§1 Overview

$X/\mathbb{F}_q$ curve, $\cdots$, F , $\mathbb{A}$, $\mathcal{O}$

$G/\mathbb{F}_q$ red gp, connected, split   Assume G semisimple, without level

$\ell \neq p$, $\hat{G}/\overline{\mathbb{Q}}_\ell$ Langlands dual gp of G

We have defined the pre-stack of shtukas $\mathrm{Sht}_{G,I}$ over X^I: $\mathrm{Sht}_{G,I}(S) = \{(x_i), \mathcal{E} \dashrightarrow {}^\tau\mathcal{E}\}$ isom outside x_i

I = finite set , W finite dim $\overline{\mathbb{Q}}_\ell$-linear rep of $\hat{G}^I$ $\in \mathrm{Rep}(\hat{G}^I)$

We have defined the stack of shtukas $\mathrm{Sht}_{G,I,W}$ over X^I:

$$\mathrm{Sht}_{G,I,W} \xrightarrow{p_G} X^I \leftarrow \eta_I \leftarrow \overline{\eta}_I$$

~~A S-pt of $\mathrm{Sht}_{G,I}$~~

$$\mathrm{Sht}_{G,I,W}(S) = \{ (x_i) \in X^I(S),\ \mathcal{E} \dashrightarrow (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{E} \}$$

isom outside x_i

relative position on x_i is bounded by W

A S-pt of $\mathrm{Sht}_{G,I,W}$ is called a shtuka on $X \times S$.

We have defined the Satake perverse sheaf on $\mathrm{Sht}_{G,I,W}$:

$$\epsilon_G : \mathrm{Sht}_{G,I,W} \xrightarrow[\text{smooth}]{} [G_{I,\infty} \backslash \mathrm{Gr}_{G,I,W}]$$

$$\mathcal{F}_{G,I,W} := \epsilon_G^* \mathcal{S}_{G,I,W} \qquad \mathcal{S}_{G,I,W}$$

We have defined the cohomology gps:

$$\mathrm{Sht}_{G,I,W} \xrightarrow{p_G} X^I \leftarrow \eta_I \leftarrow \overline{\eta}_I$$

$$\underbrace{\mathrm{Sht}^{\leq\mu}_{G,I,W}}_{\text{f.t.}} \underset{\text{open}}{\hookrightarrow} \mathrm{Sht}_{G,I,W} \qquad \mu: \underbrace{\text{dominant coweight of } G}_{\hat{\Lambda}^+_G}$$

$j \in \mathbb{Z}$

$$H^{j,\leq\mu}_{G,I,W} := \cancel{H^j_c(\mathrm{Sht}^{\leq\mu}_{G,I,W})} \left(R^j p_{G!} \left(\mathcal{F}_{G,I,W}\big|_{\mathrm{Sht}^{\leq\mu}_{G,I,W}} \right) \right) \Big|_{\overline{\eta}_I} \quad \text{finite dim } \overline{\mathbb{Q}}_\ell\text{-v.s.}$$

$$H^j_{G,I,W} := \varinjlim_{\mu} H^{j,\leq\mu}_{G,I,W}$$

$$\circlearrowleft$$

$$\mathcal{H}_{G,v} := C_c(G(\mathcal{O}_v) \backslash G(F_v) / G(\mathcal{O}_v), \overline{\mathbb{Q}}_\ell)$$

$v \in |X|$

Rem. When $I = \emptyset$, $W = \mathbb{1}$,

$$H^0_{G,\emptyset,\mathbb{1}} = C_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) = C_c(G(F) \backslash G(\mathbb{A}) / G(\mathcal{O}), \overline{\mathbb{Q}}_\ell)$$

the action of $\mathcal{H}_{G,v}$ coincides with convolution from right.

Thm ( ) 1: $\exists\, \mu_0 \in \hat{\Lambda}^+_G$ (depending on X, G, I, W, v, j), s.t. $H^j_{G,I,W} = \mathcal{H}_{G,v} \cdot H^{j,\le\mu_0}_{G,I,W}$.

Cor 1: $H^j_{G,I,W}$ is of finite type as $\mathcal{H}_{G,v}$-module.

Rem: When $I=\emptyset$, $W=\mathbb{1}$, it is a classical result

$$C_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) = \mathcal{H}_{G,v} \cdot C_c(\mathrm{Bun}_G^{\le\mu_0}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell).$$

§2. Application:

Fact: $H^j_{G,I,W}$ ~~is equi~~ $\circlearrowleft \pi_1(\eta_I, \overline{\eta_I})$, $\circlearrowleft$ partial Frobenius

Drinfeld's lemma: Let V be a finite dim $\overline{\mathbb{Q}}_\ell$-v.s, equipped with a continuous $\overline{\mathbb{Q}}_\ell$-linear action of $\pi_1(\eta_I, \overline{\eta_I})$ and a $\overline{\mathbb{Q}}_\ell$-linear action of partial Frobenius ~~compactible~~ commute between them. Then V is equipped with an action of $\mathrm{Gal}(\overline{F}/F)^I$.

Cor 2: Let $\mathfrak{M}$ be a $\mathcal{H}_{G,v}$-module of f.t., equipped 〃 〃 $\mathcal{H}_{G,v}$-linear 〃 〃. Then $\mathfrak{M}$ is equipped with an action of $\mathrm{Weil}(\overline{F}/F)^I$.

(continuous means $\mathfrak{M}$ is a union of f. dim subspace which are stable under $\pi_1^{geo}(\eta_I, \overline{\eta_I})$ and on which the action of $\pi_1^{geo}(\eta_I, \overline{\eta_I})$ is continuous)

Cor 1 + Cor 2: $H^j_{G,I,W} \circlearrowleft \mathrm{Weil}(\overline{F}/F)^I$

We can extend the excursion operators of V. L from C_c^{cusp} to C_c:

$\forall I, W$

$$\begin{tikzcd} C_c = H^0_{G,\emptyset,\mathbb{1}} \arrow[r, "x \in W^{\hat{G}}"] \arrow[rrr, bend right, "{S_{I,W,x,\xi,(\gamma_i)} = S_{I,f,(\gamma_i)}}"'] & H^0_{G,I,W} \arrow[r, "(\gamma_i) \in \mathrm{Weil}(\overline{F}/F)^I"] & H^0_{G,I,W} \arrow[r, "\xi \in (W^\vee)^{\hat{G}}"] & H^0_{G,\emptyset,\mathbb{1}} = C_c \end{tikzcd}$$

$W, x, \xi \rightsquigarrow f: \hat{G}\backslash \hat{G}^I / \hat{G} \to \overline{\mathbb{Q}}_\ell$

$(g_i) \mapsto \langle \xi, (g_i)\cdot x \rangle$

V.L.: $\underbrace{C_c^{cusp}}_{\text{finite dim}} = \bigoplus\limits_{\sigma:\, Gal(\bar F/F)\to \hat G(\bar{\mathbb{Q}}_\ell)} \mathfrak{H}_\sigma$ (*)

everywhere unramified, compatible with Satake isom

: $\underbrace{C_c}_{\infty \text{ dim}} = ?$ < Let ~~I be~~ $v \in |X|$ and I an ideal of $\mathcal{H}_{G,v}$ of finite codim.

Then $\underbrace{C_c / I \cdot C_c}_{\text{finite dim}} = \bigoplus\limits_{\rho:\, Weil(\bar F/F) \to \hat G(\bar{\mathbb{Q}}_\ell)} \mathfrak{H}_\rho$

everywhere unramified, compatible with Satake isom

$\mathfrak{H}_\rho$ included in the generalized eigenspace of $T(h_{V,v})$ for the eigenvalue $\nu(T(h_{V,v})) = \chi_V(\rho(\mathrm{Frob}_v))$.

compatible with parabolic induction:

$G \leftarrow P \twoheadrightarrow M$

Lem: The excursion operators commute with the constant term morphism

$C_G^P : \underbrace{C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}), \bar{\mathbb{Q}}_\ell)}_{= C_c(G)} \to \underbrace{C(M(F)\backslash M(\mathbb{A})/M(\mathbb{O}), \bar{\mathbb{Q}}_\ell)}_{= C(M)}$.

Cor 3:

$$\begin{tikzcd}
C_c(G)/I\cdot C_c(G) \arrow[r, equal] \arrow[d, "\overline{C_G^P}"'] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus\limits_{\rho:\, Weil(\bar F/F)\to \hat G(\bar{\mathbb{Q}}_\ell)} \mathfrak{H}_\rho^G \arrow[d] \\
C(M)/I\cdot C(M) \arrow[r, equal] & \bigoplus\limits_{\rho':\, Weil(\bar F/F)\to \hat M(\bar{\mathbb{Q}}_\ell)} \mathfrak{H}_{\rho'}^M
\end{tikzcd}$$

$$= \bigoplus_{\rho:\, Weil(\bar F/F)\to \hat G(\bar{\mathbb{Q}}_\ell)} \Big(\bigoplus_{\substack{\rho':\, Weil(\bar F/F)\to \hat M(\bar{\mathbb{Q}}_\ell) \\ j\circ\rho' = \rho}} \mathfrak{H}_{\rho'}^M \Big)$$

If $\overline{C_G^P}(\mathfrak{H}_\rho^G) \neq 0$, then ρ factors through some ρ' (à conj près)

$j : \hat M(\bar{\mathbb{Q}}_\ell) \hookrightarrow \hat G(\bar{\mathbb{Q}}_\ell)$

Open question: 1) In (*), when σ is irreducible?

2) For which σ, $\mathfrak{H}_\sigma \neq 0$?

(Rem: (V.L and X. Zhu) If σ is irreducible, then

$$(H^j_{G,I,W})_\sigma = \left(A_\sigma \underset{\overline{\mathbb{Q}}_\ell}{\otimes} W \circ \sigma^I\right)^{S_\sigma}$$ equivariant by the action of $\mathcal{H}_G \times Gal(\overline{F}/F)^I$

S_σ: centraliser of $Im(\sigma)$ in $\widehat{G}$)

Au cas où $\nearrow$

§ 3. ~~Pro~~ constant term morphism on cohomology gps

Let P be a standard parabolic subgp of G and M be its Levi quotient.

~~let~~ I = finite set.

$$\begin{tikzcd}
 & Sht_{P,I} \arrow[dl, "i"'] \arrow[dr, "\pi"] & \\
Sht_{G,I} & \{(x_i), \mathcal{P} \dashrightarrow {}^\tau\mathcal{P}\} \arrow[dl, mapsto] \arrow[dr, mapsto] & Sht_{M,I} \\
\{(x_i), \mathcal{P}\overset{P}{\times} G \dashrightarrow {}^\tau\mathcal{P}\overset{P}{\times} G\} & & \{(x_i), \mathcal{P}\overset{P}{\times} M \dashrightarrow {}^\tau\mathcal{P}\overset{P}{\times} M\}
\end{tikzcd}$$

$W \in Rep(\widehat{G}^I)$

Def: $Sht_{P,I,W} := i^{-1}(Sht_{G,I,W})$

Let $\widehat{M}/\overline{\mathbb{Q}}_\ell$ be the Langlands dual gp of M. We can view W as a rep of $\widehat{M}^I$ via $\widehat{M}^I \hookrightarrow \widehat{G}^I$. Then we define $Sht_{M,I,W}$.

Fact: $\pi(Sht_{P,I,W}) \subset Sht_{M,I,W}$. ~~$Sht_{G,I,W} \Leftarrow$~~

why? $Bun_P \to Bun_M$ est t.f.

b.c. Extension est t.f.

finite type

~~Def: $Sht^{\leq\mu}_{P,I,W}$~~

$$\begin{tikzcd}
\mathcal{F}_{G,I,W} & Sht_{G,I,W} \arrow[d, "\epsilon_G"', "sm"] & Sht_{P,I,W} \arrow[l, "i"'] \arrow[rr, "\pi"] \arrow[dr, "\pi_d"] \arrow[d, "\epsilon_P", "sm"'] & & Sht_{M,I,W} \arrow[d, "\epsilon_M"', "sm"] & \mathcal{F}_{M,I,W} := \epsilon_G^* S_{M,I,W} \\
 & & & \widetilde{Sht}_{M,I,W} \arrow[ur, "\tilde{\pi}"] \arrow[dl, "\tilde{\epsilon}_M"] \arrow[d, phantom, "\square"] & & (1) \\
S_{G,I,W} & {[G_{I,d}\backslash Gr_{G,I,W}]} & {[P_{I,d}\backslash Gr_{P,I,W}]} \arrow[l, "\bar{i}"'] \arrow[rr, "\bar{\pi}"] & & {[M_{I,d}\backslash Gr_{M,I,W}]} & S_{M,I,W}
\end{tikzcd}$$

First, we want to construct $\pi_! i^* \mathcal{F}_{G,I,W} \to \mathcal{F}_{M,I,W}$ in $D^b_c(\mathrm{Sht}_{M,I,W}, \bar{\mathbb{Q}}_\ell)$.

Let $\widetilde{\mathrm{Sht}}_{M,I,W}$ be the fiber product.

Fact: π_d is smooth.

$$\pi_! i^* \mathcal{F}_{G,I,W} = \pi_! i^* \epsilon_G^* S_{G,I,W} = \pi_! \epsilon_P^* \bar{i}^* S_{G,I,W}$$

$$= \tilde{\pi}_! \pi_{d!} \pi_d^* \tilde{\epsilon}_M^* \bar{i}^* S_{G,I,W}$$

$$= \tilde{\pi}_! \pi_{d!} \pi_d^! \tilde{\epsilon}_M^* \bar{i}^* S_{G,I,W} [-2\dim \pi_d]$$

$$\xrightarrow{\text{counit}} \tilde{\pi}_! \tilde{\epsilon}_M^* \bar{i}^* S_{G,I,W} [-2\dim \pi_d]$$

$$\overset{\text{proper base chgt}}{=} \epsilon_M^* \bar{\pi}_! \bar{i}^* S_{G,I,W} [-2\dim \pi_d]$$

by the compatibility of ~~geo Satake~~ geo Satake with constant term $\to$ $= \epsilon_M^* S_{M,I,W} = \mathcal{F}_{M,I,W}$

Second, we want to construct $H^j_{G,I,W} \to H^j_{M,I,W}$ not yet defined.

~~M is not semisimple, what is $H^j_{M,I,W}$ when ?~~

Let $\mu \in \hat{\Lambda}^+_G$. ~~We have defined~~ $\mathrm{Sht}^{\leq \mu}_{G,I,W} \xhookrightarrow{\text{open}} \mathrm{Sht}_{G,I,W}$.

Def: $\mathrm{Sht}^{\leq \mu}_{P,I,W} := i^{-1}(\mathrm{Sht}^{\leq \mu}_{G,I,W})$

Let $\lambda \in \hat{\Lambda}^+_M$, we can define $\mathrm{Sht}^{\leq^M \lambda}_{M,I,W} \xhookrightarrow{\text{open}} \mathrm{Sht}_{M,I,W}$

Def: $\mathrm{Sht}^{\leq \mu}_{M,I,W} := \bigcup_{\lambda \in \hat{\Lambda}^+_M, \lambda} \mathrm{Sht}^{\leq^M \lambda}_{M,I,W}$

For $\mu \in \hat{\Lambda}^+_G$ $\quad \lambda \leq \mu$ in $\hat{\Lambda}_G$

(i.e. $\mu - \lambda$ = linear combination of simple coroots of G with positive coef)

Fact: $\pi(\mathrm{Sht}^{\le\mu}_{P,I,W}) \subset \mathrm{Sht}^{\le\mu}_{M,I,W}$.

Fact: $\mathrm{Sht}^{\le\mu}_{M,I,W} = \coprod \mathrm{Sht}^{\le\mu,\nu}_{M,I,W}$,

$\Lambda^{\mu} := \{\nu \in \hat{\Lambda}_{Z_M},\ \nu \le pr(\mu)\}$

each $\mathrm{Sht}^{\le\mu,\nu}_{M,I,W}$ is of finite type.

$pr: \hat{\Lambda}^{\mathbb{Q}}_{G} \twoheadrightarrow \hat{\Lambda}^{\mathbb{Q}}_{Z_M}$

(2)
$$\begin{tikzcd}
 & \mathrm{Sht}^{\le\mu,\nu}_{P,I,W}\big|_{\bar\eta_I} \arrow[dl, "i"'] \arrow[dr, "\pi"] \arrow[d, "p_P"] & \\
\mathrm{Sht}^{\le\mu}_{G,I,W}\big|_{\bar\eta_I} \arrow[r, "p_G"'] & \bar\eta_I & \mathrm{Sht}^{\le\mu,\nu}_{M,I,W}\big|_{\bar\eta_I} \arrow[l, "p_M"]
\end{tikzcd}$$

$\boxed{\text{ref: J. Wang}}$

Prop (Varshavsky): $i: \mathrm{Sht}_{P,I,W}\big|_{\bar\eta_I} \to \mathrm{Sht}^{\le\mu}_{G,I,W}\big|_{\bar\eta_I}$ is proper.

pourquoi ?

Then restricted on (2), we have $p_{G!}\mathcal{F}_{G,I,W} \to p_{G!}\, i_* i^* \mathcal{F}_{G,I,W} = p_{G!}\, i_!\, i^* \mathcal{F}_{G,I,W}$

$= p_{M!}\,\pi_!\, i^* \mathcal{F}_{G,I,W} \to p_{M!}\mathcal{F}_{M,I,W}$

$\Rightarrow$ ~~$H_{G,I,W}$~~ ~~$H^j_c(\mathrm{Sht}^{\le\mu}_{G,I,W}|_{\bar\eta_I})$~~

Def: $H^{j,\le\mu,\nu}_{M,I,W} := \left(R^j p_{M!}\left(\mathcal{F}_{M,I,W}\big|_{\mathrm{Sht}^{\le\mu,\nu}_{M,I,W}}\right)\right)\Big|_{\bar\eta_I}$

$\forall \mu \in \hat{\Lambda}^{+}_{G}$, $\nu \in \Lambda^{\mu}$

$H^{j,\le\mu}_{M,I,W} := \prod_{\nu\in\Lambda^\mu} H^{j,\le\mu,\nu}_{M,I,W}$

$H^{j}_{M,I,W} := \varinjlim_{\mu} H^{j,\le\mu}_{M,I,W}$

$\Rightarrow H^{j,\le\mu}_{G,I,W} \longrightarrow H^{j,\le\mu,\nu}_{M,I,W}$

$\Rightarrow H^{j,\le\mu}_{G,I,W} \xrightarrow{C_G^{P,j,\le\mu}} H^{j,\le\mu}_{M,I,W}$

$\Rightarrow H^{j}_{G,I,W} \xrightarrow{C_G^{P,j}} H^{j}_{M,I,W}$

e.g. When $I=\emptyset$, $W=\mathbb{1}$, $C_G^{P,0}$ coincides with classical constant term morphism.

~~Prop:~~ We can define $Sht_{G,I,W}^{\leq\mu} \leftarrow Sht_{P,I,W}^{\leq\mu} \rightarrow Sht_{M,I,W}^{\leq\mu}$

Prop 1: $\exists\ C \in \mathbb{Q}_{\geq 0}$ s.t. $\forall\ \mu \in \hat{\Lambda}_G^+$, ~~satisfying~~ if $\langle \mu, \alpha \rangle > C$ for all α simple root of G but not simple root of M,

then $\forall j$ the morphism $C_G^{P,j,\leq\mu} : H_{G,I,W}^{j,\leq\mu} \rightarrow H_{M,I,W}^{j,\leq\mu}$ is an isom.

(~~pf/ use uses Prop 10.4.5 of [DG] / compact generation of the cat of D-mod on Bun_G~~). because $H^1(X \times S, \mathcal{U}_{P_M}) = 0$

§ 4. Proof of Thm 1.

From now on, $H_G^j := H_{G,I,W}^j$, $H_M^j := H_{M,I,W}^j$ $v \in |X|$, $\deg v = 1$

$\mathcal{H}_{G,v} \xrightarrow{Sat} \mathcal{H}_{M,v}$

$h_{\omega} = \mathbb{1}_{G(\mathcal{O}_v)\omega G(\mathcal{O}_v)}$

P maximal parabolic subgp, M.

α simple root of G not in M.

ω quasi-fundanere ~~wei~~ coweight of G in the center of M.

$$\begin{tikzcd}
H_G^{j,\leq\mu}/H_G^{j,\leq\mu-\alpha} \arrow[r, "h_\omega"] \arrow[d, "\overline{C_G^{P,\leq\mu}}"'] \arrow[dr, phantom, "\circlearrowright"] & H_G^{j,\leq\mu+\omega}/H_G^{j,\leq\mu+\omega-\alpha} \arrow[d, "\overline{C_G^{P,\leq\mu+\omega}}"] \\
H_M^{j,\leq\mu}/H_M^{j,\leq\mu-\alpha} \arrow[r, "Sat(h_\omega)", "\sim"'] & H_M^{j,\leq\mu+\omega}/H_M^{j,\leq\mu+\omega-\alpha}
\end{tikzcd}$$

easy

~~Pf~~ Want: For $\mu >> 0$, $\overline{C_G^{P,\leq\mu}}$ is an isom.

Difficulty:

note $Sht_G^{\leq\mu-\alpha} \underset{open}{\hookrightarrow} Sht_G^{\leq\mu} \underset{closed}{\hookleftarrow} Sht_G^{S(\mu)}$

complementary

$\mu >> 0$

not know if comm

$$
\begin{tikzcd}
\cdots \arrow[r] & H_G^{j-1,S(\mu)} \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\cancel{\text{not comm}}"] & H_G^{j,\leq\mu-\alpha} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H_G^{j,\leq\mu} \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & H_G^{j,S(\mu)} \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\cancel{\text{not comm}}"] & H_G^{j+1,\leq\mu-\alpha} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_M^{j-1,S(\mu)} \arrow[r] \arrow[dr] & H_M^{j,\leq\mu-\alpha} \arrow[r] & H_M^{j,\leq\mu} \arrow[r] & H_M^{j,S(\mu)} \arrow[r] \arrow[dr] & H_M^{j+1,\leq\mu-\alpha} \arrow[r] & \cdots \\
& & 0 \arrow[u] & & & 0 \arrow[u] &
\end{tikzcd}
$$

Question: ~~A priori~~,

Is $H_G^{j,\leq\mu} / H_G^{j,\leq\mu-\alpha} \to H_G^{j,S(\mu)}$ isom ~~surj~~ ?

e.g.

Answer: For $\mu >> 0$, Yes.

~~Solution:~~

~~Prop 2: $\exists\ C_G^0 \in \mathbb{Q}_{>0}$ s.t.~~

~~(a) Let $\mu \in \hat{\Lambda}_G^+$ s.t. $\langle \mu, \gamma \rangle \geq C_G^0$ for all simple roots γ of G. Then for α, P, M as above, the morphism~~

~~$$\ker(H_G^{j,\leq\mu-\alpha} \to H_M^j) \to \ker(H_G^{j,\leq\mu} \to H_M^j) \text{ is surj.}$$~~

~~(b) $\exists\ \mu_0 \in \hat{\Lambda}_G^+$ s.t. $\forall\ \lambda \in \hat{\Lambda}_G^+$ if $\lambda \geq \mu_0$ and $\langle \lambda, \gamma \rangle \geq C_G^0$ for all simple roots γ of G, then the morphism~~

~~$$\ker(H_G^{j,\leq\mu_0} \to \prod_M H_M^j) \to \ker(H_G^{j,\leq\lambda} \to \prod_M H_M^j) \text{ is surj.}$$~~

(c) $\exists\ C_G \in \mathbb{Q}_{>0}$, ~~$C_G \geq C_G^0$~~ s.t. $\forall\ \mu \in \hat{\Lambda}_G^+$, if $\langle \mu, \gamma \rangle > C_G$ for all simple roots γ of G, then the morphism $H_G^{j,\leq\mu} \to H_G^j$ is inj.

In particular, we have a short exact sequence

$$0 \to H_G^{j,\leq\mu-\alpha} \to H_G^{j,\leq\mu} \to H_G^{j,S(\mu)} \to 0. \quad \forall j$$

$G = SL_2$ example :

0 when $\mu >> 0$
if not, we don't know .

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[d, equal] & & & & & & 0 \arrow[d, equal] & \\
\cdots \arrow[r] & H_G^{-1,S(\mu)} \arrow[r] \arrow[d, "\simeq"] & H_G^{0,\leq\mu-\alpha} \arrow[r] \arrow[d] & H_G^{0,\leq\mu} \arrow[r] \arrow[d] & H_G^{0,S(\mu)} \arrow[r] \arrow[d, "\simeq"] & H_G^{1,\leq\mu-\alpha} \arrow[r] \arrow[d] & H_G^{1,\leq\mu} \arrow[r] \arrow[d] & H_G^{1,S(\mu)} \arrow[r] \arrow[d, "\simeq"] & \cdots \\
\arrow[r] & H_T^{-1,S(\mu)} \arrow[r] \arrow[d, equal] & H_T^{0,\leq\mu-\alpha} \arrow[r] & H_T^{0,\leq\mu} \arrow[r] & H_T^{0,S(\mu)} \arrow[r] & H_T^{1,\leq\mu-\alpha} \arrow[r] \arrow[d, equal] & H_T^{1,\leq\mu} \arrow[r] \arrow[d, equal] & H_T^{1,S(\mu)} \arrow[r] \arrow[d, equal] & \cdots \\
 & 0 & & & & 0 & 0 & 0 &
\end{tikzcd}$$

$j \neq 0$ ~~easy~~ $H_G^{j,S(\mu)} \simeq H_T^{j,S(\mu)} = 0$

In fact, for any G ,

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[d, equal] & & & & ? & & 0 \arrow[d, equal] & \\
\cdots \arrow[r] & H_G^{-1,S(\mu)} \arrow[r] \arrow[d, "\simeq"] & H_G^{0,\leq\mu-\alpha} \arrow[r] \arrow[d] & H_G^{0,\leq\mu} \arrow[r] \arrow[d] & H_G^{0,S(\mu)} \arrow[r] \arrow[d, "\simeq"] & H_G^{1,\leq\mu-\alpha} \arrow[r] \arrow[d] & H_G^{1,\leq\mu} \arrow[r] \arrow[d] & H_G^{1,S(\mu)} \arrow[r] \arrow[d, "\simeq"] & \cdots \\
\cdots \arrow[r] & H_{M_\alpha}^{-1,S(\mu)} \arrow[r] \arrow[d, equal] & H_{M_\alpha}^{0,\leq\mu-\alpha} \arrow[r] & H_{M_\alpha}^{0,\leq\mu} \arrow[r] & H_{M_\alpha}^{0,S(\mu)} \arrow[r] & H_{M_\alpha}^{1,\leq\mu-\alpha} \arrow[r] \arrow[d, equal] & H_{M_\alpha}^{1,\leq\mu} \arrow[r] \arrow[d, equal] & H_{M_\alpha}^{1,S(\mu)} \arrow[r] \arrow[d, equal] & \cdots \\
 & 0 & & & & 0 & 0 & 0 &
\end{tikzcd}$$

isom for M_α

not for T .