$E_x^{\mathcal{M}} \subset \overline{\mathbb{Q}}_\ell$ $X$ smooth projective, $\mathcal{M}$ geometrically irreducible. (Del-1) the subfield generated by the coefficients of $P_x^{\mathcal{M}}$

$E_{\leq n}^{\mathcal{M}} \subset \overline{\mathbb{Q}}_\ell$ generated by $\underbrace{E_x^{\mathcal{M}}, \deg x \leq N}_{x \in |X|}$.

The number of points of bounded degree is finite, so $[E_{\leq n}^{\mathcal{M}} : \mathbb{Q}] < \infty$ by (i).

~~Thus~~ Thus we have an increasing sequence of finite extensions of $\mathbb{Q}$, and $E^{\mathcal{M}} = \bigcup E_{\leq n}^{\mathcal{M}}$.

Want to prove that $E_{\leq n}^{\mathcal{M}} = E^{\mathcal{M}}$ if $n \geq n_0$.

If $\dim X = 1$ Lafforgue proved that $\exists\, n_0$.

First step: in the case of curves, prove a quantitative version, i.e., give a bound for $n_0$ in terms of $g$.

<u>Prop. A</u>. Let $\dim X = 1$. Then $E_{\leq n}^{\mathcal{M}} = E^{\mathcal{M}}$ if $n \geq \log_{\sqrt{q}} g + \text{const}$ $(= 2 \log_q g + \text{const})$.

<u>Prop. B.</u> ($\dim X$ arbitrary) Let $x \in |X|$, $\deg x = N$. Then $\exists$ geometrically irreducible curve $C \subset X$ such that $x \in C$, $C$ is smooth at $x$, and $g_{\hat{C}} \leq aN$, $a = a(X)$, (where $\hat{C}$ is the normalization of $C$). Moreover, one can choose such $C$ so that $\mathcal{M}|_{\hat{C}}$ is still geometrically irreducible.

<u>(ii) follows from Prop. A–B by descent.</u>

Let $x \in |X|$, $\deg x = N$. Claim: if $N$ is big enough then $E_x^{\mathcal{M}} \subset E_{<N}^{\mathcal{M}}$. Choose $C$ as in Prop. B. Apply Prop. A to its normalization $\hat{C}$ (and to the pullback of $\mathcal{M}$ to $\hat{C}$). So ~~Note that $g_{\hat{C}} \leq g_C \leq aN$~~

$E_x^{\mathcal{M}} \subset E_{\leq n}^{\mathcal{M}}$, $n \sim \log_{\sqrt{q}} (aN) + \text{const}$.

$N$ large $\Rightarrow n < N$, q.e.d.

(Del-2)

Remarks <u>for myself</u>: ① The root of the equation $x = \log x + \alpha$ approximately equals $\alpha + \log\alpha$ (if $\alpha$ is big).

② Deligne's idea was to give an estimate not for $[E^{\mathcal{M}} : \mathbb{Q}]$ but for the degree of $x$ required.

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<u>Remarks.</u> a) Deligne gives a polynomial bound for $g$ rather than a linear one. This is still enough because of the log. ("All the roads lead to Rome.") The linear bound is contained in Esnault-Kerz.

b) Deligne and Esnault-Kerz prefer not to take care of the irreducibility of the pullback $\mathcal{M}|_{\hat{C}}$. Then the proof of Prop. B becomes easier, but the proof of Prop. A becomes longer. There are many choices here, and all the roads lead to Rome.

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Remark. The linear bound for $g$ follows from <u>Hurwitz's formula</u>. First, choose a finite morphism $f: X \to \mathbb{P}^n$ etale at $x$. (Or: first choose a generically etale $f$, then choose $x$ (we can shrink $X$!)) [Note: $\deg f$ is bounded by some number depending on $X$ but not $x$. Let $D \subset \mathbb{P}^n$ be the discriminant of $f$ (this makes sense because $f$ is flat).] Both Deligne and Esnault-Kerz choose $C$ of the form $C = f^{-1}(C')$, where $C' \subset \mathbb{P}^n$ is a rational curve passing through $x$, $\deg C' \leq n + \text{const}$ ("const" is necessary to ensure the geometric irreducibility of $C$ and the pullback of $\mathcal{M}$ to $\hat{C}$). Now Hurwitz formula gives

$$g_{\hat{C}} \leq \underbrace{-2\cdot \deg f + \deg D \cdot \deg C'}_{\text{linear in } n}$$

Inequality rather than equality because we consider the normalization $\hat{C}$

Prop. A′ (dim $X = 1$, $\mathcal{M}$ geometrically irreducible).

If $\frac{g}{\sqrt{q}} \le \frac{1}{4r^2}$ then $E^{\mathcal{M}} = E^{\mathcal{M}}_{\le 1}$.

(Applying this to $X \otimes \mathbb{F}_{q^m}$ for a suitable $m$, one gets almost immediately that $E^{\mathcal{M}} = E^{\mathcal{M}}_{\le n}$ if $n \ge 1 + \log_{\sqrt{q}} (4r^2 g) = \log_{\sqrt{q}} g + \text{const}$, QED.

More precisely, one applies Prop. A′ for $m = n$ and $m = n-1$.)

Why $\frac{g}{\sqrt{q}}$? We want $E^{\mathcal{M}}$ to be generated by $E^{\mathcal{M}}_x$, $x \in X(\mathbb{F}_q)$; so we definitely want $X(\mathbb{F}_q) \ne \emptyset$.

A. Weil: $|X(\mathbb{F}_q) - \underbrace{q-1}_{\text{"principal term"}}| \le \underbrace{2g\sqrt{q}}_{\text{"error term"}}$

justified if $\frac{g}{\sqrt{q}} \ll 1$.

Want $2g\sqrt{q} \le q$, i.e., $\frac{g}{\sqrt{q}} \le \frac{1}{2}$

"strangeness of $X$"

Prop. $\tilde{A}$. Same assumptions on $X, \mathcal{M}$. Suppose we have $\tilde{\mathcal{M}}$ satisfying the same assumptions. If

$\mathrm{Tr}(F_x, \mathcal{M}) = \mathrm{Tr}(F_x, \tilde{\mathcal{M}})$, $\forall x \in X(\mathbb{F}_q)$ then $\tilde{\mathcal{M}} \simeq \mathcal{M}$.

Why Prop. $\tilde{A} \Rightarrow$ Prop. A′. By Galois theory, to prove that $E^{\mathcal{M}} = E^{\mathcal{M}}_{\le 1}$, we have to show that $\sigma(P_x) = P^{\mathcal{M}}_x$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E^{\mathcal{M}}_{\le 1})$ (here $\overline{\mathbb{Q}} < \overline{\mathbb{Q}}_\ell$). Now we use Part (iii) of Deligne's conjecture for curves, proved by Lafforgue. It says that $\sigma(P_x) = P^{\sigma\mathcal{M}}_x$, where ${}^\sigma\mathcal{M}$ is the "$\sigma$-companion" of $\mathcal{M}$. Now apply Proposition $\tilde{A}$ to $\tilde{\mathcal{M}} = {}^\sigma\mathcal{M}$.

Proof of Prop. $\tilde{A}$. Define $f, \tilde{f} : X(\mathbb{F}_q) \to \overline{\mathbb{Q}}$,
$f(x) := \mathrm{Tr}(F_x, \mathcal{M})$, $\tilde{f}(x) := \mathrm{Tr}(F_x, \tilde{\mathcal{M}})$.

Claim. If $\tilde{\mathcal{M}}$ and $\mathcal{M}$ are not "geometrically isomorphic" then $|(f, \tilde{f})| < (f, f)$ (in fact, $f$ and $\tilde{f}$ are "almost orthogonal" if the "strangeness" of $X$ is small).

Comment. The definition of scalar product involves the complex conjugation corresponding to some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. The claim holds for any such embedding. [For myself: the true statement is that the totally real number $(f,f)^2 - |(f,\tilde{f})|^2$ is totally positive.]

Proof of the claim: it is based on the Lefschetz formula and the sheaf-theoretic analog of the Weil estimate proved by Deligne in Weil-II.

$\mathcal{M}, \tilde{\mathcal{M}}$ ~~have~~ are pure of weight 0. $\overline{\tilde{f}(x)} = \mathrm{Tr}(F_x, \tilde{\mathcal{M}}^*)$

$(f, \tilde{f}) = \sum_i (-1)^i \mathrm{Tr}(F_r, H^i(X \otimes \overline{\mathbb{F}}_q, \mathcal{M} \otimes \tilde{\mathcal{M}}^*))$.

$\mathcal{M}, \tilde{\mathcal{M}}$ not geometrically isomorphic $\Rightarrow H^0 = H^2 = 0$.

$\dim H^1 = (2g-2)r^2$. The Frobenius eigenvalues on $H^1$ have abs. value $\sqrt{q}$ (by Weil-II), so $|(f,\tilde{f})| \le (2g-2)r^2\sqrt{q} < 2gr^2\sqrt{q}$

Similarly, $|(f,f) - q - 1| \le A\sqrt{q}$, $A = (2g-2)r^2 + 2 \le 2gr^2\sqrt{q}$ (here $q+1$ comes from $H^2$ and $H^0$, which are 1-dimensional)

$\frac{g}{\sqrt{q}}$ small $\Rightarrow |(f,\tilde{f})| < |(f,f)|$.

End of the proof. Thus $\tilde{\mathcal{M}}$ is geometrically isomorphic to $\mathcal{M}$, so $\tilde{\mathcal{M}}$ is a twist of $\mathcal{M}$ (by some 1-dimensional local system on $\mathrm{Spec}\,\mathbb{F}_q$). If the twist is nontrivial then $\tilde{f} = \alpha f$ for $\alpha \ne 1$. On the other hand, $\tilde{f} = f$ by assumption. So $f = 0$. But we already saw that $(f,f) > 0$,