1. Introduction

Let k be a perfect field of characteristic $p > 0$. By $W_m(k)$ we denote the p-typical Witt vectors of k truncated at the level m. The theory of cristalline cohomology discovered by Alexandre Grothendieck, in particular, gives a canonical lifting of the de Rham cohomology of algebraic varieties into mixed characteristics. Namely,

Theorem ([1]). *Fix a positive integer m. There is a functor*

$$\mathrm{R}\Gamma_{\mathrm{cris}} : Sm/k \to D(W_m(k) - mod)$$

from the category of smooth varieties over k to the derived category of $W_m(k)$-modules such that

(i) $\mathrm{R}\Gamma_{\mathrm{cris}} \otimes^L_{W_m(k)} k \simeq \mathrm{R}\Gamma_{\mathrm{dR}}$

(ii) For any smooth scheme $\tilde{X}/W_m(k)$ with an isomorphism $\tilde{X} \times_{W_m(k)} k \simeq X$ there is a canonical quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cris}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\tilde{X}/W_m(k))$.

Moreover, a version for the full Witt vectors $W(k)$ holds for proper varieties

Theorem ([1]). *There is a functor*

$$\mathrm{R}\Gamma_{\mathrm{cris}} : SmPr/k \to D(W(k) - mod)$$

from the category of smooth proper varieties over k to the derived category of $W(k)$-modules such that

(i) $\mathrm{R}\Gamma_{\mathrm{cris}} \otimes^L_{W(k)} k \simeq \mathrm{R}\Gamma_{\mathrm{dR}}$

(ii) For any smooth scheme $\tilde{X}/W(k)$ with an isomorphism $\tilde{X} \times_{W(k)} k \simeq X$ there is a canonical quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cris}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\tilde{X}/W(k))$.

For any DG algebra or DG category A over a ring R by $CP_\bullet(A/R), HP_\bullet(A/R)$ we denote the periodic cyclic complex and its cohomology respectively. By u we denote the periodicity parameter of degree 2 which makes $CP_\bullet(A/R)$ into a module over $R((u))$. We will generalize the theorem above to the setting of periodic cyclic homology of DG algebras. One reason to expect such a generalization is that periodic cyclic homology related to the de Rham cohomology by the following

Theorem. *For a smooth proper variety X over a field k of characteristic zero or greater than* $\dim X$ *there are canonical isomorphisms*

$$HP_k(D^b(Coh(X))) \simeq \bigoplus_{n\in\mathbb{Z}} H^{k+2n}_{dR}(X/k)$$

Another reason is that on the periodic cyclic homology of a family of DG algebras there exists a flat connection.

Theorem ([2, Proposition 3.1],[4, Section 3]). *For a term-wise flat DG algebra $A_\bullet$ over $k[x_1, \ldots, x_n]$ there is a canonical flat connection on the module $HP_\bullet(A_\bullet/k[x_1, \ldots, x_n])$ over* $\mathbb{A}^n = \mathrm{Spec}\ k[x_1, \ldots, x_n]$.

From now on we assume that $p > 2$. Our main result is the following

Theorem 1.1. *There exists a functor*

$$CP^{\mathrm{cris}}_\bullet : DG - alg/k \to D(W(k)((u)) - mod) \tag{1.1}$$

such that

(i) $CP^{\mathrm{cris}}_\bullet \otimes^L_{W(k)((u))} k((u)) \simeq CP_\bullet$

(ii) For any term-wise flat DG algebra $\tilde{A}$ over $W_m(k)$ for some $m \geq 0$ with a quasi-isomorphism $r : \tilde{A} \otimes^L_{W_m(k)((u))} k((u)) \simeq A$ there is a quasi-isomorphism

$$CP_\bullet(A_\bullet/W_m(k)) \simeq CP^{\mathrm{cris}}_\bullet(A) \otimes^L_{W(k)((u))} W_m(k)((u))$$

functorial in $\tilde{A}$ and r.

(iii) For any term-wise flat and term-wise p-adically complete DG algebra $\tilde{A}$ over $W(k)$ with a quasi-isomorphism $r : \tilde{A} \otimes^L_{W(k)((u))} k((u)) \simeq A$ there is a quasi-isomorphism $CP_\bullet(A_\bullet/W(k)) \simeq CP^{\mathrm{cris}}_\bullet(A)$ functorial in $\tilde{A}$ and r.

2. Gauss-Manin connection and the parallel transport

Proposition 4.1.8 in [5] gives for a DG algebra A and a k-linear derivation D of degree n of A a canonical contracting homotopy for the Lie derivative $L_D : CP_\bullet(A) \to CP_\bullet(A)$. Namely, there is an operator $\iota : CP_\bullet(A) \to CP_{\bullet+n-1}(A)$ such that

$$[\iota_D, b + B] = L_D \tag{2.1}$$

It can be used to construct the Gauss-Manin connection on a family of DG algebras. We recall this construction from [2] in a special case. Let $\mathcal{A}$ be a term-wise flat term-wise complete DG algebra over $W(k)[t]$. We replace it by a quasi-isomorphic one so that we can assume that the graded algebra structure on $\mathcal{A}$ does not deform along t, that is $\bigoplus \mathcal{A}^i \simeq W(k)[t] \otimes_{W(k)} \bigoplus A^i$ but the differential depends on t. Denote by ∇' the trivial connection on $\bigoplus \mathcal{A}^i$ given by $\nabla'(f(t) \otimes a) = f'(t) \otimes a \otimes dt$. This connection does not necessarily commute with the differential on the DG algebra itself, but it turns out that we can twist it to obtain a connection on the periodic cyclic complex.

Proposition 2.1 ([2, Proposition 3.1])**.**

$$\nabla^{GM} = \nabla' + u^{-1}\iota_{[\nabla', d_{\mathcal{A}}]}$$

is a connection on $CP_\bullet(\mathcal{A}/W(k)[t])$

Recall that periodic cyclic homology satisfies the Kunneth formula. For homotopically flat DG algebras A, B over a commutative ring R there is a quasi-isomorphism

$$CP_\bullet(A)\hat{\otimes}_{R((u))}CP_\bullet(B) \xrightarrow{\sim} CP_\bullet(A \otimes_R B)$$

where $\hat{\otimes}$ means u-adic completion. It follows that for a commutative DG algebra A the complex $CP_\bullet(A)$ carries an algebra structure and for every DG algebra B the complex $CP_\bullet(B)$ has a structure of a module over $CP_\bullet(A)$.

For a DG algebra $\mathcal{A}$ over $W(k)[t]$ for any $x \in W(k)$ denote by $\mathcal{A}_x$ the fiber $\mathcal{A}/(t - x)\mathcal{A}$.

Proposition 2.2. *(i) If $\mathcal{A}$ is a term-wise flat term-wise p-adically complete DG algebra over $W(k)[t]$, then the complexes $CP_\bullet(\mathcal{A}_0)$ and $CP_\bullet(\mathcal{A}_p)$ are isomorphic.*

(ii) We can choose quasi-isomorphisms $CP_\bullet(\mathcal{A}_0/W(k)) \simeq CP_\bullet(\mathcal{A}_p/W(k))$ for each $\mathcal{A}$ so that they commute with the Kunneth isomorphism.

Proof. (i) The parallel transport along the vector field $\frac{\partial}{\partial t}$ from the point 0 to the point p is given by the formula

$$T(a) = \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} (\nabla^{GM}_{\frac{\partial}{\partial t}})^n (1 \otimes a) \tag{2.2}$$

The series is convergent because $\mathrm{ord}_p(\frac{p^n}{n!}) \geq n\frac{p-2}{p-1}$ which tends to infinity because of the assumption $p > 2$. The fact that it induces a morphism of complexes automatically follows from the fact that ∇^{GM} commutes with the differential. It is an isomorphism since the inverse is given by parallel transport in the other direction.

(ii) As against part (i) here we only construct a quasi-isomorphism, not a genuine isomorphism of complexes. Denote by K the field $W(k)[1/p]$. Let R be the divided power envelope of the point $t = 0$ inside $\operatorname{Spec} W(k)[t]$. Note that it coincides with the envelope of $t = p$. Explicitly, R is the $W(k)$-subalgebra in $K[t]$ generated by $\frac{t^{p^n}}{n!}$ for all $n \geq 0$. Denote by $CP_{\bullet}^{red}(R/W(k))$ the image of $CP_{\bullet}(R/W(k))$ inside $CP_{\bullet}(K[t]/K)$. Put

$$\mathcal{R}_{\mathcal{A}} = CP_{\bullet}^{red}(R/W(k)) \otimes_{CP_{\bullet}(R/W(k))} CP_{\bullet}(\mathcal{A}/W(k))$$

Sending t to 0 and p gives well-defined morphisms of complexes $\rho_0 : \mathcal{R} \to CP_{\bullet}(\mathcal{A}_0/W(k))$ and $\rho_p : \mathcal{R} \to CP_{\bullet}(\mathcal{A}_p/W(k))$ respectively. We will prove that both of them are quasi-isomorphisms thus proving the statement because $\mathcal{R}_{\mathcal{A}}$ also satisfy the Kunneth formula with respect to $\mathcal{A}$.

Indeed, the kernel of ρ_i is given by $\ker(CP_{\bullet}^{red}(R/W(k)) \xrightarrow{t\mapsto i} W(k)((u))) \otimes_{CP_{\bullet}(R/W(k))} CP_{\bullet}(\mathcal{A}/W(k))$ so it is left to prove that $\ker(CP_{\bullet}^{red}(R/W(k)) \xrightarrow{t\mapsto 0} W(k)((u)))$ and $\ker(CP_{\bullet}^{red}(R/W(k)) \xrightarrow{t\mapsto p} W(k)((u)))$ both have zero cohomology because $CP_{\bullet}(\mathcal{A}/W(k))$ is homotopically flat.

Denote by $CH_{\bullet}^{red}(R/W(k))$ the image of Hochschild complex $CH_{\bullet}(R/W(k))$ in $CH_{\bullet}(K[t])$. By Theorem 3.2.2 in [5] $CH_{\bullet}(K[t])$ is homotopy equivalent to $\Omega^0_{K[t]/K} \oplus \Omega^1_{K[t]/K}[1]$ and, moreover, the homotopy equivalence can be chosen to be induced from that of between $CH_{\bullet}(W(k)[t])$ between $\Omega^0_{W(k)[t]/W(k)} \oplus \Omega^1_{W(k)[t]/W(k)}[1]$. Thus, $CH_{\bullet}^{red}(R/W(k))$ is homotopy equivalent to $\Omega^{0,red}_{R/W(k)} \oplus \Omega^{1,red}_{R/W(k)}[1]$ where $\Omega^{\bullet,red}_{R/W(k)}$ is the image of $\Omega^{\bullet}_{R/W(k)}$ in $\Omega^{\bullet}_{K[t]/K}$ ore, explicitly, the quotient by the relations $d(\frac{t^k}{k!}) = \frac{t^{k-1}}{(k-1)!}dt$.

By the Hochschild-to-cyclic spectral sequence and Theorem 3.2.5 in [5] there is a commutative diagram of complexes there the vertical arrows are quasi-isomorphisms

$$\begin{array}{ccc} CP_{\bullet}^{red}(R/W(k)) & \longrightarrow & CP_{\bullet}(K[t]/K) \\ \downarrow & & \downarrow \\ (\Omega^{\bullet,red}_{R/W(k)}, d_{\mathrm{dR}})((u)) & \longrightarrow & (\Omega^{\bullet}_{K[t]/K}, d_{\mathrm{dR}})((u)) \end{array}$$

It follows that $CP_{\bullet}^{red}(R/W(k)) \xrightarrow{t\mapsto 0} W(k)((u))$ and $CP_{\bullet}^{red}(R/W(k)) \xrightarrow{t\mapsto p} W(k)((u))$ are quasi-isomorphisms because $(\Omega^{\bullet,red}_{R/W(k)}, d_{\mathrm{dR}})$ has $H^0 = W(k), H^1 = 0$. □

3. Point

$CP_{\bullet}(\mathbb{Z}_p/\mathbb{Z}_p[t])$

In this section we compute our main example of the Gauss-Manin connection. Namely, the periodic cyclic homology of $\mathbb{Z}_p$ over $\mathbb{Z}_p[t]$. A flat resolution for $\mathbb{Z}_p$ is $\mathbb{Z}_p[t][\varepsilon]$ where $\deg \varepsilon = -1, d\varepsilon = p$

4. Proof of Theorem 1.1

Put $R = W(k) \xrightarrow{p} W(k)$ – a DG algebra in degrees $-1, 0$ which is as an algebra the trivial square-zero extension of $W(k)$ by $W(k)$. It is quasi-isomorphic to k. Applying Lemma 2.2 to $A = W(k)[t] \xrightarrow{pt} W(k)[t]$ we see that *algebras* $CP_\bullet(R/W(k))$ and $CP_\bullet(W(k)[1] \oplus W(k)/W(k))$ are quasi-isomorphic.

Now pick a term-wise complete term-wise flat resolution $\tilde{A}$ of A over $W(k)$. Then $CP_\bullet(A/W(k))$ is quasi-isomorphic to a module over $CP_\bullet(k/W(k))$, hence quasi-isomorphic to a module over $CP_\bullet(W(k)[1] \oplus W(k)/W(k))$. Next, $W(k)$ is an algebra over $W(k)[1] \oplus W(k)$, so $CP_\bullet(W(k)/W(k)) = W(k)((u))$ is a module over $CP_\bullet(W(k)[1] \oplus W(k)/W(k))$. Define $CP^{\mathrm{cris}}_\bullet$ as the following tensor product

$$CP^{\mathrm{cris}}_\bullet(A) := CP_\bullet(A/W(k)) \otimes^L_{CP_\bullet(W(k)[1]\oplus W(k)/W(k))} W(k)((u)) \tag{4.1}$$

Indeed, $CP^{\mathrm{cris}}_\bullet(A) \otimes^L_{W(k)} k = CP_\bullet(A \otimes^L_{W(k)} k/k) \otimes^L_{CP_\bullet(k[1]\oplus k/k)} CP_\bullet(k/k)$ where the structure on $CP_\bullet(A \otimes^L_{W(k)} k/k)$ of a module over $CP_\bullet(k[1] \oplus k/k)$ is given by the algebra structure on $A \otimes^L_{W(k)} k$ over $k \otimes^L_{W(k)} k \simeq k[1] \oplus k$ because quasi-isomorphism from Lemma 2.2 is identical when tensored by k. The latter $CP_\bullet(A \otimes^L_{W(k)} k/k) \otimes^L_{CP_\bullet(k[1]\oplus k/k)} CP_\bullet(k/k)$ is quasi-isomorphic to $CP_\bullet((A \otimes^L_{W(k)} k) \otimes^L_{k[1]\oplus k} k)$ by the Kunneth formula and that, in turn, is quasi-isomorphic to $CP_\bullet(A/k)$ so the propety (i) is satisfied.

Next, if $\tilde{A}$ is a lifting of A over $W(k)$, it gives the following flat resolution of A over $W(k)$. Put $B = cone(\tilde{A} \xrightarrow{p} \tilde{A})$ with algebra structure of a trivial square-zero extension of $\tilde{A}$ by $\tilde{A}[1]$. Since $\tilde{A}$ is term-wise flat, $\tilde{A} \otimes_{W(k)} k = A$ so $A = \tilde{A}/p\tilde{A}$ and B is quasi-isomorphic to A over $W(k)$. Let us use B to compute $CP^{\mathrm{cris}}_\bullet(A)$. Consider a DG algebra $cone(\tilde{A}[t] \xrightarrow{t} \tilde{A}[t])$. Lemma 2.2 gives us an isomorphism $CP_\bullet(B/W(k)) \simeq CP_\bullet(cone(\tilde{A} \xrightarrow{0} \tilde{A})/W(k)) = CP_\bullet(\tilde{A}[1] \oplus \tilde{A}/W(k))$ which, by functoriality of the Gauss-Manin connection fits into a commutative diagram

$$\begin{array}{ccc} CP_\bullet(R/W(k)) \otimes CP_\bullet(B/W(k)) & \longrightarrow & CP_\bullet(W(k)[1] \oplus W(k)/W(k)) \otimes CP_\bullet(\tilde{A}[1] \oplus \tilde{A}/W(k)) \\ \uparrow & & \uparrow \\ CP_\bullet(R/W(k)) & \longrightarrow & CP_\bullet(W(k)[1] \oplus W(k)/W(k)) \end{array} \tag{4.2}$$

where vertical arrows are the module structures over $CP_\bullet(R/W(k))$ and $CP_\bullet(W(k)[1] \oplus W(k)/W(k))$. Thus, $CP^{\mathrm{cris}}_\bullet(A)$ is quasi-isomorphic to

$$CP_\bullet(\tilde{A}[1] \oplus \tilde{A}/W(k)) \otimes^L_{CP_\bullet(W(k)[1]\oplus W(k)/W(k))} CP_\bullet(W(k)/W(k))$$

which is quasi-isomorphic to $CP_\bullet(\tilde{A}/W(k))$ by the Kunneth formula.

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