X^{HT} \arrow[rr, "F"] \arrow[dr] & & X^{dR} \\ & X \arrow[ur] & \end{tikzcd}$$

We can assume X affine. The functors $X \mapsto X^{HT}$, $X \mapsto X^{dR}$ commute with $\varprojlim$'s. So it suffices to construct a commutative diagram of ring stacks
$$\begin{tikzcd} (\mathbb{A}^1)^{HT} \arrow[rr, "F"] \arrow[dr] & & (\mathbb{A}^1)^{dR} \\ & \mathbb{A}^1 \arrow[ur] & \end{tikzcd}$$

$(\mathbb{A}^1)^{HT} = \mathrm{Cone}(W \xrightarrow{\cdot V(1)} W)$

$\mathbb{A}^1 = \mathrm{Cone}(W \xrightarrow{V} W)$

$(\mathbb{A}^1)^{dR} = \mathrm{Cone}(W \xrightarrow{\cdot p} W)$

$$\begin{tikzcd} W \arrow[r, "\cdot V(1)"] \arrow[d, "F"] \arrow[dd, bend right=60, "F"'] & W \arrow[d, "id"] \arrow[dd, bend left=60, "F"] \\ W \arrow[r, "V"] \arrow[d, "id"] & W \arrow[d, "F"] \\ W \arrow[r, "\cdot p"] & W \end{tikzcd}$$

$x \cdot V(1) = VF(x)$

$FV = p$

Let us discuss the geometry of the maps $X^{HT} \to X$ and $X \to X^{dR}$.

Let us start with X^{dR}. More generally, given $\pi: X \to S$, let $X^{dR/S}$ be the fiber product
$$\begin{tikzcd} X^{dR/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{dR} \arrow[d] \\ S \arrow[r] & S^{dR} \end{tikzcd}$$

<u>Exercise</u>. Let $S' \to S$ be a map. Then $(X \times_S S')^{dR/S'} = X^{dR/S} \times_S S'$.

Have $X \to X^{dR/S}$. ~~Let Γ...~~ Define a groupoid $\Gamma \rightrightarrows X$ by $\Gamma := X \times_{X^{dR/S}} X$.

<u>Fact</u>. Assume X/S smooth. Then

(i) $\Gamma =$ PD-hull of $X_{diag} \subset X \times_S X$,

(ii) Γ is flat (i.e., $\Gamma \rightrightarrows X$ are flat),

(iii) $X/\Gamma \xrightarrow{\sim} X^{dR/S}$.

<u>Exercise</u>. Prove if $X = \mathbb{A}^1 \times S$.

The above theorem says that $X^{dR/S}$ is similar to Simpson's X^{dR}.

~~Corollary~~ Let $\pi^{dR/S}: X^{dR/S} \to S$.

Cor. $(\pi^{dR/S})_* \mathcal{O} = \{$relative ~~der~~ dR cohomology of $X/S\}$.

Essentially explained in winter. The idea of dR cohomology is to pass from the groupoid Γ to its Lie algebroid. ~~&~~ This could be explained if $S = \mathrm{Spf}\,\mathbb{Z}_p$, $X = \mathbb{A}^1 = \mathbb{G}_a$. Then

$$R\Gamma(\mathbb{G}_a^\#, \widehat{\mathbb{Z}_p[x]}) \overset{!}{=} R\Gamma(\mathrm{Lie}\,\mathbb{G}_a, \widehat{\mathbb{Z}_p[x]}) = (\widehat{\mathbb{Z}_p[x]} \xrightarrow{d/dx} \widehat{\mathbb{Z}_p[x]}).$$

Note: $\mathrm{Lie}\,\mathbb{G}_a$ is a direct summand of $\mathrm{Lie}\,\mathbb{G}_a^\#$.

Now let us discuss $f: X^{HT} \to X$ assuming X is smooth over $\mathbb{Z}_p$ (the case of a smooth map $X \to S$ is "similar").

Fact. X^{HT} is a gerbe over X banded by $\Theta_X^\#$ (i.e., the PD-hull of $X \overset{0}{\hookrightarrow} \Theta_X$). So locally (in fact, even Zariski-locally) $X^{HT} \simeq \{$classifying stack of $\Theta_X^\#\}$

Note: the above gerbe is $\mathbb{G}_m^\#$-equivariant because one has X^{HT}.

Corollary. $R^i f_* \mathcal{O}_{X^{HT}} = \Omega^i_X$ (But $Rf_*\mathcal{O}_{X^{HT}} \neq \bigoplus_i \Omega^i[-i]$ in general).

Get $H^p(X, \Omega^q_X) \Rightarrow H^{p+q}(X^{HT}, \mathcal{O}_{X^{HT}})$, as promised.

Fact. Let $Y := X \otimes \mathbb{F}_p$. ~~&~~ Then $(Rf_*\mathcal{O}_{X^{HT}}) \otimes \mathbb{F}_p = \{0 \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \to \cdots\}$

Not surprising because $X^{HT} \otimes \mathbb{F}_p = X^{dR} \otimes \mathbb{F}_p$. The cohomology sheaves are Ω^i_Y (Cartier isomorphism).

$X^{HT} \otimes \mathbb{F}_p = ?$

$X^{HT} \otimes \mathbb{F}_p = X^{dR} \otimes \mathbb{F}_p = Y^{dR/\mathbb{F}_p} = Y/\Gamma$, where $\Gamma \rightrightarrows Y$ is the PD-hull of $Y_{diag} \subset Y \times Y$. Let $\Gamma' \subset Y \times Y$ be the equivalence relation on Y corresponding to $Y \xrightarrow{Fr} Y$:

$$\begin{tikzcd} \Gamma' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "Fr"] \\ Y \arrow[r, "Fr"] & Y \end{tikzcd}$$

It is well known that $\Gamma \to Y \times Y$ factors as $\Gamma \overset{\text{faithfully flat}}{\twoheadrightarrow} \Gamma' \subset Y \times Y$. So $Y \xrightarrow{Fr} Y$ factors as $Y \to Y/\Gamma \xrightarrow{\text{gerbe}} Y$

The map $X^{HT} \otimes \mathbb{F}_p$ is nothing but $Y/\Gamma \to Y$.