ON A CONJECTURE OF DELIGNE

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Dedicated to the memory of I.M.Gelfand

Abstract. Let X be a smooth variety over $\mathbb{F}_p$. Let E be a number field. For each nonarchimedean place λ of E prime to p consider the set of isomorphism classes of irreducible lisse $\overline{E}_\lambda$-sheaves on X with determinant of finite order such that for every closed point $x \in X$ the characteristic polynomial of the Frobenius F_x has coefficents in E. We prove that this set does not depend on λ.

The idea is to use a method developed by G. Wiesend to reduce the problem to the case where X is a curve. This case was treated by L. Lafforgue.

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1. Introduction

1.1. Main theorem.

Theorem 1.1. *Let X be a smooth scheme over $\mathbb{F}_p$. Let E be a finite extension of $\mathbb{Q}$. Let λ, λ' be nonarchimedean places of E prime to p and E_λ, $E_{\lambda'}$ the corresponding completions. Let $\mathcal{E}$ be a lisse $\overline{E}_{\lambda'}$-sheaf on X such that for every closed point $x \in X$ the polynomial $\det(1 - F_x t, \mathcal{E})$ has coefficients in E and its roots are λ-adic units. Then there exists a lisse $\overline{E}_\lambda$-sheaf on X compatible with $\mathcal{E}$ (i.e., having the same characteristic polynomials of the operators F_x for all closed points $x \in X$).*

According to a conjecture of Deligne (see §1.2 below), Theorem 1.1 should hold for *any normal scheme* of finite type over $\mathbb{F}_p$.

Remark 1.2. If $\dim X = 1$ then Theorem 1.1 is a well known corollary of the Langlands conjecture for $GL(n)$ over functional fields proved by L. Lafforgue [Laf]. More precisely, it immediately follows from [Laf, Theorem VII.6].

We will deduce Theorem 1.1 from the particular case $\dim X = 1$ using a powerful and general method developed by G. Wiesend [W1] (not long before his untimely death).

1.2. Deligne's conjecture.

Here is a part of [De3, Conjecture 1.2.10)].

Conjecture 1.3. *Let X be a normal scheme of finite type over $\mathbb{F}_p$, $\ell \neq p$ a prime, and $\mathcal{E}$ an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X whose determinant has finite order.*

(a) *There exists a subfield $E \subset \overline{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}$ such that for every closed point $x \in X$ the polynomial $\det(1 - F_x t, \mathcal{E})$ has coefficients in E.*

(b) *For a possibly bigger E and every nonarchimedean place λ of E prime to p there exists a lisse E_λ-sheaf compatible with $\mathcal{E}$.*

(c) *The roots of the polynomials $\det(1 - F_x t, \mathcal{E})$ (and therefore their inverses) are integral over $\mathbb{Z}[p^{-1}]$*

In the case of curves Conjecture 1.3 was completely proved by Lafforgue [Laf, Theorem VII.6]. Using a Bertini argument[1], he deduced from this a part of Conjecture 1.3 for $\dim X > 1$. Namely, he proved statement (c) and the following part of (a): all the coefficients of the polynomials $\det(1 - F_x t, \mathcal{E})$, $x \in X$,

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[1] The Bertini argument is clarified in [De5, §1.5-1.9]. A somewhat similar technique is explained in Appendix C below.

are algebraic numbers. The fact that the extension of $\mathbb{Q}$ generated by these algebraic numbers is finite was proved by Deligne [De5]. Combining (a), (c), Theorem 1.1, and the main result of [Ch] one gets (b) in the case where X is smooth (one needs [Ch] to pass from $\overline{E}_\lambda$-sheaves to E_λ-sheaves.)

1.3. **An open question.** We will deduce Theorem 1.1 from the particular case $\dim X = 1$ treated by L. Lafforgue [Laf] and a more technical Theorem 2.5 (in which we consider an arbitrary regular scheme X of finite type over $\mathbb{Z}[\ell^{-1}]$). Following G. Wiesend [W1], we will prove Theorem 2.5 "by pure thought" (see §2.4 for more details). Such a proof of Theorem 1.1 turns out to be possible because Wiesend's method allows to *bypass* the following problem.

Question 1.4. Let X be an irreducible smooth variety over $\mathbb{F}_q$ and let $\mathcal{E}$ be an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf of rank r on X whose determinant has finite order. Let K be the field of rational functions on X and let ρ be the ℓ-adic representation of $\mathrm{Gal}(\bar{K}/K)$ corresponding to $\mathcal{E}$. Is it true that a certain Tate twist of ρ appears as a subquotient of $H^i(Y \otimes_K \bar{K}, \overline{\mathbb{Q}}_\ell)$ for some algebraic variety Y over K and some number i?

L. Lafforgue [Laf] gave a positive answer to Question 1.4 if $\dim X = 1$. Without assuming that $\dim X = 1$, we prove in this article that ρ is a part of a compatible system of representations $\rho_\lambda : \mathrm{Gal}(\bar{K}/K) \to GL(r, \overline{E}_\lambda)$, where $E \subset \overline{\mathbb{Q}}_\ell$ is the number field generated by the coefficients of the polynomials $\det(1 - F_x t, \mathcal{E})$, $x \in X$, and λ runs through the set of all nonarchimedean places of E prime to p. Nevertheless, *Question 1.4 remains open* if $\dim X > 1$ because it is not clear how to formulate a motivic analog of the construction from §4.1.

1.4. **Application: the Grothendieck group of weakly motivic $\overline{\mathbb{Q}}_\ell$-sheaves.** Theorem 1.1 allows to associate to each scheme X of finite type over $\mathbb{F}_p$ a certain group $K_{\mathrm{mot}}(X, \overline{\mathbb{Q}})$ (where "mot" stands for "motivic") and according to a theorem of Gabber [Fuj], the "six operations" are well defined on K_{mot}. Details are explained in §1.4.1-1.4.3 below.

Morally, $K_{\mathrm{mot}}(X, \overline{\mathbb{Q}})$ should be the Grothendieck group of the "category of motivic $\overline{\mathbb{Q}}$-sheaves" on X. (Here the words in quotation marks do not refer to any precise notion of motivic sheaf.)

1.4.1. *A corollary of Theorem 1.1.* Let X be a scheme of finite type over $\mathbb{F}_p$. The set of its closed points will be denoted by $|X|$. Let ℓ be a prime different from p and let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Let $\mathrm{Sh}(X, \overline{\mathbb{Q}}_\ell)$ be the abelian category of $\overline{\mathbb{Q}}_\ell$-sheaves on X and $\mathscr{D}(X, \overline{\mathbb{Q}}_\ell) = D^b_c(X, \overline{\mathbb{Q}}_\ell)$ the bounded ℓ-adic derived category [De3, §1.2-1.3].

Now let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Suppose that we are given a map

$$\Gamma : |X| \to \{\text{subsets of } \overline{\mathbb{Q}}^\times\}, \qquad x \mapsto \Gamma_x . \tag{1.1}$$

Once we choose a prime $\ell \neq p$, an algebraic closure $\overline{\mathbb{Q}}_\ell \supset \mathbb{Q}_\ell$, and an embedding $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ we can consider the following full subcategory $\mathrm{Sh}_\Gamma(X, \overline{\mathbb{Q}}_\ell, i) \subset \mathrm{Sh}(X, \overline{\mathbb{Q}}_\ell)$: a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ is in $\mathrm{Sh}_\Gamma(X, \overline{\mathbb{Q}}_\ell)$ if for every closed point $x \in X$ all eigenvalues of the geometric Frobenius $F_x : \mathcal{F}_x \to \mathcal{F}_x$ are in $i(\Gamma_x)$. Let $\mathscr{D}_\Gamma(X, \overline{\mathbb{Q}}_\ell, i) \subset \mathscr{D}_\Gamma(X, \overline{\mathbb{Q}}_\ell)$ be the full subcategory of complexes whose cohomology sheaves are in $\mathrm{Sh}_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$. Let $K_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$ denote the Grothendieck group of $\mathrm{Sh}_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$, which is the same as the Grothendieck group of $\mathscr{D}_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$.

For any field E set

$$A(E) := \{f \in E(t)^\times \mid f(0) = 1\}.$$

A sheaf $\mathcal{F} \in \mathrm{Sh}_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$ defines a map

$$f_{\mathcal{F}} : |X| \to A(i(\overline{\mathbb{Q}})) = A(\overline{\mathbb{Q}}), \qquad x \mapsto \det(1 - F_x t, \mathcal{F}).$$

For any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $f_{\mathcal{F}} = f_{\mathcal{F}'} f_{\mathcal{F}/\mathcal{F}'}$, so we get a homomorphism $K_\Gamma(X, \overline{\mathbb{Q}}_\ell, i) \to A(\overline{\mathbb{Q}})^{|X|}$, where $A(\overline{\mathbb{Q}})^{|X|}$ is the group of all maps $|X| \to A(\overline{\mathbb{Q}})$.

Lemma 1.5. *This map is injective.*

Proof. We have to show that if $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Sh}_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$ have equal images in $A(\overline{\mathbb{Q}})^{|X|}$ then they have equal classes in $K_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$. Stratifying X, one reduces this to the case where $\mathcal{F}_1, \mathcal{F}_2$ are lisse and X is normal. Then use the Čebotarev density theorem. □

Lemma 1.5 allows to consider $K_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$ as a subgroup of $A(\overline{\mathbb{Q}})^{|X|}$. The next statement immediately follows from Theorem 1.1 and Conjecture 1.3(a) proved by Deligne [De5].

Corollary 1.6. *Suppose that for each $x \in |X|$ all elements of Γ_x are units outside of p. Then the subgroup $K_\Gamma(X, \overline{\mathbb{Q}}_\ell, i) \subset A(\overline{\mathbb{Q}})^{|X|}$ does not depend on the choice of ℓ, $\overline{\mathbb{Q}}_\ell$, and $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.* □

In the situation of Corollary 1.6 we will write simply $K_\Gamma(X, \overline{\mathbb{Q}})$ instead of $K_\Gamma(X, \overline{\mathbb{Q}}_\ell, i)$.

1.4.2. *Weakly motivic $\overline{\mathbb{Q}}_\ell$-sheaves and their Grothendieck group.* Let us consider two particular choices of the map (1.1).

Definition 1.7. *For $x \in |X|$ let $\Gamma_x^{\mathrm{mix}} \subset \overline{\mathbb{Q}}^\times$ be the set of numbers $\alpha \in \overline{\mathbb{Q}}^\times$ with the following property: there exists $n \in \mathbb{Z}$ such that all complex absolute values of α equal $q_x^{n/2}$, where q_x is the order of the residue field of x. Let Γ_x^{mot} be the set of those numbers from Γ_x^{mix} that are units outside of p.*

In other words, Γ_x^{mot} is the *group of Weil numbers* with respect to q_x.

Since Γ_x^{mix} is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the categories $\mathrm{Sh}_{\Gamma^{\mathrm{mot}}}(X, \overline{\mathbb{Q}}_\ell, i)$ and $\mathscr{D}_{\Gamma^{\mathrm{mot}}}(X, \overline{\mathbb{Q}}_\ell, i)$ do not depend on the choice of $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. We denote them by $\mathrm{Sh}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$ and $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$. We also have similar categories $\mathrm{Sh}_{\mathrm{mix}}(X, \overline{\mathbb{Q}}_\ell)$ and $\mathscr{D}_{\mathrm{mix}}(X, \overline{\mathbb{Q}}_\ell)$.

Definition 1.8. *Objects of* $\mathrm{Sh}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$ *(resp.* $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$*) are called* weakly motivic $\overline{\mathbb{Q}}_\ell$-sheaves *(resp.* weakly motivic $\overline{\mathbb{Q}}_\ell$-complexes).

Remarks 1.9.
(i) A result of L. Lafforgue [Laf, Corollary VII.8] implies that $\mathrm{Sh}_{\mathrm{mix}}(X, \overline{\mathbb{Q}}_\ell)$ is equal to the category of *mixed $\overline{\mathbb{Q}}_\ell$-sheaves* introduced by Deligne [De3, Definition 1.2.2]. Therefore $\mathscr{D}_{\mathrm{mix}}(X, \overline{\mathbb{Q}}_\ell)$ is equal to the category of *mixed $\overline{\mathbb{Q}}_\ell$-complexes* from [De3, §6.2.2].
(ii) According to [De3] and [BBD], the category $\mathscr{D}_{\mathrm{mix}}(X, \overline{\mathbb{Q}}_\ell)$ is stable under all "natural" functors (e.g., under Grothendieck's "six operations"). The same is true for $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$, see Appendix B.
(iii) Any indecomposable object of $\mathscr{D}(X, \overline{\mathbb{Q}}_\ell)$ is a tensor product of an object of $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$ and an invertible $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathrm{Spec}\, \mathbb{F}_p$ (see Theorem B.7 from Appendix B).

Corollary 1.6 is applicable to Γ_x^{mot} (but not to Γ_x^{mix}). So we have a well defined group

$$K_{\mathrm{mot}}(X, \overline{\mathbb{Q}}) := K_{\Gamma^{\mathrm{mot}}}(X, \overline{\mathbb{Q}}) \,.$$

Remark 1.10. Let C denote the union of all CM-subfields of $\overline{\mathbb{Q}}$. For any prime power q, the subfield of $\overline{\mathbb{Q}}$ generated by all Weil numbers with respect to q equals C (see Theorem D.1 from Appendix D). So for any $X \neq \emptyset$, the kernel of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $K_{\mathrm{mot}}(X, \overline{\mathbb{Q}})$ equals $\mathrm{Gal}(\overline{\mathbb{Q}}/C)$.

1.4.3. *Functoriality of $K_{\mathrm{mot}}(X, \overline{\mathbb{Q}})$.* By Remark 1.9(ii), the "six operations" preserve the class of weakly motivic $\overline{\mathbb{Q}}_\ell$-complexes. So it is clear that once you fix a prime $\ell \neq p$, an algebraic closure $\overline{\mathbb{Q}}_\ell \supset \mathbb{Q}_\ell$, and an embedding $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, you get an action of the "six operations" on K_{mot}. O. Gabber proved [Fuj, Theorem 2] that in fact, *the action of the "six operations" on K_{mot} does not depend on the choice of ℓ, $\overline{\mathbb{Q}}_\ell$, and i.* By virtue of Theorem 1.1 and Conjecture 1.3(a) proved by Deligne, another result of Gabber [Fuj, Theorem 3] can be reformulated as follows: *the basis of $K_{\mathrm{mot}}(X, \overline{\mathbb{Q}})$ formed by the classes of irreducible perverse sheaves is independent of ℓ, $\overline{\mathbb{Q}}_\ell$, and i.*

1.5. **Structure of the article.** In §2.1-2.2 we formulate Theorem 2.5, which is the main technical result of this article. It gives a criterion for the existence of a lisse E_λ-sheaf on a regular scheme X of finite type over $\mathbb{Z}[\ell^{-1}]$ with prescribed polynomials $\det(1 - F_x t, \mathcal{E})$, $x \in X$; the criterion is formulated in terms of 1-dimensional subschemes of X. In §2.3 we deduce Theorem 1.1 from Theorem 2.5.

In §2.4 we formulate three propositions which imply Theorem 2.5. They are proved in §3-5.

Following Wiesend [W1], we use the Hilbert irreducibility theorem as the main technical tool. A variant of this theorem convenient for our purposes is formulated in §2.5 (see Theorem 2.15) and proved in Appendix A. In the case of schemes over $\mathbb{F}_p$ one can use Bertini theorems instead of Hilbert irreducibility, see §2.6.

In §6 we give counterexamples showing that in Theorems 2.5 and 2.15 the regularity assumption cannot be replaced by normality.

In Appendix B we show that the category of weakly motivic $\overline{\mathbb{Q}}_\ell$-sheaves, $\mathscr{D}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell)$, defined in §1.4.2 is stable under all "natural" functors.

In Appendix C we discuss the Bertini theorem and Poonen's "Bertini theorem over finite fields".

In Appendix D we justify Remark 1.10 by proving that the union of all CM fields is generated by Weil numbers.

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2. Formulation of the main technical theorem

Fix a prime ℓ and a finite extension $E_\lambda \supset \mathbb{Q}_\ell$. Let $O \subset E_\lambda$ denote its ring of integers.

2.1. The sets $\mathrm{LS}_r(X)$ and $\widetilde{\mathrm{LS}}_r(X)$.

2.1.1. *The set* $\mathrm{LS}_r(X)$. Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. Say that lisse E_λ-sheaves on X are *equivalent* if they have isomorphic semisimplifications. Let $\mathrm{LS}_r(X) = \mathrm{LS}_r^{E_\lambda}(X)$ be the set of equivalence classes of lisse E_λ-sheaves on X of rank r. Clearly $\mathrm{LS}_r(X)$ is a contravariant functor in X.

Example 2.1. Suppose that X has a single point x. If $\mathcal{E}$ is an E_λ-sheaf on X of rank r then $\det(1-F_x t, \mathcal{E})$ is a polynomial in t of the form

$$1 + c_1 t + \ldots + c_r t^r, \quad c_i \in O,\ c_r \in O^\times. \tag{2.1}$$

Let $P_r(O)$ be the set of all polynomials of the from (2.1). Thus we get a bijection $\mathrm{LS}_r(X) \xrightarrow{\sim} P_r(O)$.

2.1.2. *The sets* $|X|$ *and* $||X||$. Let X be a scheme of finite type over $\mathbb{Z}$. We write $|X|$ for the set of closed points of X. Let $||X||$ denote the set of isomorphism classes of pairs consisting of a finite field $\mathbb{F}$ and a morphism $\alpha : \operatorname{Spec} \mathbb{F} \to X$. Associating to such α the point $\alpha(\operatorname{Spec} \mathbb{F}) \in |X|$ one gets a canonical map $||X|| \to |X|$. The multiplicative monoid of positive integers, $\mathbb{N}$, acts on $||X||$ (namely, $n \in \mathbb{N}$ acts by replacing a finite field $\mathbb{F}$ with its extension of degree n). Both $||X||$ and $|X|$ depend functorially on X. The map $||X|| \to |X|$ and the action of $\mathbb{N}$ on $||X||$ are functorial in X.

Remark 2.2. One has a canonical embedding $|X| \hookrightarrow ||X||$ (given $x \in |X|$ take $\mathbb{F}$ to be the residue field of x and take $\alpha : \operatorname{Spec} \mathbb{F} \to X$ to be the canonical embedding). Combining the embedding $|X| \hookrightarrow ||X||$ with the action of $\mathbb{N}$ on $||X||$ we get a bijection $\mathbb{N} \times |X| \xrightarrow{\sim} ||X||$. Note that the embedding $|X| \hookrightarrow ||X||$ and the bijection $\mathbb{N} \times |X| \xrightarrow{\sim} ||X||$ are *not functorial* in X.

2.1.3. *The set* $\widetilde{\mathrm{LS}}_r(X)$. The set $P_r(O)$ from Example 2.1 can be thought of as the set of O-points of a scheme P_r (which is isomorphic to $(\mathbb{G}_a)^{r-1} \times \mathbb{G}_m$). The morphism $(\mathbb{G}_m)^r \to P_r$ that takes $(\beta_1, \ldots, \beta_r)$ to the polynomial $\prod_{i=1}^{r} (1-\beta_i t)$ induces an isomorphism $(\mathbb{G}_m)^r / S_r \xrightarrow{\sim} P_r$, where S_r is the symmetric group. The ring homomorphism $\mathbb{Z} \to \operatorname{End}((\mathbb{G}_m)^r)$ defines an action of the multiplicative monoid $\mathbb{N}$ on $(\mathbb{G}_m)^r$ and therefore an action of $\mathbb{N}$ on P_r. Now let X be a scheme of finite type over $\mathbb{Z}$.

Definition 2.3. $\widetilde{\mathrm{LS}}_r(X)$ *(or* $\widetilde{\mathrm{LS}}_r^{E_\lambda}(X)$*) is the set of* $\mathbb{N}$*-equivaraint maps* $||X|| \to P_r(O)$.

Remark 2.4. By Remark 2.2, restricting an $\mathbb{N}$-equivaraint map $||X|| \to P_r(O)$ to the subset $|X| \subset ||X||$ one gets a bijection

$$\widetilde{\mathrm{LS}}_r(X) \xrightarrow{\sim} \{\text{Maps from } |X| \text{ to } P_r(O)\}. \tag{2.2}$$

It is not functorial in X if the structure of functor on the r.h.s. of (2.2) is introduced naively. Nevertheless, we will use (2.2) in order to write elements of $\widetilde{\mathrm{LS}}_r(X)$ as maps $|X| \to P_r(O)$. The value of $f \in \widetilde{\mathrm{LS}}_r(X)$ at $x \in |X|$ will be denoted by $f(x)$ or f_x (the latter allows to write the corresponding polynomial (2.1) as $f_x(t)$).

2.1.4. *The map* $\mathrm{LS}_r(X) \to \widetilde{\mathrm{LS}}_r(X)$. Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. A lisse E_λ-sheaf $\mathcal{E}$ on X defines an $\mathbb{N}$-equivaraint map $f_{\mathcal{E}} : ||X|| \to P_r(O)$: namely, if $\mathbb{F}$ is a finite field equipped with a morphism $\alpha : \operatorname{Spec}\mathbb{F} \to X$ then $f_{\mathcal{E}}(\mathbb{F}, \alpha) \in P_r(O)$ is the characteristic polynomial of the geometric Frobenius with respect to $\mathbb{F}$ acting on the stalk of $\alpha^*\mathcal{E}$. Thus we get a map $\mathrm{LS}_r(X) \to \widetilde{\mathrm{LS}}_r(X)$ functorial in X.

If X_{red} is normal this map is injective by the Čebotarev density theorem (see [S, Theorem 7]). In this case we do not distinguish an element of $\mathrm{LS}_r(X)$ from its image in $\widetilde{\mathrm{LS}}_r(X)$ and consider $\mathrm{LS}_r(X)$ as a subset of $\widetilde{\mathrm{LS}}_r(X)$.

2.2. **Formulation of the theorem.** If X is a separated curve over a field[2] there is a well known notion of tame etale covering and therefore a notion of tame lisse E_λ-sheaf ("tame" means "tamely ramified at infinity"). If two lisse E_λ-sheaves are equivalent (i.e., have isomorphic semisimplifications) and one of them is tame then so is the other. Let $\mathrm{LS}_r^{\mathrm{tame}}(X) \subset \mathrm{LS}_r(X)$ be the subset of equivalence classes of tame lisse E_λ-sheaves.

By an *arithmetic curve* we mean a scheme of finite type over $\mathbb{Z}$ of pure dimension 1.

Theorem 2.5. *Let X be a regular scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. An element $f \in \widetilde{\mathrm{LS}}_r(X)$ belongs to $\mathrm{LS}_r(X)$ if and only if it satisfies the following conditions:*

(i) *for every regular arithmetic curve C and every morphism $\varphi : C \to X$ one has $\varphi^*(f) \in \mathrm{LS}_r(C)$;*

(ii) *there exists a dominant etale morphism $X' \to X$ such that for every smooth separated curve C over a finite field and every morphism $C \to X'$ the image of f in $\widetilde{\mathrm{LS}}_r(C)$ belongs to $\mathrm{LS}_r^{\mathrm{tame}}(C)$.*

Remarks 2.6. (i) In Theorem 2.5 the regularity assumption on X cannot be replaced by normality (in §6 we give two counterexamples, in which X is a surface over a finite field). The regularity assumption allows us to use the Zariski-Nagata purity theorem in the proof of Corollary 5.2 and to apply Theorem 2.15.

(ii) I do not know if the regularity assumption in Theorem 1.1 can be replaced by normality.

(iii) The sets $\mathrm{LS}_r(X)$ and $\widetilde{\mathrm{LS}}_r(X)$ were defined in §2.1 for a fixed finite extension $E_\lambda \supset \mathbb{Q}_\ell$, so $\mathrm{LS}_r(X) = \mathrm{LS}_r^{E_\lambda}(X)$, $\widetilde{\mathrm{LS}}_r(X) = \widetilde{\mathrm{LS}}_r^{E_\lambda}(X)$. Replacing E_λ by $\overline{\mathbb{Q}}_\ell$ in these definitions one gets bigger sets, denoted by $\mathcal{LS}_r(X)$ and $\widetilde{\mathcal{LS}}_r(X)$. *If X is a smooth scheme over $\mathbb{F}_p$ then Theorem 2.5 remains valid for $\mathcal{LS}_r(X)$ and $\widetilde{\mathcal{LS}}_r(X)$ instead of* $\mathrm{LS}_r(X)$ *and* $\widetilde{\mathrm{LS}}_r(X)$. This follows from Theorem 2.5 as stated above combined with [De5, Remark 3.10] and Lemma 2.7 below (the remark and the lemma ensure that an element of $\widetilde{\mathcal{LS}}_r(X)$ satisfying conditions (i-ii) from Theorem 2.5 belongs to $\widetilde{\mathrm{LS}}_r^{E_\lambda}(X)$ for some subfield $E_\lambda \subset \overline{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}_\ell$).

(iv) Theorem 2.5 and its proof remain valid for regular *algebraic spaces* of finite type over $\mathbb{Z}[\ell^{-1}]$.

2.3. **Theorem 2.5 implies Theorem 1.1.**

Lemma 2.7. *Let G be a group. Let ρ be a semisimple representation of G over $\overline{E}_\lambda$ of dimension $r < \infty$ whose character is defined over E_λ. Let $F \subset \overline{E}_\lambda$ be any extension of E_λ such that $[F : E_\lambda]$ is divisible by r, $r-1$,..., 2. Then ρ can be defined over F.*

[2] According to [W2, KS2], there is a good notion of tameness in a much more general situation.

Proof. We can assume that ρ cannot be decomposed into a direct sum of representations of dimension $< r$ whose characters are defined over E_λ. Then $\rho = \bigoplus_{i\in I} \rho_i$, where the ρ_i's are irreducible, $\rho_i \not\simeq \rho_j$ for $i \neq j$, and the action of $\mathrm{Gal}(\overline{E}_\lambda/E_\lambda)$ on I is transitive. Clearly $\dim \rho_i = r' := r/d$, where $d = \mathrm{Card}(I)$. Let us show that for every $\mathrm{Gal}(\overline{E}_\lambda/F)$-orbit $\mathcal{O} \subset I$, the representation $\rho_{\mathcal{O}} = \bigoplus_{i\in\mathcal{O}} \rho_i$ can be defined over F. Fix $i_0 \in \mathcal{O}$, then the stabilizer of i_0 in $\mathrm{Gal}(\overline{E}_\lambda/E_\lambda)$ equals $\mathrm{Gal}(\overline{E}_\lambda/K)$ for some extension $K \supset E_\lambda$ of degree d. The character of ρ_{i_0} is defined over K. The obstruction to ρ_{i_0} being defined over K is an element $u \in \mathrm{Br}(K)$ with $r'u = 0$. We claim that

$$u \in \mathrm{Ker}(\mathrm{Br}(K) \to \mathrm{Br}(KF)), \tag{2.3}$$

where $KF \subset \overline{E}_\lambda$ is the composite field. To prove this, it suffices to check that

$$r' \,|\, [KF : K] \,. \tag{2.4}$$

But $r \,|\, [F : E_\lambda]$ (by the assumption on F), so $r \,|\, [KF : E_\lambda]$, which is equivalent to (2.4). By (2.3), ρ_{i_0} is defined over KF, i.e., $\rho_{i_0} \simeq V \otimes_{KF} \overline{E}_\lambda$, where V is a representation of G over KF. Then $V \otimes_K \overline{E}_\lambda = V \otimes_{KF} (KF \otimes_K \overline{E}_\lambda) \simeq \rho_{\mathcal{O}}$, so $\rho_{\mathcal{O}}$ is defined over K. ☐

Now let us deduce Theorem 1.1 from Theorem 2.5. Let E, E_λ, $E_{\lambda'}$, and $\mathcal{E}$ be as in Theorem 1.1. Then $\mathcal{E}$ defines an element $f \in \widetilde{\mathrm{LS}}_r^{E_\lambda}(X)$ and the problem is to show that $f \in \mathrm{LS}_r^F(X)$ for some finite extension $F \supset E_\lambda$. Let $F \supset E_\lambda$ be any extension of degree $r!$. Let us show that $f \in \widetilde{\mathrm{LS}}_r^F(X)$ satisfies conditions (i)-(ii) of Theorem 2.5. Condition (i) follows from Theorem 1.1 for curves (proved by L. Lafforgue) combined with Lemma 2.7. Condition (ii) holds for f viewed as an element of $\mathrm{LS}_r^{E_{\lambda'}}(X)$. To conclude that it holds for f viewed as an element of $\widetilde{\mathrm{LS}}_r^F(X)$, use the following corollary of [De1, Theorem 9.8]: if C is a smooth curve over a finite field, $\mathcal{F}'$ is a tame lisse $E_{\lambda'}$-sheaf on C, and $\mathcal{F}$ is a lisse F-sheaf on C compatible with $\mathcal{F}'$ then $\mathcal{F}$ is also tame. Now we can apply Theorem 2.5 and get Theorem 1.1.

2.4. **Steps of the proof of Theorem 2.5.** We follow Wiesend's work [W1] (see also the related article [KS1]).

The "only if" statement of Theorem 2.5 is easy. First of all, if $\mathcal{E}$ is a torsion-free lisse O-sheaf of rank r then the corresponding $f \in \mathrm{LS}_r(X)$ clearly satisfies (i). Property (ii) also holds for f: choose X' so that $\mathcal{E}/\ell\mathcal{E}$ is trivial and use the fact that the kernel of the homomorphism $GL(r, O) \to GL(r, O/\ell O)$ is a pro-ℓ-group, so it cannot contain nontrivial pro-p-subgroups for $p \neq \ell$.

The "if" statement of Theorem 2.5 follows from Propositions 2.12-2.14 formulated below.

Definition 2.8. *A pro-finite group is said to be* almost pro-ℓ *if it has an open pro-ℓ-subgroup.*

Remark 2.9. In this case the open pro-ℓ-subgroup can be chosen to be normal.

Definition 2.10. *We say that $f \in \widetilde{\mathrm{LS}}_r(X)$ is* trivial modulo an ideal $I \subset O$ *if for every $x \in |X|$ the polynomial $f_x \in O[t]$ is congruent to $(1-t)^r$ modulo I.*

Definition 2.11. *If X is connected then $\mathrm{LS}'_r(X)$ is the set of all $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfying condition (i) from Theorem 2.5 and the following one: there is a closed normal subgroup $H \subset \pi_1(X)$ such that*

(a) *$\pi_1(X)/H$ is almost pro-ℓ;*
(b) *for every nonzero ideal $I \subset O$ there is an open subgroup $V \subset \pi_1(X)$ containing H such that the pullback of f to X_V is trivial modulo I (here X_V is the connected covering of X corresponding to V).*

If X is disconnected then $\mathrm{LS}'_r(X)$ is the set of all $f \in \widetilde{\mathrm{LS}}_r(X)$ whose restriction to each connected component $X_\alpha \subset X$ belongs to $\mathrm{LS}'_r(X_\alpha)$.

The image of $\mathrm{LS}_r(X)$ in $\widetilde{\mathrm{LS}}_r(X)$ is clearly contained in $\mathrm{LS}'_r(X)$.

Proposition 2.12. *Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. Suppose that $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfies conditions (i)-(ii) from Theorem 2.5. Then there is a dense open $U \subset X$ such that the image of f in $\widetilde{\mathrm{LS}}_r(U)$ belongs to $\mathrm{LS}'_r(U)$.*

Proposition 2.12 will be proved in §3 using only standard facts about fundamental groups. The next statement is the key step of the proof of Theorem 2.5.

Proposition 2.13. *Let X be a regular scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. Then $\mathrm{LS}'_r(X) = \mathrm{LS}_r(X)$.*

Proposition 2.14. *Let X be a regular scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. Suppose that $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfies condition (i) of Theorem 2.5. If there exists a dense open $U \subset X$ such that $f|_U \in \mathrm{LS}_r(U)$ then $f \in \mathrm{LS}_r(X)$.*

Propositions 2.13 and 2.14 will be proved in §4 and §5 using the Hilbert irreducibility theorem.

2.5. **Hilbert irreducibility.** We prefer the following formulation of Hilbert irreducibility, which is very close to [W1, Lemma 20] or [KS1, Proposition 1.5].

Theorem 2.15. *Let X be an irreducible regular scheme of finite type over $\mathbb{Z}$, $U \subset X$ a non-empty open subset, $H \subset \pi_1(U)$ an open normal subgroup, and $S \subset X$ a finite reduced subscheme.*

- (i) *There exists an irreducible regular arithmetic curve C with a morphism $\varphi : C \to X$ and a section $\sigma : S \to C$ such that $\varphi(C) \cap U \neq \emptyset$ and the homomorphism $\pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective.*
- (ii) *Suppose that for each $s \in S$ we are given a 1-dimensional subspace $l_s \subset T_s X$, where $T_s X = (\mathfrak{m}_s/\mathfrak{m}_s^2)^*$ is the tangent space. Then one can find C, φ, σ as above so that for each $s \in S$ one has $\mathrm{Im}(T_{\sigma(s)} C \to T_s X) = l_s$.*

In Appendix A we deduce Theorem 2.15 from a conventional formulation of Hilbert irreducibility.

Remarks 2.16. (i) In Theorem 2.15 the regularity assumption cannot be replaced by normality, see Remark 6.1 at the end of §6.
(ii) Theorem 2.15 and its proof remain valid for regular *algebraic spaces* of finite type over $\mathbb{Z}$.

Proposition 2.17. *Let ℓ be a prime such that $X \otimes \mathbb{Z}[\ell^{-1}] \neq \emptyset$. Then Theorem 2.15 remains valid for every closed normal subgroup $H \subset \pi_1(U)$ such that $\pi_1(U)/H$ is almost pro-ℓ.*

Proof. Set $G := \pi_1(U)/H$. It suffices to find an open normal subgroup $V \subset G$ with the following property: every closed subgroup $K \subset G$ such that the map $K \to G/V$ is surjective equals G (then one can apply Theorem 2.15 to V instead of H).

Let $V' \subset G$ be an open normal pro-ℓ-subgroup and let $V \subset V'$ be the normal subgroup such that $V'/V = H_1(V', \mathbb{Z}/\ell\mathbb{Z})$. Then V has the required properties. Let us check that V is open. Let $\tilde{U} \to U$ be the covering corresponding to V'. Set $\tilde{U}[\ell^{-1}] := \tilde{U} \otimes \mathbb{Z}[\ell^{-1}]$. Then

$$H^1(V', \mathbb{Z}/\ell\mathbb{Z}) \subset H^1(\tilde{U}, \mathbb{Z}/\ell\mathbb{Z}) \subset H^1(\tilde{U}[\ell^{-1}], \mathbb{Z}/\ell\mathbb{Z}).$$

The group $H^1(\tilde{U}[\ell^{-1}], \mathbb{Z}/\ell\mathbb{Z})$ is finite (see [Milne, ch.II, Proposition 7.1]). Therefore $H^1(V', \mathbb{Z}/\ell\mathbb{Z})$ is finite. Thus the group $V'/V = H_1(V', \mathbb{Z}/\ell\mathbb{Z})$ is finite and therefore V is open. □

2.6. **The characteristic p case.**

Remarks 2.18. (i) If X is an irreducible smooth projective scheme over $\mathbb{F}_p$ and $U = X$ then Theorem 2.15 remains valid for $H = \{1\}$. This follows from B. Poonen's "Bertini theorem over finite fields" [Po, Theorems 1.1-1.2] combined with the usual Bertini theorem [Ha, Ch.III, Corollary 7.9].

(ii) More generally, suppose that X is a quasiprojective irreducible smooth scheme over $\mathbb{F}_p$ and $U \subset X$ any non-empty open subset. Represent X as an open subvariety of an irreducible normal projective variety $\bar{X}$ over $\mathbb{F}_p$. Let $D_1, \ldots, D_n$ be the irreducible components of $\bar{X} \setminus U$ of dimension $\dim X - 1$. Etale coverings of U tamely ramified at the generic points of $D_1, \ldots, D_n$ are classified by $\pi_1(U)/H$, where H is a certain normal subgroup of $\pi_1(U)$. Then Theorem 2.15 remains valid for this H. This follows from Proposition C.1 (see Appendix C).

Remark 2.19. If X is a manifold over $\mathbb{F}_p$ one can slightly modify the proof of Theorem 2.5. First, by Remarks 2.18, it suffices to prove a weaker version of Proposition 2.12, with $\mathrm{LS}'_r(U)$ being replaced by the set of all $f \in \widetilde{\mathrm{LS}}_r(U)$ satisfying the following condition: for every nonzero ideal $I \subset O$ there exists a surjective finite etale morphism $\pi : U' \to U$ such that $\pi^* f$ is trivial modulo I. Another possible modification is indicated in Remark 5.4 below.

3. Proof of Proposition 2.12 (after G. Wiesend)

Proposition 2.12 clearly follows from the next lemma, in which condition (ii) is stronger than condition (ii) from Theorem 2.5. The lemma and its proof only slightly differs from [W1, Proposition 17] or [KS1, Proposition 3.6].

Lemma 3.1. *Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$ and $f \in \widetilde{\mathrm{LS}}_r(X)$. Assume that*

(i) *for every regular arithmetic curve C and every morphism $\varphi : C \to X$ one has $\varphi^*(f) \in \mathrm{LS}_r(C)$;*
(ii) *if C is a separated curve over a finite field then $\varphi^*(f) \in \mathrm{LS}_r^{\mathrm{tame}}(C)$.*

Then there is a dense open $U \subset X$ such that the image of f in $\widetilde{\mathrm{LS}}_r(U)$ belongs to $\mathrm{LS}'_r(U)$.

Proof. We can assume that X is reduced, irreducible, and normal. If $\dim X \le 1$ the statement is obvious. Now assume that $\dim X > 1$ and the lemma holds for all schemes whose dimension is less than that of X. Replacing X by an open subset we can assume that there is a smooth morphism from X to some scheme S such that the geometric fibers of the morphism are non-empty connected curves. After shrinking S we can assume that one of the following holds:

(i) the morphism $X \to S$ has a factorization
$$X = \bar{X} \setminus D \subset \bar{X} \xrightarrow{\pi} S,$$
where π is a smooth projective morphism, $X \setminus D$ is fiberwise dense in X, and D is finite and etale over S;
(ii) S is a scheme over $\mathbb{F}_p$ and for some $n \in \mathbb{N}$ the morphism $X^{(p^n)} \to S$ has a factorization as in (i) (here $X^{(p^n)}$ is obtained from X by base change with respect to $\mathrm{Fr}^n : S \to S$).

If we are in situation (ii) then it suffices to prove the statement for $X^{(p^n)}$ instead of X. So we can assume that we are in situation (i). (Another way to conclude that is suffices to consider situation (i) is to use M. Artin's theorem on the existence of "elementary fibrations" [SGA4, exposé XI, Propostion 3.3].)

We can also assume that the morphism $X \to S$ has a section $\sigma : S \to X$ and that $\sigma^*(f) \in \mathrm{LS}'_r(S)$ (otherwise replace S by S' and X by $X \times_S S'$, where S' is an appropriate scheme etale over S).

Let $\mathfrak{m} \subset O$ be the maximal ideal. For every $n \in \mathbb{N}$ set $G_n = GL(r, O/\mathfrak{m}^n)$ and consider the functor that associates to an S-scheme S' the set of isomorphism classes of tame[3] G_n-torsors on $X \times_S S'$ trivialized over $S' \overset{\sigma'}{\hookrightarrow} X \times_S S'$. It follows from [SGA1, exposé XIII, Corollaries 2.8-2.9] that

(i) this functor is representable by a scheme T_n etale and of finite type over S,
(ii) the morphism $T_{n+1} \to T_n$ is finite for each $n \ge 1$.

So after shrinking S we can assume that the morphism $T_n \to S$ is finite for each n. We will prove that in this situation $f \in \mathrm{LS}'_r(X)$.

By Definition 2.11, to show that $f \in \mathrm{LS}'_r(X)$ we have to construct surjective finite etale morphisms $X_n \to X$, $n \in \mathbb{N}$, so that

(a) the image of f in $\widetilde{\mathrm{LS}}_r(X_n)$ is trivial modulo $\mathfrak{m}^n$,
(b) for some (or any) geometric point $\bar{x}$ of X, the quotient of the group $\pi_1(X, \bar{x})$ by the intersection of the kernels of its actions on the fibers $(X_n)_{\bar{x}}$, $n \in \mathbb{N}$, is almost pro-ℓ.

[3]Here "tame" means "tamely ramified along D relatively to S".

Since $\sigma^*(f) \in \mathrm{LS}'_r(S)$ we have a sequence of surjective finite etale morphisms $S_n \to S$, $n \in \mathbb{N}$, that satisfies the analogs of (a) and (b) for S. On the other hand, we will construct surjective finite etale morphisms $Y_n \to X$, $n \in \mathbb{N}$, with the following properties:

- $(*)$ for every geometric point $\bar{s} \to S$, every tame locally constant sheaf of r-dimensional free $(O/\mathfrak{m}^n)$-modules on the fiber $X_{\bar{s}}$ has constant pullback to $(Y_n)_{\bar{s}}$;
- $(**)$ for some (or any) geometric point $\bar{x}$ of X, the quotient of the group $\pi_1(X, \bar{x})$ by the intersection of the kernels of its actions on the fibers $(Y_n)_{\bar{x}}$, $n \in \mathbb{N}$, is almost pro-ℓ.

Then we can take $X_n := S_n \times_S Y_n$.

To construct Y_n, consider the universal G_n-torsor $\mathfrak{T}_n \to T_n \times_S X$. Now set $Y_n := \mathrm{Res}(\mathfrak{T}_n)$, where

$$\mathrm{Res} : \{\text{schemes over } T_n \times_S X\} \to \{\text{schemes over } X\}$$

is the Weil restriction functor. In other words, Y_n is the scheme over X such that for any X-scheme X' one has

$$\mathrm{Mor}_X(X', Y_n) := \mathrm{Mor}_{T_n \times_S X}(T_n \times_S X', \mathfrak{T}_n).$$

The fiber of Y_n over any geometric point $\bar{s} \to S$ equals the fiber product of the G_n-torsors over $X_{\bar{s}}$ corresponding to all points of $(\mathfrak{T}_n)_{\bar{s}}$, so Y_n has property $(*)$. We will show that property $(**)$ also holds.

Let $\eta \in S$ be the generic point and $\bar{\eta} \to \eta$ a geometric point. In property $(**)$ we take $\bar{x}$ to be the composition $\bar{\eta} \to \eta \hookrightarrow S \xrightarrow{\sigma} X$. Let $\pi_1^t(X_\eta, \bar{x})$ denote the tame fundamental group ("tame" means "tamely ramified along D_η"). Let $H_n \subset \pi_1^t(X_\eta, \bar{x})$ be the kernel of the action of $\pi_1^t(X_\eta, \bar{x})$ on $(X_n)_{\bar{x}}$. Since X was assumed normal the map $\pi_1(X_\eta, \bar{x}) \to \pi_1(X, \bar{x})$ is surjective, so to prove property $(**)$ it suffices to check that the quotient $\pi_1^t(X_\eta, \bar{x})/\bigcap_n H_n$ is almost pro-ℓ.

We have an exact sequence

$$0 \to K \to \pi_1^t(X_\eta, \bar{x}) \to \Gamma \to 0, \quad K := \pi_1^t(X_{\bar{\eta}}, \bar{x}), \quad \Gamma := \pi_1(\eta, \bar{\eta})$$

and a splitting $\sigma_* : \Gamma \to \pi_1^t(X_\eta, \bar{x})$. Thus $\pi_1^t(X_\eta, \bar{x})$ identifies with the semidirect product $\Gamma \ltimes K$. The subgroup $H_n \subset \pi_1^t(X_\eta, \bar{x})$ identifies with $\Gamma_n \ltimes K_n$, where K_n is the intersection of the kernels of all homomorphisms $K \to GL(r, O/\mathfrak{m}^n)$ and Γ_n is the kernel of the action of Γ on K/K_n.

The group K is topologically finitely generated, so the group $K' := K/\bigcap_n K_n$ is almost pro-ℓ. Thus it remains to show that the quotient $\Gamma/\bigcap_n \Gamma_n$ is almost pro-ℓ. Since $\bigcap_n \Gamma_n \supset \mathrm{Ker}(\Gamma \to \mathrm{Aut}\, K')$ it suffices to show that the group $\mathrm{Aut}\, K'$ equipped with the compact-open topology is an almost pro-ℓ-group. This follows from the fact that K' is topologically finitely generated and almost pro-ℓ. Indeed, if $V \subset K'$ is the maximal open normal pro-ℓ-subgroup then the automorphisms of K' that act as identity on the finite groups K'/V and $H_1(V, \mathbb{Z}/\ell\mathbb{Z})$ form an open pro-ℓ-subgroup of $\mathrm{Aut}\, K'$. □

Remark 3.2. Suppose that X is irreducible. Lemma 3.1 says that for each $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfying certain conditions there exists a non-empty open $U \subset X$ and a normal subgroup $H \subset \pi_1(X)$ such that $f|_U$ saitisfies the condition from Definition 2.11. The proof of Lemma 3.1 shows that one can choose U and H to be independent of f. This is not surprising: Theorem 2.5 will show that one can take U to be the set of regular points of X_{red} and H to be the intersection of the kernels of all homomorphisms $\pi_1(X) \to GL(r, O)$ satisfying a tameness condition.

4. Proof of Proposition 2.13 (after Moritz Kerz)

We can assume that X is irreducible. The problem is to show that if $f \in \mathrm{LS}'_r(X)$ then $f \in \mathrm{LS}_r(X)$, i.e., f comes from a representation $\rho : \pi_1(X) \to GL(r, E_\lambda)$. The original proof of Proposition 2.13 can be found in version 5 of the e-print [Dr] (the idea was to first construct the character of ρ using elementary representation theory and a compactness argument). The simpler proof given in §4.1-4.2 is due to M. Kerz. In §4.3 we give a variant of his proof, in which Lemma 4.3 is replaced with a compactness argument.

Recall that the set of closed points of X is denoted by $|X|$. For each $x \in |X|$ we have the geometric Frobenius F_x, which is a conjugacy class in $\pi_1(X)$.

4.1. **Constructing a representation of $\pi_1(X)$.** Let $H \subset \pi_1(X)$ be a closed normal subgroup satisfying properties (a)-(b) from Definition 2.11 with respect to our $f \in \mathrm{LS}'_r(X)$. By Proposition 2.17, there exists an irreducible regular arithmetic curve C with a morphism $\varphi : C \to X$ such that the map $\varphi_* : \pi_1(C) \to \pi_1(X)/H$ is surjective. Take any such pair (C, φ). Then $\varphi^*(f)$ comes from a semisimple representation $\rho_C : \pi_1(C) \to GL(r, E_\lambda)$. After an appropriate conjugation, ρ_C becomes a homomorphism $\pi_1(C) \to GL(r, O)$.

Lemma 4.1. $\operatorname{Ker} \rho_C \supset H_C$*, where* $H_C := \operatorname{Ker}(\varphi_* : \pi_1(C) \to \pi_1(X)/H)$.

Proof. Since ρ_C is semisimple and H_C is normal the restriction of ρ_C to H_C is semismple. So it remains to show that $\rho_C(h)$ is unipotent for all $h \in H_C$.

Property (b) from Definition 2.11 implies that for every nonzero ideal $I \subset O$ there exists an open normal subgroup $U_I \subset \pi_1(C)$ such that for every $c \in |C|$ with $F_c \in U_I$ the polynomial $\det(1 - t\rho_C(F_c))$ is congruent to $(1-t)^r$ modulo I. So by Čebotarev density and continuity of ρ_C,

$$\det(1 - t\rho_C(h)) \equiv (1-t)^r \bmod I \ \text{ for all } h \in U_I \,. \tag{4.1}$$

But H_C is contained in each of the U_I's, so (4.1) implies that $\rho_C(h)$ is unipotent for all $h \in H_C$. ☐

By Lemma 4.1, we can consider ρ_C as a homomorphism $\pi_1(X)/H \to GL(r, O)$. By construction, the equality

$$\det(1 - t\rho_C(F_x)) = f_x(t) \tag{4.2}$$

holds if $x \in \varphi(|C|)$ and $\varphi^{-1}(x)$ contains a point whose residue field is equal to that of x. To prove Proposition 2.13, we will now show that (4.2) holds for *all* $x \in |X|$.

4.2. **Using a lemma of Faltings.**

Remark 4.2. The proof of Proposition 2.17 shows that the group $\pi_1(X)/H = \pi_1(C)/H_C$ is topologically finitely generated.

Lemma 4.3. (i) *There exists a finite subset $T \subset |C|$ such that any semisimple representations $\rho_1, \rho_2 : \pi_1(C)/H_C \to GL(r, E_\lambda)$ with*

$$\operatorname{Tr} \rho_1(F_c) = \operatorname{Tr} \rho_2(F_c) \ \text{ for all } c \in T$$

are isomorphic.

(ii) *If C is flat over $\mathbb{Z}$ this is true for $\pi_1(C)$ instead of $\pi_1(C)/H_C$.*

Proof. Statement (ii) is due to Faltings (see [Fa, Satz 5] or [De4, Theorem 3.1]). The proof of (ii) uses only the finiteness of the set of homomorphisms from $\pi_1(C)$ to any fixed finite group (which holds if C is flat over $\mathbb{Z}$). So by Remark 4.2, the same argument proves (i). ☐

Let T be as in Lemma 4.3(i). We have to show that (4.2) holds for any $x \in |X|$. By Proposition 2.17, there exists an irreducible regular arithmetic curve C' with a morphism $\varphi' : C' \to X$ such that the map $\varphi'_* : \pi_1(C') \to \pi_1(X)/H$ is surjective and for each $y \in T \cup \{x\}$ there exists a point $z \in (\varphi')^{-1}(y)$ whose residue field is equal to that of y. Applying the argument of §4.1 to (C', φ') we get a semisimple representation $\rho_{C'} : \pi_1(X)/H \to GL(r, E_\lambda)$ such that $\det(1 - t\rho_C(F_y)) = f_y(t)$ for each $y \in T \cup \{x\}$. By the above choice of T, this implies that the representations $\pi_1(C) \to GL(r, E_\lambda)$ corresponding to ρ_C and $\rho_{C'}$ are isomorphic. So $\rho_C \simeq \rho_{C'}$ and therefore $\det(1 - t\rho_C(F_x)) = f_x(t)$, QED.

4.3. **Using a compactness argument.** Instead of referring to the lemma of Faltings, one can finish the proof of Proposition 2.13 as follows. We have to prove the existence of a homomorphism $\rho : \pi_1(X) \to GL(r, O)$ such that

$$\det(1 - t\rho(F_x)) = f_x(t) \tag{4.3}$$

for all $x \in |X|$. By Remark 4.2, the group $\pi_1(X)/H$ is topologically finitely generated. So the set

$$Z := \operatorname{Hom}(\pi_1(X)/H, GL(r, O)) = \varprojlim_{I \neq 0} \operatorname{Hom}(\pi_1(X)/H, GL(r, O/I))$$

equipped with the topology of projective limit is compact.

For each $x \in |X|$ let $Z_x \subset Z$ denote the set of all homomorphisms $\rho : \pi_1(X)/H \to GL(r, O)$ satisfying (4.3). Then each Z_x is a closed subset of Z. We have to show that the intersection of these subsets is non-empty. But Z is compact, so it suffices to prove that for any finite subset $S \subset |X|$ the set

$$\bigcap_{x \in S} Z_x \tag{4.4}$$

is non-empty. By Proposition 2.17, there exists an irreducible regular arithmetic curve C with a morphism $\varphi : C \to X$ and a scheme-theoretical section $S \to X$ such that the map $\varphi_* : \pi_1(C) \to \pi_1(X)/H$ is surjective. The set (4.4) contains the homomorphism $\rho_C : \pi_1(X)/H \to GL(r, O)$ constructed in §4.1, so it is non-empty.

5. Proof of Proposition 2.14

5.1. **A specialization lemma.** We will need the following elementary lemma. (For deeper statements in this spirit, see [W1, §2.4], [KS1, Lemma 2.1 and Proposition 2.3], [KS2, Lemma 2.4].)

Lemma 5.1. *Let X be a regular scheme of finite type over $\mathbb{Z}$, $D \subset X$ an irreducible divisor, G a finite group, and $\pi : Y \to X \setminus D$ a G-torsor ramified at D. Then there exists a closed point $x \in D$ and a 1-dimensional subspace l of the tangent space $T_x X := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ with the following property:*

- (★) *if $C \subset X_{\bar{x}}$ is any regular 1-dimensional closed subscheme tangent to l such that $C \not\subset D_{\bar{x}}$ then the pullback of $\pi : Y \to X \setminus D$ to $C \setminus \{\bar{x}\}$ is ramified at $\bar{x}$.*

Here $\bar{x}$ is a a geometric point corresponding to x and $X_{\bar{x}}$, $D_{\bar{x}}$ are the strict Henselizations.

Proof. We will show that *at least one* of the following statements (a) and (b) is true:

- (a) property (★) holds for any closed point x of some non-empty open subset $U \subset D$ and for any $l \not\subset T_x D$;
- (b) D is a variety over a finite field and there exists a conic closed subset F of the tangent bundle TD such that $F \neq TD$ and property (★) holds whenever $l \subset T_x D$ and $l \not\subset F$.

Let $I \subset G$ be the inertia subgroup at the generic point of D. To prove that (a) or (b) holds, we can replace X by any scheme X' etale over X with $D \times_X X' \neq \emptyset$ and replace Y by $Y \times_X X'$. So we can assume that $I = G$. Then G is solvable (because I is). So we can assume that $|G|$ is a prime number p (otherwise replace Y by Y/H, where $H \subset G$ is a normal subgroup of prime index).

Let $\bar{Y}$ be the normalization of X in the ring of fractions of Y. We have a finite morphism $\bar{\pi} : \bar{Y} \to X$ and an action of G on $\bar{Y}$ such that $\bar{Y}/G = X$. After shrinking X we can assume that $\bar{Y}$ is regular.

Set $\tilde{D} := (\bar{\pi}^{-1}(D))_{\mathrm{red}}$. The assumption $I = G$ means that the action of G on $\tilde{D}$ is trivial and the morphism $\bar{\pi}_D : \tilde{D} \to D$ is purely inseparable. Let e_1 be its degree and let e_2 be the multiplicity of $\tilde{D}$ in the divisor $\bar{\pi}^{-1}(D)$. Then $e_1 e_2 = |G| = p$, so e_1 equals 1 or p.

Case 1: $e_1 = 1$, $e_2 = p$. Let us show that (a) holds for $U = D$. Since $e_1 = 1$ the morphism $\bar{\pi}_D : \tilde{D} \to D$ is an isomorphism. So if l is as in (a) and $C \subset X_{\bar{x}}$ is any regular 1-dimensional closed subscheme tangent to l then $T_{\bar{x}} C$ is transversal to the image of the tangent map $T_{\bar{\pi}^{-1}(x)} \bar{Y} \to T_x X$. Therefore $C \times_X \bar{Y}$ is regular. On the other hand, the fiber of $\bar{\pi}$ over $\bar{x}$ has a single point. So the pullback of $\pi : Y \to X \setminus D$ to $C \setminus \{\bar{x}\}$ is indeed ramified at $\bar{x}$.

Case 2: $e_1 = p$, $e_2 = 1$. Then D and $\tilde{D}$ are varieties over $\mathbb{F}_p$. After shrinking X we can assume that D and $\tilde{D}$ are smooth and the differential of the morphism $\bar{\pi}_D : \tilde{D} \to D$ has constant corank δ. Since $e_1 = p$ one has $\delta = 1$. The images of the differential of $\bar{\pi}_D$ at various points of $\tilde{D}$ form a vector subbundle of $(TD) \times_D \tilde{D}$ of codimension 1. Let $F \subset TD$ be its image; then F is a conic closed subset. We will show that (b) holds for this F. Let $x \in D$ be a closed point and $H_x := \mathrm{Im}\,(T_{\bar{\pi}^{-1}(x)} \bar{Y} \to T_x X)$. Since $\delta = 1$ and $e_2 = 1$ one has $\operatorname{codim} H_x = 1$. So if l is as in (b) then l is transversal to H_x. Now one can finish the argument just as in Case 1. □

Corollary 5.2. *Let X be a regular scheme of finite type over $\mathbb{Z}[\ell^{-1}]$ and $U \subset X$ a dense open subset. Let $\mathcal{E}$ a lisse E_λ-sheaf on U. Suppose that $\mathcal{E}$ does not extend to a lisse E_λ-sheaf on X. Then there exists a closed point $x \in X \setminus U$ and a 1-dimensional subspace $l \subset T_x X$ with the following property:*

- ($*$) *if C is a regular arithmetic curve, $c \in C$ a closed point, and $\varphi : (C, c) \to (X, x)$ a morphism with $\varphi^{-1}(U) \neq \emptyset$ such that the image of the tangent map $T_c C \to T_x X \otimes_{k_x} k_c$ equals $l \otimes_{k_x} k_c$ then the pullback of $\mathcal{E}$ to $\varphi^{-1}(U)$ is ramified at c.*

(Here k_x and k_c are the residue fields.)

Proof. By the Zariski-Nagata purity theorem [SGA2, exposé X, Theorem 3.4], $\mathcal{E}$ is ramified along some irreducible divisor $D \subset X$, $D \cap U = \emptyset$. Now use Lemma 5.1. □

5.2. **Proof of Proposition 2.14.** Proposition 2.14 is equivalent to the following lemma.

Lemma 5.3. *Let X be be a regular scheme of finite type over $\mathbb{Z}[\ell^{-1}]$. Let $U \subset X$ be a dense open subset, $\mathcal{E}_U$ a semisimple lisse E_λ-sheaf on U, and $f_U \in \mathrm{LS}_r(U) \subset \widetilde{\mathrm{LS}}_r(U)$ the class of $\mathcal{E}_U$. Let $f \in \widetilde{\mathrm{LS}}_r(X)$ be such that $f|_U = f_U$ and $\varphi^*(f) \in \mathrm{LS}_r(C)$ for every regular arithmetic curve C and every morphism $\varphi : C \to X$. Then*

- (i) *$\mathcal{E}_U$ extends to a lisse E_λ-sheaf $\mathcal{E}$ on X;*
- (ii) *the class of $\mathcal{E}$ in $\mathrm{LS}_r(X) \subset \widetilde{\mathrm{LS}}_r(X)$ equals f.*

Proof. (i) Assume the contrary. We can assume that X is irreducible. Choose a closed point $x \in X \setminus U$ and a 1-dimensional subspace $l \subset T_{\bar{x}} X$ satisfying property ($*$) from Corollary 5.2. Let $H \subset \pi_1(U)$ be the kernel of the representation $\rho : \pi_1(U) \to GL(r, E_\lambda)$ corresponding to $\mathcal{E}$. The group $\pi_1(U)/H \simeq \operatorname{Im} \rho$ has an open pro-ℓ-subgroup, so by Proposition 2.17, there is an irreducible regular arithmetic curve C with a closed point $c \in C$ whose residue field is isomorphic to that of x and a morphism $\varphi : (C, c) \to (X, x)$ such that $\varphi^{-1}(U) \neq \emptyset$, the homomorphism $\varphi_* : \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, and the image of the tangent map $T_c C \to T_x X$ equals l. The surjectivity of $\varphi_* : \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ implies that the pullback of $\mathcal{E}$ to $\varphi^{-1}(U)$ is semisimple. So the assumptions on f ensure that the pullback of $\mathcal{E}$ to $\varphi^{-1}(U)$ has no ramification at c. This contradicts property ($*$) from Corollary 5.2.

(ii) Let $f^? \in \mathrm{LS}(X) \subset \widetilde{\mathrm{LS}}_r(X)$ be the class of $\mathcal{E}$. We have to show that $f^?(x) = f(x)$ for every closed point $x \in X$. Choose a triple (C, c, φ), where C is a regular arithmetic curve, $c \in C$ is a closed point whose residue field is isomorphic to that of x and $\varphi : (C, c) \to (X, x)$ is a morphism such that $\varphi^{-1}(U) \neq \emptyset$. Then $\varphi^* f^?, \varphi^* f \in \mathrm{LS}_r(C)$ have equal images in $\mathrm{LS}_r(\varphi^{-1}(U))$. So $\varphi^* f = \varphi^* f$ and therefore $f^?(x) = f(x)$. □

Remark 5.4. If X is over $\mathbb{F}_p$ then in the proof of Lemma 5.3(i) the surjectivity of $\varphi_* : \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is not essential. Indeed, for our purpose it suffices to know that the pullback of $\mathcal{E}$ to $\varphi^{-1}(U)$ is *geometrically* semisimple, and this follows from [De3, Theorem 3.4.1 (iii)] combined with [De3, Remark 1.3.6] and [Laf, Proposition VII.7 (i))].

6. Counterexamples

In this section we give two examples showing that in Theorem 2.5 the regularity assumption on X cannot be replaced by normality. In both of them one has a normal scheme X of finite type over $\mathbb{Z}[\ell^{-1}]$ with a unique singular point $x_0 \in X$ and a desingularization $\pi : \hat{X} \to X$ inducing an isomorphsim $\pi^{-1}(X \setminus \{x_0\}) \xrightarrow{\sim} X \setminus \{x_0\}$. One also has an element $f \in \widetilde{\mathrm{LS}}_r(X)$ such that $\pi^* f \in \mathrm{LS}_r(\hat{X}) \subset \widetilde{\mathrm{LS}}_r(\hat{X})$ but $f \notin \mathrm{LS}_r(X)$. In the first example $f \notin \mathrm{LS}_r(X)$ because the semisimple lisse E_λ-sheaf $\mathcal{E}$ on $\hat{X}$ corresponding to $\pi^* f$ does not descend to X. In the second example $\mathcal{E}$ is constant and therefore descends to X, but $f(x_0)$ is "wrong". In both examples the key point is that $(\pi^{-1}(x_0))_{\mathrm{red}}$ is not normal.

6.1. **First example.**

6.1.1. *The idea.* Let X, x_0, and $\pi : \hat{X} \to X$ be as above. Let $C := (\pi^{-1}(x_0))_{\mathrm{red}}$. Let $i : C \hookrightarrow \hat{X}$ be the embedding.

Now let $\mathcal{E}$ be an E_λ-sheaf on $\hat{X}$ of rank r. It defines an element $f_{\mathcal{E}} \in \widetilde{\mathrm{LS}}_r(\hat{X})$. Suppose that

(a) $\mathcal{E}$ is semisimple;
(b) $i^*\mathcal{E}$ is not geometrically constant;
(c) for every $c \in |C|$ the polynomial $f(c) \in P_r(O)$ equals the "trivial" polynomial $(1-t)^r \in P_r(O)$.

Define $f \in \widetilde{\mathrm{LS}}_r(X)$ as follows: $f|_{X\setminus\{x_0\}} := f_{\mathcal{E}}|_{X\setminus\{x_0\}}$ and $f(x_0)$ is the polynomial $(1-t)^r \in P_r(O)$. Then it is easy to see that f satisfies the conditions of Theorem 2.5 but $f \notin \mathrm{LS}_r(X)$.

It remains to construct X, $\hat{X}$, π, $\mathcal{E}$ with the above properties. Note that C *cannot be normal:* otherwise (c) would imply (b) by Čebotarev density.

6.1.2. *A construction of* X, $\hat{X}$, π, $\mathcal{E}$ *for* $r = 1$. The following construction was communicated to me by A. Beilinson.

Let $n \in \mathbb{N}$. Over $\mathbb{F}_q$ consider the curve $\mathbb{P}^1 \times (\mathbb{Z}/n\mathbb{Z})$ (i.e., a disjoint union of n copies of $\mathbb{P}^1$). Gluing $(\infty, i) \in \mathbb{P}^1 \times (\mathbb{Z}/n\mathbb{Z})$ with $(0, i+1) \in \mathbb{P}^1 \times (\mathbb{Z}/n\mathbb{Z})$ for all $i \in \mathbb{Z}/n\mathbb{Z}$ one gets a curve C_n equipped with a free action of $\mathbb{Z}/n\mathbb{Z}$. At least, if $n = 3$ or $n = 4$ it is easy to embed C_n into a smooth quasiprojective surface $\hat{Y}$ over $\mathbb{F}_q$ so that the action of $\mathbb{Z}/n\mathbb{Z}$ on C_n extends to $\hat{Y}$ (take $\hat{Y} = \mathbb{P}^2$ if $n = 3$ and $\hat{Y} = \mathbb{P}^1 \times \mathbb{P}^1$ if $n = 4$). We can assume that the action of $\mathbb{Z}/n\mathbb{Z}$ on $\hat{Y}$ is free (otherwise replace $\hat{Y}$ by an open subset). We can also assume that the intersection matrix of the curve $C_n \subset \hat{Y}$ is negative definite (otherwise pick a sufficiently big $(\mathbb{Z}/n\mathbb{Z})$-stable finite reduced subscheme S of the nonsingular part of C_n and replace $\hat{Y}$ by its blow-up at S). Blowing down $C_n \subset \hat{Y}$ we get an algebraic surface[4] Y with a unique singular point y_0.

Now let $\hat{X}$ and X be the quotients of $\hat{Y}$ and Y by the action of $\mathbb{Z}/n\mathbb{Z}$. The morphism $\hat{Y} \to Y$ induces a birational morphism $\pi : \hat{X} \to X$. Since the action of $\mathbb{Z}/n\mathbb{Z}$ on $\hat{Y}$ is free the surface $\hat{X}$ is smooth. Finally, let $\mathcal{E}$ be the rank 1 local system on $\hat{X}$ corresponding to the $(\mathbb{Z}/n\mathbb{Z})$-torsor $\hat{Y} \to \hat{X}$ and a nontrivial character $\mathbb{Z}/n\mathbb{Z} \to \overline{\mathbb{Q}}_\ell^\times$.

6.2. **Second example.** Let $\hat{Y}$, Y and y_0 be as in §6.1.2. Let $\alpha \in H^1(\mathbb{F}_q, \mathbb{Z}/n\mathbb{Z})$, $\alpha \neq 0$. The group $\mathbb{Z}/n\mathbb{Z}$ acts on $(\hat{Y}, Y, y_0)$. Now let $(\hat{X}, X, x_0)$ denote the form of $(\hat{Y}, Y, y_0)$ corresponding to α. Then

(a) X is a normal surface over $\mathbb{F}_q$, $x_0 \in X(\mathbb{F}_q)$ is a singular point, and $\hat{X}$ is a desingularization of X;
(b) the residue field of any closed point of the preimage of x_0 in $\hat{X}$ contains $\mathbb{F}_{q^m}$, where $m > 1$ is the order of α.

Now we will define an element $f \in \widetilde{\mathrm{LS}}_1(X)$, i.e., a function $f : |X| \to O^\times$, where $O \subset E_\lambda$ is the ring of integers. Namely, set

$$f(x) := 1 \text{ if } x \neq x_0, \quad f(x_0) := \zeta,$$

where $\zeta \in O$, $\zeta^m = 1$, $\zeta \neq 1$ (we assume that E_λ is big enough so that ζ exists). Properties (a)-(b) above imply that f satisfies the conditions of Theorem 2.5 but $f \notin \mathrm{LS}_1(X)$.

Remark 6.1. Property (b) implies that there are no nonconstant morphisms $(C, c) \to (X, x_0)$ with C being a smooth curve over $\mathbb{F}_q$ and $c \in C(\mathbb{F}_q)$. So Theorem 2.15 does not hold for our surface X and $S = \{x_0\}$, and Lemma A.6 does not hold for the local ring of X at x_0. Therefore in Theorem 2.15 and Lemma A.6 the regularity assumption cannot be replaced by normality.

Appendix A. Proof of Theorem 2.15

We follow the proof of [Bl, Lemmas 3.2-3.3] and [W1, Lemma 20] (with slight modifications).

[4]Since our ground field is finite, this surface is a quasiprojective scheme rather than merely an algebraic space, see [Ar, Theorem 4.6].

A.1. **A conventional formulation of Hilbert irreducibility.** Let k be a global field, i.e., a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_p(t)$. The following version of the Hilbert irreducibility theorem is proved in [Lang2, Ch. 9, Theorem 4.2] and in [FJ, Ch. 13, Proposition 13.4.1].

Theorem A.1. *Let U be an open subscheme of the affine space $\mathbb{A}^n_k$. Let $\tilde{U}$ be an irreducible scheme finite and etale over U. Let $U(k)_{\mathrm{Hilb}} \subset U(k)$ denote the set of those k-points of U that do not split in $\tilde{U}$. Then $U(k)_{\mathrm{Hilb}} \neq \emptyset$. Moreover, for every finitely generated subring $R \subset k$ with field of fractions k one has $U(k)_{\mathrm{Hilb}} \cap R^n \neq \emptyset$.* □

Let T be a finite set of nonarchmidean places of k. Set

$$\hat{k} := \prod_{v \in T} k_v,$$

where k_v is the completion of k. The topology on $\hat{k}$ induces a topology on the set of $\hat{k}$-points of any algebraic variety over k. The following corollary of Theorem A.1 is standard.

Corollary A.2. *In the situation of Theorem A.1 the image of $U(k)_{\mathrm{Hilb}}$ in $U(\hat{k})$ is dense.*

Proof. We follow the proof of [Lang2, Ch. 9, Corollary 2.5]. Let $R \subset k$ be a finitely generated subring with field of fractions k such that $|x|_v \leq 1$ for all $x \in R$, $v \in T$. Choose a nonzero $\pi \in R$ so that $|\pi|_v < 1$ for all $v \in T$. Then every nonempty open subset of $\hat{k}^n$ contains a subset of the form $\pi^m R^n + u$, where $m \in \mathbb{Z}$, $u \in k^n$. Theorem A.1 implies that $U(k)_{\mathrm{Hilb}} \cap g(R^n) \neq \emptyset$ for any $g \in \mathrm{Aut}(\mathbb{A}^n_k)$. Setting $g(x) = \pi^m x + u$ we see that $U(k)_{\mathrm{Hilb}} \cap (\pi^m R^n + u) \neq \emptyset$. □

A.2. **Bloch's lemmas.** We follow [Bl, §3] (with minor modfications).

Lemma A.3. *Let k and T be as in §A.1. Let V be an irreducible smooth k-scheme and $\tilde{V}$ an irreducible scheme finite and etale over V. Then for any non-empty open subset $W \subset V(\hat{k})$ there exists a finite extension $k' \supset k$ equipped with a k-morphism $k' \to \hat{k}$ and a point $\alpha \in V(k')$ such that α does not split in $\tilde{V}$ and the image of α in $V(\hat{k})$ belongs to W.*

Proof. Choose a nonempty open subscheme $V' \subset V$ so that there exists a finite etale morphism $V' \to U$, where U is an open subscheme of an affine space $\mathbb{A}^n_k$. Since $V'(\hat{k})$ is dense in $V(\hat{k})$ and the map $V'(\hat{k}) \to U(\hat{k})$ is open the image of $V'(\hat{k}) \cap W$ in $U(\hat{k})$ is nonempty and open. So by Corollary A.2, it contains a point $\beta \in U(k)$ that does not split in $\tilde{V}$. Let V'_β be the fiber of $V' \to U$ over β. Since β does not split $V'_\beta = \operatorname{Spec} k'$ for some field $k' \supset k$. By construction, V'_β has a $\hat{k}$-point in W. This $\hat{k}$-point yields a k-morphism $f : k' \to \hat{k}$. On the other hand, the embedding $V'_\beta \hookrightarrow V$ defines a point $\alpha \in V(k')$. Clearly k', f, and α have the desired properties. □

Let k, T, and $\hat{k}$ be as in §A.1. Set $\hat{O} := \prod_{v \in T} O_v$, where $O_v \subset k_v$ is the ring of integers. Set

$$O := \hat{O} \times_{\hat{k}} k = \{x \in k \mid |x|_v \leq 1 \text{ for } v \in T\}. \tag{A.1}$$

If $T \neq \emptyset$ then O is a semilocal ring whose completion equals $\hat{O}$, and if $T = \emptyset$ then $O = k$.

Let A be an etale $\hat{k}$-algebra (i.e., A is a product of fields each of which is a finite separable extension of one of the fields k_v, $v \in T$). Let $\hat{O}_A \subset A$ be the integral closure of $\hat{O}$ in A.

Lemma A.4. *Let Y be a scheme of finite type over O. Let $V \subset Y \otimes_O k$ be an irreducible open subscheme smooth over k and $\tilde{V}$ an irreducible scheme finite and etale over V. For any field $K \supset k$ let $V(K)_{\mathrm{Hilb}} \subset V(K)$ denote the set of those K-points of V that do not split in $\tilde{V}$. Let $\alpha_0 \in Y(\hat{O}_A) \times_{Y(A)} V(A) \subset Y(\hat{O}_A)$ and let $I \subset \hat{O}_A$ be an open ideal. Then there exists a finite extension $k' \supset k$ equipped with a k-morphism $k' \to A$ and an element*

$$\alpha \in Y(O') \times_{Y(k')} V(k')_{\mathrm{Hilb}}, \quad O' := k' \times_A \hat{O}_A$$

such that the image of α in $Y(\hat{O}_A)$ is congruent to α_0 modulo I.

Proof. By weak approximation, there exists a finite extension $\tilde{k} \supset k$ and a finite set $\tilde{T}$ of nonarchimedian places of $\tilde{k}$ such that $A = \prod_{w \in \tilde{T}} \tilde{k}_w$. Setting $\tilde{O} := \{x \in \tilde{k} \mid |x|_w \leq 1 \text{ for } w \in \tilde{T}\}$ and replacing Y and V with $Y \otimes_O \tilde{O}$ and $V \otimes_k \tilde{k}$ we reduce the lemma to the particular case where $A = \hat{k}$ (and therefore $\hat{O}_A = \hat{O}$).

In this case let W be the set of all elements of $Y(\hat{O})$ congruent to α_0 modulo I. Let $W_1 \subset Y(\hat{k})$ be the image of W and $W_2 := W_1 \cap V(\hat{k})$. The subsets $W_1, W_2 \subset Y(\hat{k})$ are open and nonempty. Applying Lemma A.3 to V, $\tilde{V}$, and $W_2 \subset V(\hat{k})$ we get a finite extension $k' \supset k$ equipped with a k-morphism $k' \to \hat{k}$ and an element $\alpha \in Y(\hat{O}) \times_{Y(\hat{k})} V(k')_{\mathrm{Hilb}}$ such that the image of α in $Y(\hat{O})$ belongs to W. It remains to note that if one sets $O' := k' \times_{\hat{k}} \hat{O}$ then

$$Y(\hat{O}) \times_{Y(\hat{k})} V(k')_{\mathrm{Hilb}} = Y(O') \times_{Y(k')} V(k')_{\mathrm{Hilb}}$$

because the square

$$\begin{array}{ccc} \operatorname{Spec} \hat{k} & \longrightarrow & \operatorname{Spec} \hat{O} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} O' \end{array}$$

is co-Cartesian in the category of all schemes. □

A.3. **On regular local rings.** In this subsection we prove Lemma A.6, which will be used in §A.4.

Lemma A.5. *Let R be a ring and $J \subset R$ an ideal. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals not containing J. Then for every $r_0 \in R$ there exists $r \in r_0 + J$ such that $r \notin \mathfrak{p}_i$ for each i.*

Proof. We can assume that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$. By induction, we can also assume that $r \notin \mathfrak{p}_i$ for $i < n$. By assumption, $J\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1} \not\subset \mathfrak{p}_n$. Let $u \in J\mathfrak{p}_1 \ldots \mathfrak{p}_{n-1}$, $u \notin \mathfrak{p}_n$. Then set $r := r_0 + u$ if $r_0 \in \mathfrak{p}_n$ and $r := r_0$ if $r_0 \notin \mathfrak{p}_n$. □

Lemma A.6. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$. Let $l \subset (\mathfrak{m}/\mathfrak{m}^2)^*$ be a 1-dimensional subspace and $D \subset \operatorname{Spec} R$ a divisor. Then there is a regular 1-dimensional closed subscheme $C \subset \operatorname{Spec} R$ tangent to l such that $C \not\subset D$.*

Proof. It suffices to show that if $\dim R > 1$ then there is a regular closed subscheme $Y \subset \operatorname{Spec} R$ such that $Y \not\subset D$ and the tangent space of Y at its closed point contains l (then one can replace $\operatorname{Spec} R$ with Y and proceed by induction).

To construct Y, choose $r_0 \in \mathfrak{m}$ so that the image of r_0 in $\mathfrak{m}/\mathfrak{m}^2$ is nonzero and orthogonal to l. By Lemma A.5, there exists $r \in r_0 + \mathfrak{m}^2$ such that r does not vanish on any irreducible component of D. Now set $Y := \operatorname{Spec} R/(r)$. □

Note that in Lemma A.6 and Theorem 2.15 the regularity assumption cannot be replaced by normality, see Remark 6.1 at the end of §6.

A.4. **Proof of Theorem 2.15.** Let $\tilde{U}$ be the covering of U corresponding to $H \subset \pi_1(U)$. Then the surjectivity condition in Theorem 2.15(i) means that $C \times_X \tilde{U}$ is irreducible. Let us consider two cases.

A.4.1. *The case where $X \otimes \mathbb{Q} \neq \emptyset$.* If $S = \emptyset$ then applying Lemma A.4 for $k = \mathbb{Q}$, $T = \emptyset$, and $Y = V = U \otimes \mathbb{Q}$ one gets a finite extension $k' \supset \mathbb{Q}$ and a point $\alpha \in U(k')$ that does not split in $\tilde{U}$. Then it remains to choose C and $\varphi : C \to X$ so that the generic point of C equals $\operatorname{Spec} k'$ and the restriction of φ to $\operatorname{Spec} k'$ equals α.

If $S \neq \emptyset$ then apply Lemma A.4 to the following k, T, Y, V, A, α_0, and I. As before, set $k = \mathbb{Q}$. Define T to be the image of S in $\operatorname{Spec} \mathbb{Z}$; then the ring O defined by (A.1) is the ring of rational numbers whose denominators do not contain primes from T. Set $Y := X \otimes O$, $V := U \otimes \mathbb{Q}$.

To define A, α_0, and I, proceed as follows. Let $\hat{O}_{X,s}$ denote the completed local ring of X at $s \in S$. Lemma A.6 and the regularity assumption[5] on X allow to choose for each $s \in S$ a 1-dimensional regular

[5]This place (and a similar one in §A.4.2) is the only part of the proof of Theorem 2.15 where we use that X is regular.

closed subscheme $\hat{C}_s \subset \operatorname{Spec} \hat{O}_{X,s}$ tangent to l_s such that $\hat{C}_s \times_X U \neq \emptyset$ and $\hat{C}_s \otimes \mathbb{Q} \neq \emptyset$. Let $\hat{O}_A$ be the ring of regular functions on $\coprod_{s\in S} \hat{C}_s$. Let A be the ring of fractions of $\hat{O}_A$. The morphism

$$\operatorname{Spec} \hat{O}_A = \coprod_{s\in S} \hat{C}_s \to X \tag{A.2}$$

defines an element $\alpha_0 \in Y(\hat{O}_A)$. Let $I \subset \hat{O}_A$ be the square of the Jacobson radical of $\hat{O}_A$.

Now let k', O', and α be as in Lemma A.4. Let $\tilde{C}$ be the spectrum of the integral closure of $\mathbb{Z}$ in k'. Then there exists an open subscheme $C \subset \tilde{C}$ containing $\operatorname{Spec} O'$ such that $\alpha : \operatorname{Spec} O' \to X$ extends to a morphism $\varphi : C \to X$. The pair (C, φ) has the desired properties.

A.4.2. *The case where X is over $\mathbb{F}_p$.* After shrinking U we can assume that there is a smooth morphsim $t : U \to \mathbb{P}^1$, where $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{F}_p}$.

Just as in §A.4.1, we will apply Lemma A.4 for certain k, T, Y, V, A, α_0, and I. Take k to be the field of rational functions on $\mathbb{P}^1$ and let V be the generic fiber of $t : U \to \mathbb{P}^1$. Choose formal curves $\hat{C}_s$ as in §A.4.1 but instead of requiring $\hat{C}_s \otimes \mathbb{Q} \neq \emptyset$ require the pullback of dt to $\hat{C}_s$ to be nonzero. Define A, $\hat{O}_A$, I, and the morphism (A.2) just as in §A.4.1.

It remains to define T, Y, and α_0. The rational map $t : \operatorname{Spec} A \to \mathbb{P}^1$ extends to a regular map

$$\operatorname{Spec} \hat{O}_A = \coprod_{s\in S} \hat{C}_s \to \mathbb{P}^1. \tag{A.3}$$

Let $T \subset \mathbb{P}^1$ be the image of the composition $S \hookrightarrow \coprod_{s\in S} \hat{C}_s \to \mathbb{P}^1$. Define $O \subset k$ by (A.1). Set $Y := \hat{X} \times_{\mathbb{P}^1} \operatorname{Spec} O$, where $\hat{X} \subset X \times \mathbb{P}^1$ is the closure of the graph of $t : U \to \mathbb{P}^1$. The morphisms (A.2) and (A.3) define a morphism $\alpha_0 : \operatorname{Spec} \hat{O}_A \to Y$.

Now apply Lemma A.4 similarly to §A.4.1. □

Appendix B. Weakly motivic $\overline{\mathbb{Q}}_\ell$-sheaves and $\overline{\mathbb{Q}}_\ell$-complexes

In §1.4.2 we defined the category of weakly motivic $\overline{\mathbb{Q}}_\ell$-sheaves $\operatorname{Sh}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$ and the category of weakly motivic $\overline{\mathbb{Q}}_\ell$-complexes $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$, see Definition 1.8. These are full subcategories of the corresponding mixed categories $\operatorname{Sh}_{\mathrm{mix}}(X, \overline{\mathbb{Q}}_\ell)$ and $\mathscr{D}_{\mathrm{mix}}(X, \overline{\mathbb{Q}}_\ell)$, see Remark 1.9(i). In this appendix we show that the category $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$ is stable under all "natural" functors (similarly to the well known results about $\mathscr{D}_{\mathrm{mix}}$).

Lemma B.1. *Let $f : X \to Y$ be a morphism between schemes of finite type over $\mathbb{F}_p$. Suppose that a $\overline{\mathbb{Q}}_\ell$-sheaf M on X has the following property: the eigenvalues of the geometric Frobenius acting on each stalk of M are algebraic numbers which are units outside of p. Then this property holds for the sheaves $R^i f_! M$ and $R^i f_* M$.*

Proof. The statement about $R^i f_! M$ immediately follows from Theorems 5.2.2 and 5.4 of [SGA7, exposé XXI] (which were proved by Deligne in a very nice way *before* his works on the Weil conjecture).

To prove the statement about $R^i f_* M$, use the arguments from Deligne's proof of Theorem 5.6 of [SGA7, exposé XXI]. This theorem was conditional, under the assumption that Hironaka's theorem holds over $\mathbb{F}_p$. But instead of this assumption one can use de Jong's result on alterations [dJ, Theorem 4.1]. □

Remark B.2. The statement about $R^i f_! M$ from Lemma B.1 can also be deduced from [Laf, Theorem VII.6].

Theorem B.3. *Let $f : X \to Y$ be a morphism between schemes of finite type over $\mathbb{F}_p$. Then*

(i) *the functor $f_! : \mathscr{D}(X, \overline{\mathbb{Q}}_\ell) \to \mathscr{D}(Y, \overline{\mathbb{Q}}_\ell)$ maps $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$ to $\mathscr{D}_{\mathrm{mot}}(Y, \overline{\mathbb{Q}}_\ell)$;*
(ii) *the functor $f_* : \mathscr{D}(X, \overline{\mathbb{Q}}_\ell) \to \mathscr{D}(Y, \overline{\mathbb{Q}}_\ell)$ maps $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$ to $\mathscr{D}_{\mathrm{mot}}(Y, \overline{\mathbb{Q}}_\ell)$;*
(iii) *the functors f^* and $f^!$ map $\mathscr{D}_{\mathrm{mot}}(Y, \overline{\mathbb{Q}}_\ell)$ to $\mathscr{D}_{\mathrm{mot}}(X, \overline{\mathbb{Q}}_\ell)$.*

Proof. (i) Combine Lemma B.1 with Theorem 3.3.1 from [De3], which says that $f_!$ maps $\mathscr{D}_{\mathrm{mix}}(X,\overline{\mathbb{Q}}_\ell)$ to $\mathscr{D}_{\mathrm{mix}}(Y,\overline{\mathbb{Q}}_\ell)$.

(ii) Combine Lemma B.1 with Theorem 6.1.2 from [De3], which says that f_* maps $\mathscr{D}_{\mathrm{mix}}(X,\overline{\mathbb{Q}}_\ell)$ to $\mathscr{D}_{\mathrm{mix}}(Y,\overline{\mathbb{Q}}_\ell)$. Alternatively, one can deduce (ii) from (i) using the same argument as in Deligne's proof of the above-mentioned theorem (see [De3, Theorem 6.1.2] or [KW, ch. II, Theorem 9.4]).

(iii) For f^* the statement is obvious. For $f^!$ it follows from (ii), just as in the proof of [De2, Corollary 1.5]. ☐

Theorem B.4. *For any scheme X of finite type over $\mathbb{F}_p$, the full subcategory $\mathscr{D}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell)\subset\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ is stable with respect to the functor $\otimes$, the Verdier duality functor $\mathbb{D}$, and the internal $\underline{\mathrm{Hom}}$ functor.*

Proof. The statement for $\otimes$ is obvious. The other two statements follow from Theorem B.3, just as in [KW, §II.12] and in the proof of [De2, Corollary 1.6]. ☐

Definition B.5. *Say that two invertible $\overline{\mathbb{Q}}_\ell$-sheaves A and A' on $\operatorname{Spec}\mathbb{F}_p$ are equivalent if $A'A^{-1}$ is weakly motivic. Let S denote the set of equivalence classes.*

Remark B.6. The set S equipped with the operation of tensor product is an abelian group. It is easy to see that the abelian group S is a vector space over $\mathbb{Q}$.

Theorem B.7. *Let $\pi: X\to\operatorname{Spec}\mathbb{F}_p$ be a morphism of finite type. For any invertible $\overline{\mathbb{Q}}_\ell$-sheaf A on $\operatorname{Spec}\mathbb{F}_p$ let $\mathscr{D}_A(X,\overline{\mathbb{Q}}_\ell)\subset\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ be the essential image of $\mathscr{D}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell)$ under the functor of tensor multiplication by π^*A (clearly $\mathscr{D}_A(X,\overline{\mathbb{Q}}_\ell)$ depends only on the class of A in S). Then*

$$\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)=\bigoplus_{A\in S}\mathscr{D}_A(X,\overline{\mathbb{Q}}_\ell)\,. \tag{B.1}$$

Proof. (i) Let us prove that the triangulated category $\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ is generated by the subcategories $\mathscr{D}_A(X,\overline{\mathbb{Q}}_\ell)$. Clearly the triangulated category $\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ is generated by objects of the from $i_!\mathcal{E}$, where $i: Y\hookrightarrow X$ is a locally closed embedding with Y normal connected and $\mathcal{E}$ is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf on Y. So it remains to show that for any such Y and $\mathcal{E}$ there exists an invertible $\overline{\mathbb{Q}}_\ell$-sheaf A on $\operatorname{Spec}\mathbb{F}_p$ such that $\mathcal{E}\otimes\pi^*A^{-1}$ is weakly motivic. By [De3, §1.3.6], there exists A such that the determinant of $\mathcal{E}\otimes\pi^*A^{-1}$ has finite order. Since $\mathcal{E}\otimes\pi^*A^{-1}$ is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant has finite order it is weakly motivic (and pure) by a result of Lafforgue [Laf, Proposition VII.7].

(ii) It remains to show that the subcategories $\mathscr{D}_A(X,\overline{\mathbb{Q}}_\ell)$ are orthogonal to each other. In other words, we have to prove that if $M_1,M_2\in\mathscr{D}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell)$, A is an invertible $\overline{\mathbb{Q}}_\ell$-sheaf on $\operatorname{Spec}\mathbb{F}_p$ and $\operatorname{Ext}^i(M_1\otimes\pi_*A,M_2)\neq 0$ for some i then A is weakly motivic. But

$$\operatorname{Ext}^i(M_1\otimes\pi_*A,M_2)=\operatorname{Ext}^i(A,\pi_*\underline{\mathrm{Hom}}(M_1,M_2))\,, \tag{B.2}$$

and $\pi_*\underline{\mathrm{Hom}}(M_1,M_2)$ is weakly motivic by Theorems B.4 and B.3(ii). So if the r.h.s. of (B.2) is nonzero then A has to be weakly motivic. ☐

As before, we write $\mathrm{Sh}(X,\overline{\mathbb{Q}}_\ell)$ for the category of $\overline{\mathbb{Q}}_\ell$-sheaves on X. Let $\mathrm{Perv}(X,\overline{\mathbb{Q}}_\ell)\subset\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ denote the category of perverse $\overline{\mathbb{Q}}_\ell$-sheaves.

Corollary B.8. *One has*

$$\mathrm{Sh}(X,\overline{\mathbb{Q}}_\ell)=\bigoplus_{A\in S}\mathrm{Sh}_A(X,\overline{\mathbb{Q}}_\ell),\qquad \mathrm{Sh}_A(X,\overline{\mathbb{Q}}_\ell):=\mathrm{Sh}(X,\overline{\mathbb{Q}}_\ell)\cap\mathscr{D}_A(X,\overline{\mathbb{Q}}_\ell), \tag{B.3}$$

$$\mathrm{Perv}(X,\overline{\mathbb{Q}}_\ell)=\bigoplus_{A\in S}\mathrm{Perv}_A(X,\overline{\mathbb{Q}}_\ell),\qquad \mathrm{Perv}_A(X,\overline{\mathbb{Q}}_\ell):=\mathrm{Perv}(X,\overline{\mathbb{Q}}_\ell)\cap\mathscr{D}_A(X,\overline{\mathbb{Q}}_\ell), \tag{B.4}$$

where S is as in Definition B.5.

Proof. This follows from Theorem B.7 because the subcategories $\mathrm{Sh}(X,\overline{\mathbb{Q}}_\ell)\subset\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ and $\mathrm{Perv}(X,\overline{\mathbb{Q}}_\ell)\subset\mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ are closed under direct sums and direct summands. ☐

Remark B.9. The category $\mathrm{Perv}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell) := \mathrm{Perv}(X,\overline{\mathbb{Q}}_\ell) \cap \mathscr{D}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell)$ is one of the direct summands in the decomposition (B.4) (it corresponds to the trivial A). Similarly, $\mathrm{Sh}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell)$ is one of the summands in (B.3) and $\mathscr{D}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell)$ is one of the summands in (B.1).

Proposition B.10. (i) *The full subcategory $\mathscr{D}_{\mathrm{mot}}(X,\overline{\mathbb{Q}}_\ell) \subset \mathscr{D}(X,\overline{\mathbb{Q}}_\ell)$ is stable with respect to the perverse truncation functors $\tau_{\le i}$ and $\tau_{\ge i}$.*

(ii) *A perverse $\overline{\mathbb{Q}}_\ell$-sheaf is weakly motivic if and only if each of its irreducible subquotients is.*

Proof. This follows from (B.1), (B.4), and Remark B.9. On the other hand, the proposition follows from Theorem B.3, just as in the proof of the corresponding statements for mixed sheaves [BBD, 5.1.6-5.1.7]. ☐

Finally, let us consider the nearby cycle functor Ψ. Let S be a smooth curve over $\mathbb{F}_p$, $\bar{s}$ a geometric point of S corresponding to a closed point $s \in S$, and $\Gamma := \mathrm{Gal}(\bar{s}/s)$. Let η be the generic point of the Henselization S_s, $\bar{\eta}$ a geometric point over η of the strict Henselization $S_{\bar{s}}$, and $G := \mathrm{Gal}(\bar{\eta}/\eta)$. Then for any scheme X of finite type over S one has the nearby cycle functor $\Psi : \mathscr{D}(X \setminus X_s, \overline{\mathbb{Q}}_\ell) \to \mathscr{D}_G(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$, where $\mathscr{D}_G$ stands for the equivariant derived category. Once we fix a splitting of the epimorphism $G \twoheadrightarrow \Gamma$ we get the restriction functor $\mathscr{D}_G(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell) \to \mathscr{D}_\Gamma(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell) = \mathscr{D}(X_s, \overline{\mathbb{Q}}_\ell)$. It is easy to check that the preimage of $\mathscr{D}_{\mathrm{mot}}(X_s, \overline{\mathbb{Q}}_\ell)$ in $\mathscr{D}_G(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$ does not depend on the choice of the splitting. Denote it by $\mathscr{D}_{G,\mathrm{mot}}(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$.

Theorem B.11. *The nearby cycle functor $\Psi : \mathscr{D}(X \setminus X_s, \overline{\mathbb{Q}}_\ell) \to \mathscr{D}_G(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$ maps $\mathscr{D}_{\mathrm{mot}}(X \setminus X_s, , \overline{\mathbb{Q}}_\ell)$ to $\mathscr{D}_{G,\mathrm{mot}}(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$.*

The proof below is parallel to Deligne's proof of a similar statement for $\mathscr{D}_{\mathrm{mix}}$, see [De3, Theorem 6.1.13]. To slightly simplify the argument, we use the theory of perverse sheaves (which did not exist when the article [De3] was written).

Proof. The idea is to use the formula

$$i^* j_* M = R\Gamma(I, \Psi M), \qquad M \in \mathscr{D}(X \setminus X_s, , \overline{\mathbb{Q}}_\ell). \tag{B.5}$$

Here $i : X_s \to X$ and $j : X \setminus X_s \to X$ are the embeddings, $I := \mathrm{Ker}(G \twoheadrightarrow \Gamma)$ is the inertia group, and $R\Gamma(I, ?)$ stands for the derived I-invariants.

We have to prove that if $M \in \mathscr{D}(X \setminus X_s, \overline{\mathbb{Q}}_\ell)$ is weakly motivic then so is ΨM. By the monodromy theorem, after a quasi-finite base change $S' \to S$ (which does not change ΨM) we can assume that the action of I on ΨM is unipotent. Then I acts via its maximal pro-ℓ quotient $\mathbb{Z}_\ell(1)$.

By Proposition B.10(i), we can also assume that M is perverse. By [Ill, Corollary 4.5], then ΨM is perverse. So formula (B.5) implies that

$$\mathscr{H}^0(i^* j_* M) = (\Psi M)^I, \tag{B.6}$$

where $\mathscr{H}^0$ stands for the 0-th perverse cohomology sheaf. By Theorem B.3 and Proposition B.10(i), the l.h.s. of (B.5) is weakly motivic. So $(\Psi M)^I$ is weakly motivic.

Let $(\Psi M)_r$ be the maximal perverse subsheaf of ΨM on which the action of I is unipotent of degree r. The operators $(\sigma - 1)^{r-1}$, $\sigma \in I$, define a monomorphism $(\Psi M)_r/(\Psi M)_{r-1} \hookrightarrow (\Psi M)^I(1-r)$. Since $(\Psi M)^I$ is weakly motivic, ΨM is weakly motivic by Proposition B.10(ii). ☐

Appendix C. A corollary of Poonen's "Bertini theorem over finite fields"

In this appendix we justify Remark 2.18(ii) by proving the following proposition.

Proposition C.1. *Let $Y \subset \mathbb{P}^n_{\mathbb{F}_q}$ be an absolutely irreducible closed subvariety of dimension $d \ge 2$. Let $F' \subset F \subset Y$ be closed subschemes such that $\dim F' \le d-2$, the subscheme $Y \setminus F'$ is smooth, and $F \setminus F'$ is a smooth divisor in $Y \setminus F'$. Then there exists an irreducible curve $C \subset Y \setminus F'$ such that $C \not\subset F$ and for any connected etale covering $W \to Y \setminus F$ tamely ramified along the irreducible components of $F \setminus F'$ the scheme $W \times_Y C$ is connected. Moreover, given closed points $y_i \in Y \setminus F'$ and 1-dimensional subspaces $l_i \subset T_{y_i} Y$, $1 \le i \le m$, one can choose C so that $y_i \in C$ and $T_{y_i} C = l_i$ for all $i \in \{1, \dots, m\}$.*

The proof is given in §§C.1-C.2 below.

C.1. **Applying a geometric Bertini theorem.** We will use the notation Pol_n for the set of homogeneous polynomials in $n+1$ variables over $\mathbb{F}_q$ (of all possible degrees). The scheme of zeros of $f \in \mathrm{Pol}_n$ in $\mathbb{P}^n_{\mathbb{F}_q}$ will be denoted by $V(f)$.

Lemma C.2. *Let Y, F, and F' be as in Proposition C.1. Let $C := Y \cap V(f_1) \cap \ldots V(f_{d-1})$, where $f_1, \ldots, f_{d-1} \in \mathrm{Pol}_n$. Suppose that C is a smooth curve contained in $Y \setminus F'$ and meeting $F \setminus F'$ transversally. Then for any connected etale covering $W \to Y \setminus F$ tamely ramified along the irreducible components of $F \setminus F'$ the scheme $W \times_Y C$ is connected.*

Proof. Let $\overline{W} \supset W$ be the normalization of Y in the field of rational functions on W. We can assume that the field of constants of W equals $\mathbb{F}_q$ (if it equals $\mathbb{F}_{q^n}$ then consider W as a covering of $Y \otimes \mathbb{F}_{q^n}$). Then W and $\overline{W}$ are absolutely irreducible. So by Theorem 2.1(A) from [FL] (which is a Bertini theorem), $\overline{W} \times_Y C$ is (geometrically) connected. On the other hand, the tameness and transversality assumptions imply that $\overline{W} \times_Y C$ is smooth. So any open subset of $\overline{W} \times_Y C$ is connected. In particular, $W \times_Y C$ is connected. □

C.2. **Applying Poonen's theorem.** Lemma C.2 shows that to prove Proposition C.1, it suffices to construct $f_1, \ldots, f_{d-1} \in \mathrm{Pol}_n$ such that $Y \cap V(f_1) \cap \ldots V(f_{d-1})$ is a smooth curve contained in $Y \setminus F'$, meeting $F \setminus F'$ transversally, and passing through the points y_i in the given directions l_i. By induction, it suffices to prove the following statement.

Lemma C.3. *Let $Y \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a closed subcheme of pure dimension $d \geq 2$. Let $F' \subset F \subset Y$ be closed subschemes such that $\dim F' \leq d-2$, the subscheme $Y \setminus F'$ is smooth, and $F \setminus F'$ is a smooth divisor in $Y \setminus F'$. Then there exists $f \in \mathrm{Pol}_n$ such that $V(f) \cap Y$ has pure dimension $d-1$, $\dim(V(f) \cap F) \leq d-2$, $\dim(V(f) \cap F') \leq d-3$, and the schemes $V(f) \cap (Y \setminus F')$ and $V(f) \cap (F \setminus F')$ are smooth. Moreover, given closed points $y_i \in Y \setminus F'$ and hyperplanes $H_i \subset T_{y_i}Y$, $1 \leq i \leq m$, one can choose f so that $y_i \in V(f)$ and $T_{y_i}(V(f) \cap Y) = H_i$ for all $i \in \{1, \ldots, m\}$.*

Proof. Let A denote the set of $f \in \mathrm{Pol}_n$ satisfying our conditions with a possible exception of the condition

$$\dim(V(f) \cap F') \leq d-3\,. \tag{C.1}$$

Let $B \subset \mathrm{Pol}_n$ be the set of $f \in \mathrm{Pol}_n$ such that (C.1) does not hold (i.e., such that f vanishes on some irreducible component of F' of dimension $d-2$). We have to show that $A \setminus B \neq \emptyset$. In fact, $A \setminus B$ has positive density in the sense of [Po, §1]. If $d = 2$ this directly follows from [Po, Theorem 1.3] (because F' is finite). If $d > 2$ then B has density 0; on the other hand, A has positive density by [Po, Theorem 1.3]. □

Appendix D. Weil numbers and CM-fields

In this appendix we justify Remark 1.10.

D.1. **Formulation of the result.** Fix an algebraic closure $\overline{\mathbb{Q}} \supset \mathbb{Q}$. Let $R \subset \overline{\mathbb{Q}}$ be the maximal totally real subfield. Let $C \subset \overline{\mathbb{Q}}$ be the union of all CM-subfields.[6] Then $C = R(\sqrt{-d})$ for any totally positive $d \in R$.

Let p be a prime and q a power of p. A number $\alpha \in \overline{\mathbb{Q}}$ is said to be a *Weil number* if it is a unit outside of p and there exists $n \in \mathbb{Z}$ such that all complex absolute values of α equal $q^{n/2}$.

Theorem D.1. *The subfield of $\overline{\mathbb{Q}}$ generated by all Weil numbers equals C.*

The fact that all Weil numbers are in C is easy and well known (e.g., see [Ho, Proposition 4]). So Theorem D.1 follows from the next proposition, which will be proved in §D.2.

Proposition D.2. *Any CM-field K_0 is contained in a CM-field K which is generated by Weil numbers in K.*

[6] We include finiteness over $\mathbb{Q}$ in the definition of a CM-field.

D.2. **Proof of Proposition D.2.**

Lemma D.3. *Let K be a CM-field and $k \subset K$ a totally real subfield. Suppose that there exists a place $\mathfrak{p}$ of k and a place $\mathfrak{p}'$ of K such that $\mathfrak{p}|p$, $\mathfrak{p}'|\mathfrak{p}$, and the map $k_{\mathfrak{p}} \to K_{\mathfrak{p}'}$ is an isomorphism. Then Weil numbers in K generate K over k.*

Proof. Since the ideal class group of K is finite there exists $a \in K$ such that $|a|_{\mathfrak{p}'} < 1$ and all other nonarchimedean absolute values of a equal 1. In particular, $|\bar{a}|_{\mathfrak{p}'} = |a|_{\bar{\mathfrak{p}}'} = 1$ (note that the assumption $K_{\mathfrak{p}'} = k_{\mathfrak{p}}$ implies that $\bar{\mathfrak{p}}' \neq \mathfrak{p}'$). The number $\alpha := a/\bar{a}$ is a Weil number. Let $\tilde{K} \subset K$ be the subfield generated by α over k. We will show that $\tilde{K} = K$.

Let $\tilde{\mathfrak{p}}$ be the place of $\tilde{K}$ corresponding to $\mathfrak{p}'$. Then $\mathfrak{p}'$ is the only place of K over $\tilde{\mathfrak{p}}$ (this follows from the fact that $|\alpha|_{\mathfrak{p}'} < 1$ and all other nonarchimedean absolute values of α are ≥ 1). On the other hand, $K_{\mathfrak{p}'} = \tilde{K}_{\tilde{\mathfrak{p}}} = k_{\mathfrak{p}}$. So $[K : \tilde{K}] = [K_{\mathfrak{p}'} : \tilde{K}_{\tilde{\mathfrak{p}}}] = 1$. □

Proof of Proposition D.2. Let K_0 be a CM-field. There exists a finite extension $F \supset \mathbb{Q}_p$ such that K_0 admits an embedding $i : K_0 \hookrightarrow F$. Fix F and i. By weak approximation, there exists a totally real field k with $k \otimes \mathbb{Q}_p = \mathbb{Q}_p \times F$. Note that both k and K_0 are subfields of F. The composite field $K_1 := k \cdot K_0 \subset F$ is a CM-field. Applying Lemma D.3 to the place of K_1 corresponding to the embedding $K_1 \hookrightarrow F$, we see that Weil numbers in K_1 generate K_1 over k.

On the other hand, let $d \in k$ be a totally positive element such that the $\mathbb{Q}_p$-algebra $k(\sqrt{-d}) \otimes \mathbb{Q}_p$ has $\mathbb{Q}_p$ as a factor. Then by Lemma D.3, Weil numbers in $k(\sqrt{-d})$ generate $k(\sqrt{-d})$ over $\mathbb{Q}$.

Now set $K := K_1(\sqrt{-d})$. Then K is a CM-field such that Weil numbers in K generate K over $\mathbb{Q}$. □

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