Singularity properties of convolutions of algebraic morphisms and applications

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Joint work with Itay Glazer

October 18, 2018
Motivation: convolution in analysis

Let $f, g \in L^1(\mathbb{R})$ be two real valued functions and recall their convolution is defined by

$$f \ast g(x) = \int_{\mathbb{R}} f(t) g(x - t) \, dt.$$  

This operation improves smoothness properties of functions.

1. We have $(f \ast g)' = f' \ast g = f \ast g'$,
2. and if $f \in C^k(\mathbb{R})$ and $g \in C^l(\mathbb{R})$ then $f \ast g \in C^{k+l}(\mathbb{R})$.

In particular, if either $f$ or $g$ is smooth then $f \ast g$ is a smooth function.

Question: Is there a geometric analogue to this phenomenon?
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Convolution in algebraic geometry: definition

From here onwards we assume our varieties and groups are defined over a field $K$ of characteristic 0.

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Let $X_1$ and $X_2$ be algebraic varieties, $G$ an algebraic group and let $\phi_1: X_1 \to G$ and $\phi_2: X_2 \to G$ be algebraic morphisms. Define their convolution $\phi_1 \ast \phi_2: X_1 \times X_2 \to G$ by

$$\phi_1 \ast \phi_2(\mathbf{x}_1, \mathbf{x}_2) = \phi_1(\mathbf{x}_1) \cdot \phi_2(\mathbf{x}_2).$$

**Example 1**

Take $\phi: \mathbb{A}^1 \to \mathbb{A}^1$ with $\phi(\mathbf{x}) = \mathbf{x}^3$. Then $\phi \ast 2(\mathbf{x}, \mathbf{y}) = \mathbf{x}^3 + \mathbf{y}^3$.

**Example 2**

Let $G$ be any algebraic group and let $[,] : G \times G \to G$ be the commutator map $[x, y] = xyx^{-1}y^{-1}$. Then $[,] \ast [,](\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) = [\mathbf{x}_1, \mathbf{y}_1] \cdot [\mathbf{x}_2, \mathbf{y}_2] = \mathbf{x}_1 \mathbf{y}_1 \mathbf{x}_2^{-1} \mathbf{y}_2^{-1}$. 

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Example 1: Take $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ with $\varphi(x) = x^3$. Then $\varphi \ast \varphi : \mathbb{A}^2 \to \mathbb{A}^1$ is $\varphi \ast \varphi(x_1, x_2) = x_1^3 + x_2^3$.

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Take $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ with $\varphi(x) = x^3$. Then $\varphi^2 := \varphi \ast \varphi(x, y) = x^3 + y^3$. 
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Convolution in algebraic geometry: definition

Let $\varphi_i: X_i \to G, i = 1, 2$ be morphisms, and consider the functions $F \varphi_i: G \to \text{Schemes}$ by $F \varphi_i(g) = \varphi_i^{-1}(g)$.

Note that as sets we have the following:

$$F \varphi_1 \ast F \varphi_2(\mathcal{s}) = (F \varphi_1 \ast F \varphi_2)(\mathcal{s}) = \bigcup g \in G \varphi_1^{-1}(g) \times \varphi_2^{-1}(g - 1 \mathcal{s}).$$

Observation (convolution commutes with counting points over finite rings)

Let $A$ be a finite ring, and consider the maps $(\varphi_i)^A: X^A \to G^A$.

Define $|F(\varphi_i)^A|: G^A \to \mathbb{N}$ by $|F(\varphi_i)^A|(g) = \left| \varphi_i^{-1}(g) \right|$. By $(\ddagger)$ we have,

$$|F(\varphi_1)^A| \ast |F(\varphi_2)^A|(\mathcal{s}) = \sum g \in G^A |F(\varphi_1)^A|(g) \cdot |F(\varphi_2)^A|(g - 1 \mathcal{s}) = |F(\varphi_1)^A \ast (\varphi_2)^A|(\mathcal{s}).$$
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Convolution in algebraic geometry: properties

Question
1. Does our convolution operation improve singularity properties of morphisms?
2. Which properties of morphisms are preserved under convolution?

Fact
1. A morphism $\phi: X \to Y$ between smooth irreducible varieties is flat at $x \in X$ if and only if $\dim \phi - 1 \circ \phi(x) = \dim X - \dim Y$.
2. A flat morphism $\phi: X \to Y$ is smooth $\iff$ all its fibers are smooth.

Properties preserved under convolutions: dominance, flatness, flatness with reduced or normal fibers, smoothness.

If $S$ is a property of morphisms which is preserved under base change and compositions and $X \to \text{Spec}(K)$ satisfies $S$, then it is preserved under convolutions.
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Proposition (Glazer-H. 2018)

Let $X$ be a smooth, absolutely irreducible variety, $G$ an algebraic group, and $\varphi : X \to G$ a dominant morphism.

1. $\varphi^* \dim G : X \to G$ is flat.

2. $\varphi^* ((\dim G + 1)) : X \to G$ is flat with reduced fibers.

3. $\varphi^* ((\dim G + 2)) : X \to G$ is flat with normal fibers.

4. These bounds are tight.

Remark 1

To see (4), take $\varphi(x_1,...,x_m) = (x_2^1, (x_1x_2)^2, (x_1x_3)^2, ..., (x_1x_m)^2)$.

2. In general, we should not expect to get a smooth morphism if we start from a non-smooth morphism (e.g. $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $\varphi(x) = x^2$, then $d\varphi^* n(0,...,0) = 0$ for all $n \in \mathbb{N}$).
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Rational singularities

Definition
A variety $X$ has rational singularities if it is normal and for every resolution of singularities $\pi: \tilde{X} \to X$ we have $R^i \pi_* (O_{\tilde{X}}) = 0$ for $i \geq 1$.

Locally, this is equivalent to the following:

Definition
An affine variety $X$ has rational singularities if it is Cohen-Macaulay, normal and if for every strong resolution of singularities $p: \tilde{X} \to X$ and regular top differential form $\omega \in \Omega_{\text{top}}^*(X_{\text{sm}})$ there exists a regular top differential form $\tilde{\omega} \in \Omega_{\text{top}}^*(\tilde{X})$ such that $\omega = \tilde{\omega}|_{X_{\text{sm}}}$.

Example
Consider the variety $X = \{ \sum_{k} x^n_i = 0 \} \subseteq \mathbb{A}^k(k > 1)$. $X$ has rational singularities if $\sum_{k} x^n_i > 1$ (and $(0, \ldots, 0)$ is not a rational singularity if $\sum_{k} x^n_i < 1$).
A variety $X$ has rational singularities if it is normal and for every resolution of singularities $\pi : \tilde{X} \to X$ we have $R^i \pi_*(O_{\tilde{X}}) = 0$ for $i \geq 1$. Locally, this is equivalent to the following:

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Example

Consider the variety $X = \{\sum_{i=1}^k x_i^{n_i} = 0\} \subseteq \mathbb{A}^k$ ($k > 1$).

$X$ has rational singularities if $\sum_{i=1}^k \frac{1}{n_i} > 1$ (and $(0, \ldots, 0)$ is not a rational singularity if $\sum_{i=1}^k \frac{1}{n_i} < 1$).
The (FRS) property and our main result

Definition
We say that a morphism $\varphi$ between smooth varieties satisfies the (FRS) property if $\varphi$ is flat with reduced fibers of rational singularities.

Remark
The (FRS) property is preserved under convolutions.

Theorem (Glazer-H. 2018)
Let $X$ be a smooth, absolutely irreducible variety, $G$ be an algebraic group and let $\varphi : X \to G$ be a dominant morphism. Then there exists $N \in \mathbb{N}$ such that for any $n > N$ the $n$-th convolution power $\varphi^*$ is (FRS).
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Example

Let $\varphi : \mathbb{A}^1 \rightarrow G = (\mathbb{A}^1, +)$ be the map $\varphi(x) = x^3$. It is flat, its only non-smooth point is 0, and as we've seen before $\varphi^* n$ is not smooth at $(0, \ldots, 0)$ for every $n \in \mathbb{N}$. Consider the $n$-fold self convolution $\varphi^* n := \varphi^* \cdots \varphi^*$ of $\varphi$:

$\varphi^*\{x^3 = 0\}$

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\begin{align*}
\varphi^* & : (\mathbb{A}^1, +) \to (\mathbb{A}^1, +) \\
\varphi^* & \text{ is flat,}
\end{align*}
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$\Rightarrow$ $\varphi^4$ is (FRS).
Let $X$ be a finite type $\mathbb{Z}$-scheme such that $X_{\mathbb{Q}} := X \times \text{Spec}(\mathbb{Z}) \to \text{Spec}(\mathbb{Q})$ is absolutely irreducible.

We have, by the Lang-Weil bounds:

$$|X(F_p^k)| = p^{k \cdot \dim X_{\mathbb{Q}}} \left(1 + O(p^{-k/2})\right)$$

for $p \gg 0$.

In particular, the asymptotics of $|X(F_p^k)|$ in $p$ only depend on $\dim X_{\mathbb{Q}}$.

If $X$ is smooth, we have for almost all primes,

$$|X(\mathbb{Z}/p^k\mathbb{Z})| = |X(F_p)|^{p^{(k-1) \cdot \dim X_{\mathbb{Q}}}} \Rightarrow \lim_{p \to \infty} \frac{|X(\mathbb{Z}/p^k\mathbb{Z})|}{p^{k \cdot \dim X_{\mathbb{Q}}}} = 1.$$

If $X$ is singular, we might get a much larger point count over $\mathbb{Z}/p^k\mathbb{Z}$:

Example: Let $X = \text{Spec}(\mathbb{Z}[x]/(x^2))$, then $|X(\mathbb{Z}/p^2\mathbb{Z})| = p^k$ but $\dim X_{\mathbb{Q}} = 0$. 

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Rational singularities and counting points over finite rings
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Proposition

Let $\varphi : X \to Y$ be a $\mathbb{Z}$-morphism with absolutely irreducible fibers. Then $\varphi_{\mathbb{Q}}$ is (FRS) ⇒ for every $k$ we have $\lim_{p \to \infty} \sup_{y \in Y(\mathbb{Z}/p^k\mathbb{Z})} \frac{|\varphi_{\mathbb{Z}/p^k\mathbb{Z}}^{-1}(y)|}{p^{k(\dim X - \dim Y)}} = 1$. 
The (FRS) property and random walks

Let $\phi: X \to G$ be a dominant morphism from a smooth irreducible variety $X$ to an algebraic group $G$ and set $X_g := \phi^{-1}(g)$.

1. Pushing-forward the uniform probability measure $\nu_X(Z/p^kZ)$ on $X(Z/p^kZ)$, we get a family of probability measures $\{\mu_{p^k} := \phi^*(\nu_X(Z/p^kZ))\}$ on $\{G(Z/p^kZ)\}$ s.t. $\mu_{p^k}(\{g\}) = |X_g(Z/p^kZ)|/|X(Z/p^kZ)|$.

2. Consider the family of random walks $R_{p^k} = \{\mu_{p^k}, G(Z/p^kZ)\}$.

The probability distribution of the $n$-th step of $R_{p^k}$ is as follows: $\mu_{p^k}^n = \mu_{p^k} \ast \ldots \ast \mu_{p^k} = \phi^*(\nu_X(Z/p^kZ)) \ast \ldots \ast \phi^*(\nu_X(Z/p^kZ)) = \phi^*(\nu_X(Z/p^kZ)^n)$.

3. Now: $\phi^*(\text{FRS})$ for some $n \in \mathbb{N} \Rightarrow$ good point count of fibers of $\phi^*$ over $Z/p^kZ$ for $p^k \gg 0 \iff$ $n$-th step of $R_{p^k}$ is uniformly close to the stationary distribution for $p^k \gg 0$. 

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Let $\varphi : X \to G$ be a dominant morphism from a smooth irreducible variety $X$ to an algebraic group $G$ and set $X_g := \varphi^{-1}(g)$.\[1\]

Pushing-forward the uniform probability measure $\nu_{X/(\mathbb{Z}/p^k\mathbb{Z})}$ on $X/(\mathbb{Z}/p^k\mathbb{Z})$, we get a family of probability measures $\{\mu_{p^k} := \varphi^*\left(\nu_{X/(\mathbb{Z}/p^k\mathbb{Z})}\right)\}_{p, k}$ on $\{G/(\mathbb{Z}/p^k\mathbb{Z})\}_{p, k}$ such that $\mu_{p^k}(\{g\}) = |X_g/(\mathbb{Z}/p^k\mathbb{Z})|/|X/(\mathbb{Z}/p^k\mathbb{Z})|$.\[2\]

Consider the family of random walks $R_{p, k} = \{\varphi^*\mu_{p^k}, G/(\mathbb{Z}/p^k\mathbb{Z})\}_{p, k}$. The probability distribution of the $n$-th step of $R_{p, k}$ is as follows: $\mu^*_{p^k} = \mu_{p^k} \ast \cdots \ast \mu_{p^k} = \varphi^*\left(\nu_{X/(\mathbb{Z}/p^k\mathbb{Z})} \ast \cdots \ast \nu_{X/(\mathbb{Z}/p^k\mathbb{Z})}\right) = \varphi^*\left(\nu_X \times \cdots \times \nu_X\right)$.\[3\]

Now: $\varphi^*n (\text{FRS})$ for some $n \in \mathbb{N} \Rightarrow \text{good point count of fibers of } \varphi^*n \text{ over } \mathbb{Z}/p^k\mathbb{Z} \text{ for } p \gg 0 \iff \text{n-th step of } R_{p, k} \text{ is uniformly close to the stationary distribution for } p \gg 0.$
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Now: \( \varphi^*_n \) (FRS) for some \( n \in \mathbb{N} \)
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3. Now: \( \varphi^n \) (FRS) for some \( n \in \mathbb{N} \)

\( \Rightarrow \) good point count of fibers of \( \varphi^n \) over \( \mathbb{Z}/p^k\mathbb{Z} \) for \( p >> 0 \).
Let $\varphi : X \to G$ be a dominant morphism from a smooth irreducible variety $X$ to an algebraic group $G$ and set $X_g := \varphi^{-1}(g)$.

1. Pushing-forward the uniform probability measure $\nu_X(\mathbb{Z}/p^k\mathbb{Z})$ on $X(\mathbb{Z}/p^k\mathbb{Z})$, we get a family of probability measures
   
   \[
   \{\mu_{p^k} := \varphi_*(\nu_X(\mathbb{Z}/p^k\mathbb{Z}))\}_{p,k} \text{ on } \{G(\mathbb{Z}/p^k\mathbb{Z})\}_{p,k} \text{ s.t. } \mu_{p^k}(\{g\}) = \frac{|X_g(\mathbb{Z}/p^k\mathbb{Z})|}{|X(\mathbb{Z}/p^k\mathbb{Z})|}.
   \]

2. Consider the family of random walks $R_{p,k} = \{(\mu_{p^k}, G(\mathbb{Z}/p^k\mathbb{Z}))\}_{p,k}$.

   The probability distribution of the $n$-th step of $R_{p,k}$ is as follows:

   \[
   \mu_{p^k}^\ast n = \mu_{p^k} \ast \ldots \ast \mu_{p^k} = \varphi_*(\nu_X(\mathbb{Z}/p^k\mathbb{Z})) \ast \ldots \ast \varphi_*(\nu_X(\mathbb{Z}/p^k\mathbb{Z})) = \varphi_*(\nu_X \times \ldots \times X(\mathbb{Z}/p^k\mathbb{Z})).
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3. Now: $\varphi^\ast n$ (FRS) for some $n \in \mathbb{N}$
   
   $\Rightarrow$ good point count of fibers of $\varphi^\ast n$ over $\mathbb{Z}/p^k\mathbb{Z}$ for $p >> 0$

   $\iff$ $n$-th step of $R_{p,k}$ is uniformly close to the stationary distribution for $p >> 0$. 

Yotam Hendel

Singularity properties of convolutions of algebraic morphisms
The (FRS) property and representation growth

Theorem (Aizenbud-Avni)
Let $G$ be a semi-simple group and let $[,] : G \times G \to G$ be the commutator map $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. Then $[,]^* : (G \times G)^{\times 2} \to G$ is (FRS).

Corollary (Aizenbud-Avni)
Let $G$ be a semi-simple algebraic group defined over $\mathbb{Z}$, and let $\Gamma$ be either of the following:

1. A compact open subgroup of $G$ ($\mathbb{Q}_p$) for some prime $p$; or
2. of the form $\Gamma = G(\mathbb{Z})$ with rank at least 2.

Then there exists a constant $C$ such that for all integers $N$, $r_N(\Gamma) := \# \{ \text{irreducible } N\text{-dimensional } \mathbb{C}\text{-reps of } \Gamma \} < C \cdot N^{41}$.

Conjecture
Let $G$ be a semi-simple group, then $[,]^* : (G \times G)^{\times 2} \to G$ is (FRS).
The (FRS) property and representation growth

Theorem (Aizenbud-Avni)

Let $G$ be a semi-simple group and let $[ , ] : G \times G \to G$ be the commutator map $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. Then $[ , ]^{*21} : (G \times G)^{21} \to G$ is (FRS).
Theorem (Aizenbud-Avni)

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Corollary (Aizenbud-Avni)

Let $G$ be a semi-simple algebraic group defined over $\mathbb{Z}$, and let $\Gamma$ be either of the following:

1. A compact open subgroup of $G(\mathbb{Q}_p)$ for some prime $p$;
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Conjecture

Let $G$ be a semi-simple group, then $[\ , \ ] \ast [\ , \ ] : (G \times G)^2 \to G$ is (FRS).
Let $K$ be a field of characteristic 0. Recall we want to show the following:

Theorem (Glazer-H. 2018)

Let $X$ be a smooth, absolutely irreducible $K$-variety, $G$ be an algebraic $K$-group and let $\phi: X \to G$ be a dominant morphism. Then there exists $N \in \mathbb{N}$ such that for any $n > N$ the $n$-th convolution power $\phi^* n$ is (FRS).
Proof of main theorem

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Proof of main theorem: Step 1 - reduce to a $\mathbb{Q}$-morphism

1. Can assume $K$ is finitely generated.

2. $\phi : X \to G$ is defined over a ring $A$ which is of finite type over $\mathbb{Q}$ and whose generic point is $K$. Denote by $\phi_A : X_A \to G_A$ the morphism $\phi$ considered as a family of $\mathbb{Q}$-morphisms parametrized over $\text{Spec}(A)$.

3. Proposition: Assume a $K$-morphism $\psi : X^* \to G$ is (FRS) at $(x, \ldots, x)$ for every $x \in X(K)$, then $\psi^* : X^\circ \to G$ is (FRS).

4. It is enough to show that for each $a \in \text{Spec}(A)(\mathbb{Q})$ there exists $n_a \in \mathbb{N}$ such that $\phi^* n_a : X^a \to G^a$ is (FRS); consider the collection $U_n := \{x \in X_A(\mathbb{Q}) : \phi^a n_a \text{ is (FRS)}\}$, then $\bigcup_{i=1}^{\infty} U_n = X_A(\mathbb{Q})$.

5. Can assume $K/\mathbb{Q}$ is a Galois extension.

6. Restrict scalars to get a $\mathbb{Q}$-morphism $\text{Res}_{K/\mathbb{Q}}(\phi)$. Now, if the morphism $\text{Res}_{K/\mathbb{Q}}(\phi^a) = \text{Res}_{K/\mathbb{Q}}(\phi^a)$ is (FRS) then so is $\phi_a$ by noting the structure of $\text{Res}_{K/\mathbb{Q}}(\phi^a) \times \text{Spec}(\mathbb{Q}) \times \text{Spec}(K)$.
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### Proposition

Assume a $K$-morphism $\psi : X^*^N \to G$ is (FRS) at $(x, \ldots, x)$ for every $x \in X(K)$, then $\psi^{*2N} : X^{2N} \to G$ is (FRS).
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**Proposition**

Assume a $K$-morphism $\psi : X^* \to G$ is (FRS) at $(x, \ldots, x)$ for every $x \in X(K)$, then $\psi^* : X^G \to G$ is (FRS).

3. It is enough to show that for each $a \in \text{Spec}(A)(\overline{\mathbb{Q}})$ there exists $n_a \in \mathbb{N}$ such that $\varphi_a^{*n_a} : X_a^{n_a} \to G_a$ is (FRS); consider the collection

$$U_n := \left\{ x \in X_A(\overline{\mathbb{Q}}) : \varphi_A^n \text{ is (FRS) at } (x, \ldots, x) \right\}, \text{ then } \bigcup_{i=1}^{\infty} U_n = X_A(\overline{\mathbb{Q}}).$$
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Definition
Let $X$ be a smooth algebraic variety, $F$ be a non-archimedean local field and $\mu$ be a measure on $X(F)$. $\mu$ is smooth if every point $x \in X(F)$ has an analytic neighborhood $x \in U_x$ and an analytic diffeomorphism $D : U_x \rightarrow O_{\dim X F}$ such that $D^*(\mu|_{U_x})$ is a Haar measure on $O_{\dim X F}$.

$\mu$ has continuous density if $\mu = f \nu$ where $f : X(F) \rightarrow \mathbb{C}$ is continuous and $\nu$ is a smooth measure on $X(F)$.

Theorem (Aizenbud-Avni)
Let $\phi : X \rightarrow Y$ be a map between smooth varieties defined over a finitely generated field $K$ of characteristic 0, and let $x \in X(K)$. Then TFAE:

1. $\phi$ is (FRS) at $x$.
2. For any finite extension $K'/K$, there exists a non-Archimedean local field $F \supseteq K'$ and a non-negative Schwartz measure $\mu$ on $X(F)$ that does not vanish at $x$ such that $(\phi|_{X(F)})^*(\mu)$ has continuous density.
Proof of main theorem: Step 2 - the analytic criterion

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Proof of main theorem: Step 3 - finding a good collection of measures

Enough to show the following:

**Theorem**

Let $\phi: X \to G$ be as before with $K = Q$.

Then there exists a collection $\{\mu_{Q^p}\}_{p > M}$ of smooth measures on $\{X(Q^p)\}_{p > M}$ where $\text{supp} (\mu_{Q^p}) = X(Z^p)$, and a number $n \in \mathbb{N}$ such that the measure $\phi^* n^* (\mu_{Q^p} \times \ldots \times \mu_{Q^p}) = (\phi^* (\mu_{Q^p}))^n$ has continuous density with respect to a Haar measure on $G(Z^p)$.

**Fact**

Let $h: G(Z^p) \to \mathbb{C}$ be a function. If the Fourier transform $F(h)$ of $h$ is absolutely integrable, then $h$ is continuous.

**Question**

Can we find a collection of measures $\{\mu_{Q^p}\}_{p > M}$ as in the theorem and an integer $N$ such that $F(\phi^* (\mu_{Q^p})^N) = (F(\phi^* (\mu_{Q^p})))^N$ is absolutely integrable for every $p$?
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---

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Digression: motivic functions
A language $L$ is a set consisting of all logical symbols (and, or, not, implies, iff, $\exists$, $\forall$, = and variables) and can have constant symbols, function symbols and relation symbols.

Example 1
The language of rings $(+, -, \cdot, 0, 1)$.

Example 2
The language of ordered abelian groups $(+, -, \leq, 0)$.

A structure of a language is a set which interprets this language.

A formula in the language $L$ is defined recursively using equalities and relation symbols in variables and constant symbols (and function symbols applied to these) and by using logical symbols (i.e. if $\eta$ and $\chi$ are formulas then so are $\neg \eta$, $\forall x \eta$, $\eta \to \chi$, $\eta \land \chi$ etc.).

A formula without free variable is called a sentence, and a theory is a consistent set of sentences which contain all its logical implications. A model of a theory is a structure which satisfies all its sentences.
Languages, formulas and theories

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A formula in the language \( \mathcal{L} \) is defined recursively using equalities and relation symbols in variables and constant symbols (and function symbols applied to these) and by using logical symbols (i.e. if \( \eta \) and \( \chi \) are formulas then so are \( \neg \eta \), \( \forall x \eta \), \( \eta \rightarrow \chi \), \( \eta \land \chi \) etc.).
Languages, formulas and theories

- A language $\mathcal{L}$ is a set consisting of all logical symbols (and, or, not, implies, iff, $\exists$, $\forall$, $=$ and variables) and can have constant symbols, function symbols and relation symbols.

Example

1. The language of rings $(+, -, \cdot, 0, 1)$.
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Languages, formulas and theories

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An example of a theory

Let $L = (+, -, \cdot, 0, 1)$ be the language of rings. The theory $TF$ of fields consists of the following sentences (along with their logical implications):

1. $\forall x, y, z [(x + y) + z = x + (y + z)]$
2. $\forall x [x + 0 = x]$
3. $\forall x [x + (−x) = 0]$
4. $\forall x [x + y = y + x]$
5. $\forall x, y, z [(x \cdot y) \cdot z = x \cdot (y \cdot z)]$
6. $\forall x [x \cdot 1 = x]$
7. $\forall x, y [x \cdot y = y \cdot x]$
8. $\forall x, y, z [x \cdot (y + z) = x \cdot y + x \cdot z]$
9. $\forall x, 0, 1 [\exists y [xy = 1]]$

Models of this theory are fields.
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The Denef-Pas language $\mathcal{L}_{DP}$ consists of the following:

1. The language of rings $\mathcal{L}_{Val} = (+, -, \cdot, 0, 1)$ for the valued field sort $VF$.
2. The language of rings $\mathcal{L}_{Res} = (+, -, \cdot, 0, 1)$ for the residue field sort $RF$.
3. The language $\mathcal{L}_{\infty Pres} = \mathcal{L}_{Pres} \cup \{\infty\}$ for the value group sort $VG$, where $\infty$ is a constant, and $\mathcal{L}_{Pres} = (+, -, \leq, \{\equiv_{mod n}\}_{n > 0}, 0, 1)$ is the Presburger language consisting of the language of ordered abelian groups along with constants $0, 1$ and a family of $2^n$-relations $\{\equiv_{mod n}\}_{n > 0}$ of congruence modulo $n$.

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$L_{DP}$-definable sets and $L_{DP}$-definable functions

Let $M$ be the collection of non-archimedean local fields with residual characteristic $> M$.

**Definition**
A definable set $X = \{X_F \mid F \in \text{Loc}_M\}$ is a collection of sets such that there exists an $L_{DP}$ formula $\eta$ and $X_F = \eta(F) \subseteq F^n \times k_m F \times \mathbb{Z}_l$ for all $F \in \text{Loc}_M$ where $M$ is large enough.

A definable function $f : X \rightarrow Y$ between $L_{DP}$-definable sets is a collection of functions $(f_F : X_F \rightarrow Y_F)_{F \in \text{Loc}_M}$ such that the collection of their graphs is an $L_{DP}$-definable set.

**Example**
The following are $L_{DP}$-definable sets.

Let $\eta = (\text{val}(x) = 2) \lor (\text{ac}(y) = 3)$ and $X = \{(x, y) \in V_F^2 : \eta(x, y)\}.$

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- Let $X \subset \mathbb{A}^n$ be an affine $\mathbb{Z}$-scheme of finite type. Then $X$ has a natural structure of an $\mathcal{L}_{\text{DP}}$-definable set where $X_F = X(F)$ for every $F \in \text{Loc}$. 

Example

The following are $\mathcal{L}_{\text{DP}}$-definable functions.

- $\{ P(x) : F^n \to F \text{ is a polynomial with coefficients in } \mathbb{Z} \text{ and } s \text{ an integer.} \}$
- $\{ \text{val}_F(P(x)) : F^n \to F \text{ is a polynomial with coefficients in } \mathbb{Z} \}$
- $\{ 1_X(O_F) : F \in \text{Loc} \}$ where $X$ is a $\mathbb{Q}$-variety.
More examples

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Motivic functions

Definition

Let \( X \) be an \( \mathbb{L} \mathbb{D} \mathbb{P} \)-definable set. A motivic function is a collection \( h = (h_F : X_F \to \mathbb{R}) \) for every \( F \in \text{Loc} \mathbb{M} \) such that for every \( x \in X_F \) it can be written as

\[
h_F(x) = \sum_{i=1}^{N} \prod_{j=1}^{\beta_{ij}} \alpha_i F(x) \prod_{l=1}^{\eta_l} \eta_l^{1-q_a i l} F(x),
\]

where \( \{\alpha_i\} \) and \( \{\beta_{ij}\} \) are \( \mathbb{Z} \)-valued \( \mathbb{L} \mathbb{D} \mathbb{P} \)-definable functions, \( q_F = |O_F/_{m_F}| \) is the size of the residue field of \( O_F \) and \( Y_{iF}, x = \{y \in k_{riF} : (x, y) \in Y_{iF}\} \) is the fiber over \( x \) where \( Y_{iF} \subseteq X \times \mathbb{R} F_{ri} \) are \( \mathbb{L} \mathbb{D} \mathbb{P} \)-definable sets.

We denote the ring of motivic functions on \( X \) by \( \mathbb{C}(X) \).

Every definable function \( f : X \to \mathbb{V} \mathbb{G} \) is motivic.
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where $\{\alpha_i\}$ and $\{\beta_{ij}\}$ are $\mathbb{Z}$-valued $\mathcal{L}_{DP}$-definable functions, $q_F = |\mathcal{O}_F/\mathcal{m}_F|$ is the size of the residue field of $\mathcal{O}_F$ and $Y_{i,F,x} = \{y \in k_{\mathcal{r}_F} : (x, y) \in Y_i \cap X_F \times F\}$ is the fiber over $x$ where $Y_i \subseteq X_F \times F$ are $\mathcal{L}_{DP}$-definable sets.
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$$h_F(x) = \sum_{i=1}^{N} |Y_{i,F,x}| q_{F}^{\alpha_{i,F}}(x) \left( \prod_{j=1}^{N'} \beta_{ij,F}(x) \right) \left( \prod_{l=1}^{N''} \frac{1}{1 - q_{F}^{\alpha_{il}}} \right),$$

where $\{\alpha_i\}$ and $\{\beta_{ij}\}$ are $\mathbb{Z}$-valued $\mathcal{L}_{\text{DP}}$-definable functions,

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Motivic functions

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Every definable function $f : X \to \text{VG}$ is motivic.
Integration theorem for motivic functions

Example

$$\int_{\mathbb{P}^1} |x|^k \, dx = \sum_{n=0}^{\infty} p^{-1} \left( p^n - p^n - k \right).$$

The ring of motivic functions is preserved under integration.

Theorem (Cluckers-Loeser, Cluckers-Gordon-Halupczok)

Let $X$ and $Y$ be $L_{DP}$-definable sets and let $f \in C(X \times Y)$ be a motivic function.

Then there exists a function $g \in C(Y)$ and $M \in \mathbb{N}$ such that for every $F \in \text{Loc}_M$ we have

$$g_F(y) = \int_X F \cdot f_F(x, y) \, dx$$

for every $y \in Y$ such that $f_F(x, y) \in L_1(X_F)$. 

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Singularity properties of convolutions of algebraic morphisms
Example

\[
\int_{\mathbb{Z}_p} |x|^k_p \, dx = \sum_{n=0}^{\infty} \frac{p-1}{p} p^{-n} p^{-nk} = \frac{p-1}{p} \frac{1}{1 - p^{-(1+k)}}.
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The theory $\mathcal{T}_{H,ac,0}$ and elimination of quantifiers

Denote by $\mathcal{T}_{H,ac,0}$ the $\mathcal{L}_{DP}$-theory of Henselian valued fields $F$ of residue characteristic zero with an angular component map $ac : F \rightarrow k_F$.

Lemma
Let $\phi$ be a sentence in $\mathcal{L}_{DP}$. Assume that $\phi$ holds in all models of $\mathcal{T}_{H,ac,0}$. Then there exists an integer $M(\phi)$ such that $\phi$ holds in all non-Archimedean local fields with residue characteristic larger than $M(\phi)$.

Theorem (Denef-Pas)
Let $\eta$ be an $\mathcal{L}_{DP}$-formula. Then there exists an $\mathcal{L}_{DP}$-formula $\eta'$ without quantifiers of the valued field sort and an integer $M$ such that $\eta$ and $\eta'$ are equivalent for every non-Archimedean local field of residue characteristic larger than $M$. 
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proof of the main theorem: Step 3.5 - defining a motivic measure

Back to our question:

Question

Can we find a collection of smooth measures \( \{ \mu_F \} \) \( F \in \text{Loc}_M \) such that

\[
\text{supp}(\mu_F) = X(O_F)
\]

for every \( F \in \text{Loc}_M \) and \( F(\phi^*(\mu_F) \ast N) = F(\phi^*(\mu_F)) \) is absolutely integrable for some \( N \) (which does not depend on \( F \))?

Set \( \mu_F := 1_{X(O_F)} \) and consider the collection \( \mu = \{ \mu_F \} \) \( F \in \text{Loc}_M \).

The collection \( \mu \) forms a motivic function.

Since the ring of motivic functions is preserved under integration, the collection \( \sigma = \{ \phi^*(\mu_F) \} \) \( F \in \text{Loc}_M \) is motivic as well.

Claim

Let \( h \in C(G) \) be an absolutely integrable, compactly supported motivic function. Then there exists \( N \in \mathbb{N} \) such that \( h \ast N_F \) has continuous density for every \( F \in \text{Loc}_M \).
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Proof of the main theorem: final step - prove a theorem on motivic functions

Theorem (Glazer-H. 2018)

Let $h$ be a compactly supported, absolutely integrable, motivic function on $A_{\mathbb{k}}$. Then there exists a real constant $\alpha < 0$ and $M \in \mathbb{N}$ such that $|F(h)(y)| < d(F) \min\{\|y\|^{\alpha}, 1\}$ for every $F \in \text{Loc}_M$, where $d(F)$ depends only on $F$.

Theorem (L1 \Rightarrow L1 + \epsilon, Glazer-H. 2018)

Let $X$ be a smooth algebraic variety, let $\mu$ be a motivic measure on $X$, and let $h$ be a compactly supported motivic function on $X$ such that $hF \in L^1(X(F), \mu_F)$ for every $F \in \text{Loc}_M$. Then there exists $\epsilon > 0$ such that $hF \in L^1 + \epsilon(X(F), \mu_F)$ for every $F \in \text{Loc}_M'$ for some $M' \in \mathbb{N}$. 
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Proof of the main theorem: $L^1 \Rightarrow L^{1+\epsilon}$

Assume $h$ is a definable function and $X = \mathbb{A}^n$. 
Proof of the main theorem: $L^1 \Rightarrow L^{1+\epsilon}$

Assume $h$ is a definable function and $X = \mathbb{A}^n$. Let $h \in L^1(X, \mu)$ be a definable function on $X$ where $\mu$ is a motivic measure on $X$, and set $I_h(s, F) = \int_{X(F)} |h_F|^s d\mu_F$. 

Using elimination of quantifiers and certain uniformization theorems, we can write the above expression as finitely many sums of the form $\sum (e_1, \ldots, e_l) \in \mathbb{N}^l p^b_1(s) e_1 + \ldots + p^b_l(s) e_l$ where $b_i(s)$ are simple functions.
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Assume $h$ is a definable function and $X = \mathbb{A}^n$.

- Let $h \in L^1(X, \mu)$ be a definable function on $X$ where $\mu$ is a motivic measure on $X$, and set $I_h(s, F) = \int_{X(F)} |h_F|^s d\mu_F$.

- Write $I_h(s, F) = \sum_{k \in \mathbb{Z}} a_k q_F^{-ks}$ where

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- Each $a_k$ can be simplified, and $I_h(s, F)$ can be written as
  
  $$q_F^{-n} \sum_{\eta \in k_F} \sum_{l_1, \ldots, l_n, k \in \mathbb{Z}} q_F^{-ks-l_1-\ldots-l_n}$$

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Proof of the main theorem: $L^1 \Rightarrow L^{1+\epsilon}$

Assume $h$ is a definable function and $X = \mathbb{A}^n$.
- Let $h \in L^1(X, \mu)$ be a definable function on $X$ where $\mu$ is a motivic measure on $X$, and set $l_h(s, F) = \int_{\mathcal{X}(F)} |h_F|^s d\mu_F$.
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- Each $a_k$ can be simplified, and $l_h(s, F)$ can be written as $q_F^{-n} \sum_{\eta \in k_F^c} \sum_{l_1, \ldots, l_n, k \in \mathbb{Z}} q_F^{-ks-l_1-\ldots-l_n}$ where $\sigma$ is an $\mathcal{L}_{DP}$-formula.
- Using elimination of quantifiers and certain uniformization theorems, we can write the above expression as finitely many sums of the form $\sum_{(e_1, \ldots, e_l) \in \mathbb{N}^l} p^{b_1(s)e_1+\ldots+b_l(s)e_l}$ where $b_i(s)$ are simple functions.
Questions?