

Bhatt-Scholze computed the prismatic cohomology of a p-adic scheme étale over A^n . Let me formulate their answer in the stacky language in the particular case of $A^1 \setminus \{0\}$.

Crystalline cohomology.

$$\mathbb{Z}_p[x, x^{-1}] \xrightarrow{d} \mathbb{Z}_p[x, x^{-1}] \cdot \frac{dx}{x}, \quad \varphi(x) = x^p$$

Basis:

$$H_{\text{cris}}^i(A_{\mathbb{F}_p}^1 \setminus \{0\}) = \begin{cases} 0, & i \neq 0, 1 \\ \mathbb{Z}_p, & i = 0 \\ \bigoplus_n M_n, & i = 1 \end{cases}$$

$$\varphi: M_n \rightarrow M_{pn}$$

$$M_0 = \mathbb{Z}_p, \quad \varphi|_{M_0} = p, \text{ so } (M_0, \varphi) = \mathbb{Z}_p(-1)$$

$$n \neq 0, \quad n = p^k m, \quad p \nmid m \Rightarrow \begin{matrix} M_n = p^{-k} \mathbb{Z}/\mathbb{Z} \\ \varphi \downarrow \\ M_{pn} = p^{-k-1} \mathbb{Z}/\mathbb{Z} \end{matrix} \leftarrow \text{natural inclusion.}$$

Prismatic cohomology.

$$A^1 \setminus \{0\} \xrightarrow{\pi} \text{Spt } \mathbb{Z}_p, \quad (A^1 \setminus \{0\})^\Delta \xrightarrow{\pi^\Delta} \text{Spt } \mathbb{Z}_p \times \Sigma \quad R^i \pi_* \mathcal{O} = ?$$

$$A_{\mathbb{F}_p}^1 \setminus \{0\} \xrightarrow{\bar{\pi}} \text{Spec } \mathbb{F}_p \quad (A_{\mathbb{F}_p}^1 \setminus \{0\})^\Delta \xrightarrow{\bar{\pi}^\Delta} \text{Spt } \mathbb{Z}_p \quad \begin{matrix} P^* R^i \pi_* \mathcal{O} \\ \text{crystalline cohomology.} \end{matrix}$$

$$R^i \pi_* \mathcal{O} = \begin{cases} 0, & i \neq 0, 1 \\ \mathcal{O}_\Sigma, & i = 0 \\ \bigoplus_n M_n, & i = 1 \end{cases}$$

$$\varphi: F^* M_n \rightarrow M_{pn}$$

$$(M_0, \varphi: F^* M_0 \rightarrow M_0) = \mathcal{O}_\Sigma(-1)$$

Recall: $\Delta_i = (F^i)^* \Delta_0$

$$n \neq 0, \quad n = p^k m, \quad p \nmid m \Rightarrow M_n = \mathcal{O}_\Sigma(\Delta_0 + \dots + \Delta_{k-1}) / \mathcal{O}_\Sigma$$

$$F^* M_n = \mathcal{O}_\Sigma(\Delta_1 + \dots + \Delta_k) / \mathcal{O}_\Sigma$$

$$p: \text{Spt } \mathbb{Z}_p \rightarrow \Sigma$$

$$P^* \mathcal{O}_\Sigma(-1) = \mathbb{Z}_p(-1)$$

$$P^* \Delta_i = \text{Spec } \mathbb{F}_p \subset \text{Spt } \mathbb{Z}_p.$$

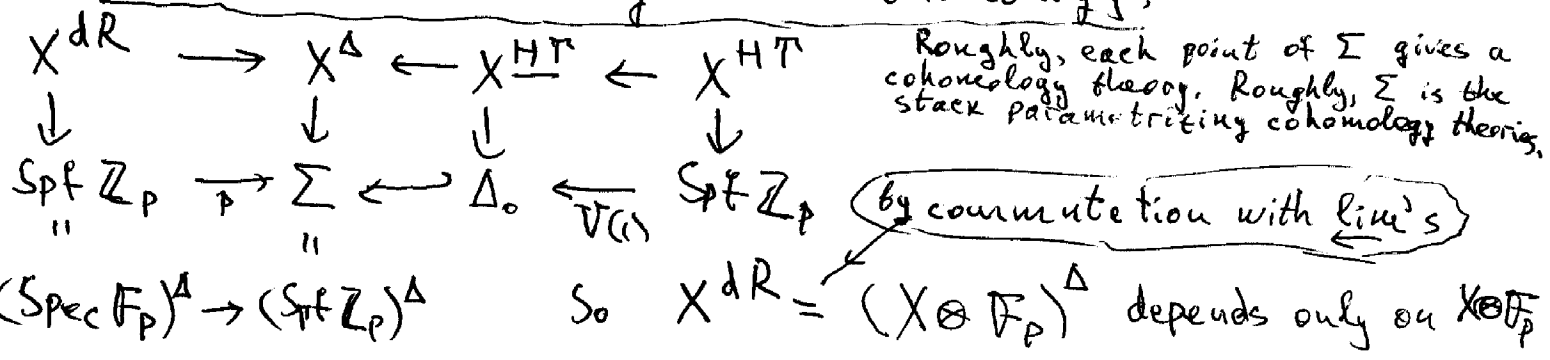
Notice that

$$\text{Supp Coker}(F^* M_n \xrightarrow{\varphi} M_{n+1}) \subset \Delta_0.$$

Bhatt-Scholze: Let $\pi: X \rightarrow S$ be smooth, $X, S \in \text{Sch } \mathbb{Z}_p$, then $\text{Cone}(F^*(\pi_\Delta)_* \mathcal{O}_{X^\Delta} \xrightarrow{\varphi} \mathcal{O}_{X^\Delta})$ is supported on $n \cdot (S^\Delta \times_\Sigma \Delta_0)$, $n \in \mathbb{N}$. (Probably $n = \dim X$, Probably something along these lines can be formulated without S smoothness and even without assuming finite type.

Prismatic cohomology is a cohomology theory for $\text{Sch}_{\mathbb{Z}_p}$, which knows about "all other" cohomology theories. Most important, it knows the étale cohomology of the generic fiber, but I don't understand this. Let's discuss something easier:

de Rham and Hodge-Tate cohomology,



$X^{dR} \otimes \mathbb{F}_p = X^{HT} \otimes \mathbb{F}_p$ because $V(1), P \in \Sigma(\mathbb{Z}_p)$ have equal images in $\Sigma(\mathbb{F}_p)$. Here $\Gamma = \mathbb{R}\Gamma$

Fact $\mathbb{R}\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) = \mathbb{R}\Gamma_{dR}(X)$ (derived de Rham if X/\mathbb{Z}_p is not smooth)

"Proof". $X^{dR} = (X \otimes \mathbb{F}_p)^\Delta$, so $\mathbb{R}\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) =$ crystalline cohomology of $X \otimes \mathbb{F}_p$

$\mathbb{R}\Gamma(X^{HT}, \mathcal{O}_{X^{HT}}) =$ "Hodge-Tate cohomology" of X . $\mathbb{R}\Gamma_{dR}(X)$

The name is due to Bhatt-Scholze; they have a good reason for it, but I don't understand ~~it~~. More important, I don't understand ~~this theory~~. $\mathbb{R}\Gamma_{dR}/\mathbb{F}_p(X \otimes \mathbb{F}_p, \mathcal{O})$.

$X^{HT} \otimes \mathbb{F}_p = X^{dR} \otimes \mathbb{F}_p$

Good news: $\Gamma(X^{HT}, \mathcal{O}_{X^{HT}}) \otimes \mathbb{F}_p = \Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) \otimes \mathbb{F}_p$ because $X^{dR} \otimes \mathbb{F}_p = X^{HT} \otimes \mathbb{F}_p$.

Fact. Let X/\mathbb{Z}_p be smooth. Then $E_2^{p,q} = H^p(X, \mathbb{Z}_X^q) \Rightarrow H^{p+q}(X^{HT}, \mathcal{O}_{X^{HT}})$ whose reduction mod p is the conjugate spectral sequence for the dR/\mathbb{F}_p cohomology of \mathbb{F}_p (the one that comes from Cartier isomorphism). So ~~the good news is~~ This statement is ~~not clear~~ about reduction mod p somewhat sloppy; to correct it, one has to talk about a filtration on $\mathbb{R}\Gamma(X^{HT}, \mathcal{O})$ rather than a spectral sequence. (but converges to something mysterious).

Good news: the conjugate spectral sequence exists not only mod p

(-3-)

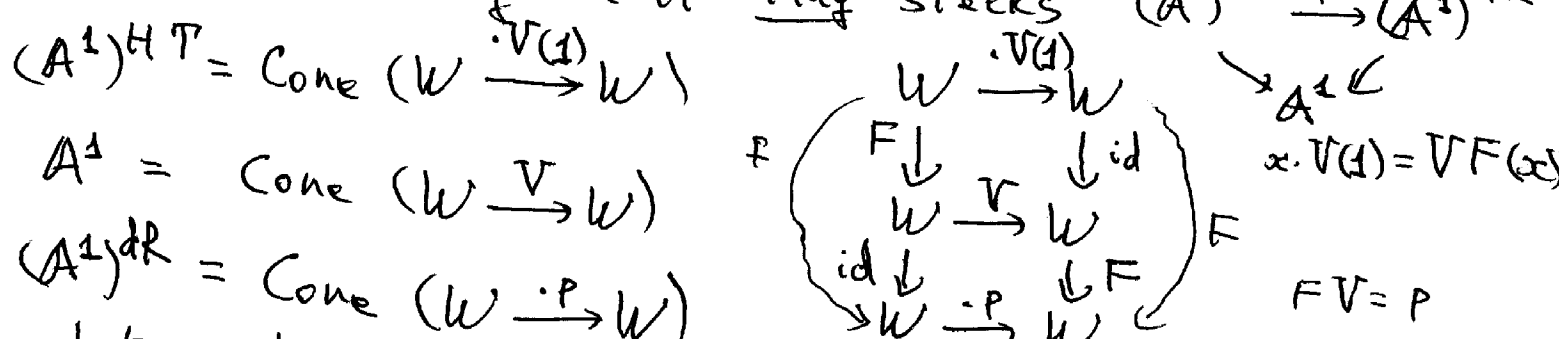
Geometry behind cohomological statements.

$P, V(1) \in \Sigma(\mathbb{Z}_p), F(p) = P, F(V(1)) = P.$

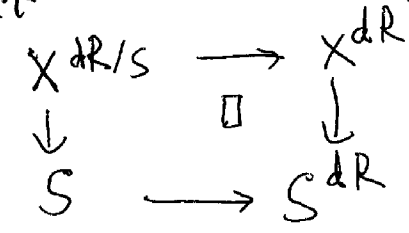
So we ~~have~~ $F: X^\Delta \rightarrow X^\Delta$ induces $F: X^{dR} \rightarrow X^{dR}, F: X^{HT} \rightarrow X^{dR}$
 (Equivalently, $F: X^{dR} \rightarrow X^{dR}$ comes from $X^{dR} = (X \otimes_{\mathbb{F}_p})^\Delta$ and $F: X \otimes_{\mathbb{F}_p} \rightarrow X \otimes_{\mathbb{F}_p}$).

We'll construct a canonical factorization $X^{HT} \xrightarrow{F} X^{dR}$

We can assume X affine. The functors $X \mapsto X^{HT}, X \mapsto X^{dR}$ commute with \lim 's. So it suffices to construct a commutative diagram of ring stacks



Let us discuss the geometry of the maps $X^{HT} \rightarrow X$ and $X \rightarrow X^{dR}$.
 Let us start with X^{dR} . More generally, given $\pi: X \rightarrow S$, let $X^{dR/S}$ be the fiber product



Exercise. Let $S' \rightarrow S$ be a map. Then $(X \times_S S')^{dR/S'} = X^{dR/S} \times_S S'$.

Have $X \rightarrow X^{dR/S}$. ~~Let $\Gamma \rightrightarrows X$~~ . Define a groupoid $\Gamma \rightrightarrows X$ by $\Gamma := X \times_{X^{dR/S}} X$.

Fact. Assume X/S smooth. Then

- (i) $\Gamma = \mathbb{P}^1$ -hull of $X_{\text{diag}} \subset X \times_S X$,
- (ii) Γ is flat (i.e., $\Gamma \rightrightarrows X$ are flat),
- (iii) $X/\Gamma \cong X^{dR/S}$.

Exercise. Prove if $X = A^1 \times S$.

The above theorem says that $X^{dR/S}$ is similar to Simpson's X^{dR} .

Corollary Let $\pi^{dR/S}: X^{dR/S} \rightarrow S$.

Cor. $(\pi^{dR/S})_* \mathcal{O} = \{ \text{relative } dR \text{ cohomology of } X/S \}$.

Essentially explained in winter. The idea of dR cohomology is to pass from the groupoid Γ to its Lie algebroid. This could be explained if $S = \text{Spt } \mathbb{Z}_p$, $X = A^1 = \mathbb{G}_a$. Then

$$R\Gamma(\mathbb{G}_a, \widehat{\mathbb{Z}_p[x]}) = R\Gamma(\text{Lie } \mathbb{G}_a, \widehat{\mathbb{Z}_p[x]}) = (\widehat{\mathbb{Z}_p[x]} \xrightarrow{d/dx} \widehat{\mathbb{Z}_p[x]}).$$

Note: Lie \mathbb{G}_a is a direct summand of \mathfrak{g}_a .
 Now let us discuss $f: X^{HT} \rightarrow X$ assuming X is smooth over \mathbb{Z}_p (the case of a smooth map $X \rightarrow S$ is "similar").

Fact. X^{HT} is a gerbe over X banded by $\mathbb{H}^\#$ (i.e., the PD-hull of $X \hookrightarrow \mathbb{H}^0$). So locally (in fact, even Zariski-locally) $X^{HT} \simeq \{ \text{classifying stack of } \mathbb{H}^\# \}$.
 Note: the above gerbe is $\mathbb{G}_m^\#$ -equivariant because one has $X^{HT} \simeq \{ \text{classifying stack of } \mathbb{H}^\# \}$.

Corollary. $R^i f_* \mathcal{O}_{X^{HT}} = \Omega_X^i$ (but $R^i f_* \mathcal{O}_{X^{HT}} \neq \bigoplus_i \Omega^i[-i]$ in general).

Get $H^p(X, \Omega_X^q) \Rightarrow H^{p+q}(X^{HT}, \mathcal{O}_{X^{HT}})$, as promised.

Fact. Let $Y := X \otimes \mathbb{F}_p$. Then $(R^i f_* \mathcal{O}_{X^{HT}}) \otimes \mathbb{F}_p = \{ 0 \rightarrow \mathcal{O}_Y \xrightarrow{d} \Omega_Y^1 \rightarrow \dots \}$.
 Not surprising because $X^{HT} \otimes \mathbb{F}_p = X^{dR} \otimes \mathbb{F}_p$. The cohomology sheaves are Ω_Y^i (Cartier isomorphism).

$X^{HT} \otimes \mathbb{F}_p = ?$
 $X^{HT} \otimes \mathbb{F}_p = X^{dR} \otimes \mathbb{F}_p = Y^{dR}/\mathbb{F}_p = Y/\Gamma$, where $\Gamma \rightrightarrows Y$ is the PD-hull of $Y \text{ diag} \subset Y \times Y$. Let $\Gamma' \subset Y \times Y$ be the equivalence relation on Y corresponding to $Y \xrightarrow{Fr} Y$:

$$\begin{array}{ccc} \Gamma' & \rightarrow & Y \\ \downarrow & \square & \downarrow Fr \\ Y & \xrightarrow{Fr} & Y \end{array}$$

It is well known that $\Gamma \rightarrow Y \times Y$ factors as $\Gamma \xrightarrow{\text{faithfully flat}} \Gamma' \subset Y \times Y$. So $Y \xrightarrow{Fr} Y$ factors as $Y \rightarrow Y/\Gamma \xrightarrow{\text{gerbe}} Y$.
 The map $X^{HT} \otimes \mathbb{F}_p$ is nothing but $Y/\Gamma \rightarrow Y$.