Particular lightface and boldface complexities somewhat above Σ^1_1

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$\mathcal{A}\Pi^1_1$

Definition

$\mathcal{A}\Pi^1_1$ is the complexity of Σ^1_1 questions about Π^1_1 objects.

Some examples:

- Let $(T_n)_{n\in\omega}$ be an effective listing of Π^1_1 subtrees of $\omega^{<\omega}$.

$$\{n : T_n \text{ is ill-founded}\}.$$

- Let $(B_n)_{n\in\omega}$ be an effective listing of Π^1_1 subsets of $\mathbb{Q}$.

$$\{n : B_n \text{ is ill-founded}\}.$$

- Let Φ be a Π^1_1-operator giving subsets of $\mathbb{Q}$.

$$\{X \in 2^\omega : \Phi(X) \text{ is ill-founded}\}.$$

$\mathcal{A}\Pi^1_1$

These are actually Δ^1_2.
Consider

$$Y = \{n : T_n \text{ is ill-founded}\}.$$

$$n \in Y \iff \exists f \in \omega^\omega \, \forall k \, [(f \restriction k) \in T_n]$$

$$n \notin Y \iff \exists S \subseteq \omega^{<\omega} \, [T_n \subseteq S \;\wedge\; S \text{ is well-founded}]$$

Complexity class notation

Suppose Γ is a complexity class.

- $\neg\Gamma = \{A : A^c \in \Gamma\}$. Example: If $\Gamma = \Sigma^0_\alpha$, then $\neg\Gamma = \Pi^0_\alpha$.
- $\Delta(\Gamma) = \Gamma \cap \neg\Gamma$. Example: If $\Gamma = \Sigma^0_\alpha$, then $\Delta(\Gamma) = \Delta^0_\alpha$.

$$\Sigma^0_1 \vee \Pi^0_1 = \{A \cup B : A \in \Sigma^0_1, B \in \Pi^0_1\}$$
$$\Sigma^0_1 \wedge \Pi^0_1 = \{A \cap B : A \in \Sigma^0_1, B \in \Pi^0_1\}$$

Note: $\Sigma^0_1 \vee \Pi^0_1 = \neg(\Sigma^0_1 \wedge \Pi^0_1)$.

Gale-Stewart games

For $A \subseteq \omega^\omega$, the game G_A is played:

$$\begin{array}{c|ccccccc} \text{I} & a_0 \in \omega & & a_2 \in \omega & & a_4 \in \omega & & \cdots \\ \text{II} & & a_1 \in \omega & & a_3 \in \omega & & a_5 \in \omega & \cdots \end{array}$$

Player I wins iff $a_0 a_1 a_2 \cdots \in A$.

A *strategy* is a function $\Theta : \omega^{<\omega} \to \omega$.
"If it's your turn and the plays so far have been σ, then play $\Theta(\sigma)$."

A *winning strategy* for a player will always win, no matter how the opponent plays.

The game quantifier

For $A \subseteq \omega^\omega$, define $\Game A$ to be the statement:

"Player I has a winning strategy for G_A."

If Γ is a complexity class, $\Game\Gamma$ is the class made by game-quantifying sets from Γ.

Examples:

- Let $(A_n)_{n\in\omega}$ be an effective listing of Γ sets in ω^ω.

$$\{n : \Game A_n\}.$$

- Let Φ be a Γ-operator giving sets in ω^ω.

$$\{X : \Game\Phi(X)\}.$$

$\mathcal{A}\Pi^1_1$ and $\Game(\Sigma^0_1 \vee \Pi^0_1)$

Proposition

$\mathcal{A}\Pi^1_1 = \Game(\Sigma^0_1 \vee \Pi^0_1)$.

Proof ($\Rightarrow$).

Consider $B \subseteq \mathbb{Q}$ which is Π^1_1.
The game:

1. Player I builds a descending sequence $q_0 > q_1 > \ldots$ while Player II passes.
2. After a q_i is played, Player II can object. Then Player I passes while Player II builds an f witnessing that $q_i \notin B$.

Player I wins if:

- PII never objects; or $\leftarrow$ closed
- PII objects, but then their f fails. $\leftarrow$ open

$\mathcal{A}\Pi^1_1$ and $\Game(\Sigma^0_1 \vee \Pi^0_1)$

Proposition

$\mathcal{A}\Pi^1_1 = \Game(\Sigma^0_1 \vee \Pi^0_1)$.

Proof ($\Leftarrow$).

Take $A \in \Sigma^0_1 \vee \Pi^0_1$. Say $A = A_0 \cup A_1$, $A_0 \in \Sigma^0_1$, $A_1 \in \Pi^0_1$.

Let $T \subseteq \omega^{<\omega}$ be the set of finite partial strategies Θ for Player I such that against any finite play by Player II:

- Θ will stay on A_1; or $\leftarrow$ computable
- Θ will reach a point from which there is a strategy to guarantee reaching A_0. $\leftarrow \Pi^1_1$ (see below)

Π^1_1: Against any strategy of Player II continuing the game from that point, there is a finite play by Player I to reach A_0. □

Dual and Δ

Proposition

$\mathcal{A}\Pi^1_1 = \reflectbox{∂}(\Sigma^0_1 \vee \Pi^0_1)$.

Proposition

$\neg\mathcal{A}\Pi^1_1 = \reflectbox{∂}(\Sigma^0_1 \wedge \Pi^0_1)$.

Proposition

$\Delta(\mathcal{A}\mathbf{\Pi}^1_1)$ *is the* σ*-algebra generated by* $\mathbf{\Sigma}^1_1$.

Proof ($\Leftarrow$).

Easy to check that $\reflectbox{$\partial$}(\mathbf{\Sigma^0_1} \vee \mathbf{\Pi^0_1})$ contains $\mathbf{\Sigma^1_1}$, $\mathbf{\Pi^1_1}$, and is closed under countable intersections and unions. $\square$

Muchnik and Medvedev

Definition

Let M and N be first order structures.

M is *Muchnik reducible* to N, written $M \leq_w N$, if every copy of N computes a copy of M.

M is *Medvedev reducible* to N, written $M \leq_s N$, if it is uniformly Muchnik reducible.
That is, there is a Turing functional Φ such that if $N_0 \cong N$, then $\Phi(N_0) \cong M$.

Question

What are the complexities of $\{(M, N) : M \leq_w N\}$ and $\{(M, N) : M \leq_s N\}$?

Upper bounds

Theorem (Greenberg, Harrison-Trainor, Scott, Shafer)

Muchnik and Medvedev reducibility are both in $\neg\mathcal{A}\mathbf{\Pi}^1_1$.

Proof ($M \not\leq_w N$ is $\mathcal{A}\mathbf{\Pi}^1_1$).

The game:

1. Player I builds f a permutation of ω. Let $N_0 = f(N)$. Player I also builds $\mathcal{O}^M$ and plays descending sequences witnessing each $n \notin \mathcal{O}^M$.
2. For each e, Player I either plays a witness to $\Phi_e(N_0)$ being partial or a strategy for the $(M, \Phi_e(N_0), \omega_1^{ck}(M) + 2)$-back-and-forth game.
3. At any point Player II can object and build either a descending sequence contradicting some claimed $n \in \mathcal{O}^M$ or a finite sequence defeating the strategy at some e.

Upper bounds

Theorem (Greenberg, Harrison-Trainor, Scott, Shafer)

Muchnik and Medvedev reducibility are both in $\neg\mathcal{A}\mathbf{\Pi}^1_1$.

Proof ($M \not\leq_w N$ is $\mathcal{A}\mathbf{\Pi}^1_1$).

Player I wins if:

- Player II never objects and no partiality witness is wrong; or
  $\nwarrow$ closed
- Player II objects, but their objection fails. $\leftarrow$ open $\square$

Upper bounds

Theorem (Greenberg, Harrison-Trainor, Scott, Shafer)

Muchnik and Medvedev reducibility are both in $\neg\mathcal{A}\mathbf{\Pi}^1_1$.

Proof ($M \not\leq_s N$ is $\mathcal{A}\mathbf{\Pi}^1_1$).

Same as $\leq_w$, but instead of Player I dealing with every e, Player II names a single e before anything else happens. □

Lower bounds

Definition

For $\gamma < \omega_1$, a set A is $D^\gamma \boldsymbol{\Sigma}^1_1$ if there is an increasing sequence $(A_\alpha)_{\alpha \leq \gamma}$ of $\boldsymbol{\Sigma^1_1}$ sets with $A_\gamma = \omega^\omega$ and such that

$$A = \{x : \text{parity}(\min\{\alpha : x \in A_\alpha\}) \neq \text{parity}(\gamma)\}.$$

Recall: $\Delta(\mathcal{A}\boldsymbol{\Pi}^1_1)$ is the σ-algebra generated by $\boldsymbol{\Sigma}^1_1$.

This difference hierarchy reaches partway up this σ-algebra.

Theorem (Aguilera, Greenberg, T)

Muchnik and Medvedev reducibility are both hard for every level of the $\boldsymbol{\Sigma}^1_1$ difference hierarchy.

A tool

Theorem (Harrison-Trainor, Kalimullin, Melnikov)

The set $\{i : M_i^{\emptyset'}$ has a computable copy$\}$ is Σ^1_1-complete.

Theorem (T)

The above result holds for every non-computable degree, with all possible uniformity.
That is, there is a uniform process to take a Σ^1_1 sentence ψ and produce a Turing functional Φ such that:

- *If ψ holds, then there is a structure M such that for every oracle X, $\Phi(X) \cong M$.*
- *If ψ fails, then there is a structure M such that for every non-computable oracle X, $\Phi(X) \cong M$, and further M has no computable copy.*

Lower bounds

Theorem (Aguilera, Greenberg, T)

Muchnik and Medvedev reducibility are both hard for every level of the $\mathbf{\Sigma}^1_1$ difference hierarchy.

Proof.

Suppose A is $D^\gamma\Sigma^1_1$, given by $(A_\alpha)_{\alpha\leq\gamma}$. Suppose we are given x.
Let ψ_α be the Σ^1_1-sentence $x \in A_\alpha$.
Construct M_α as given by the theorem, relativized to $0^{(\alpha)}$.
Let $\hat{M}_\alpha$ be an α-level marker extension of M_α.

If ψ_α holds, $\hat{M}_\alpha$ has a computable copy. If it fails, $\hat{M}_\alpha$ has copies in precisely the non-low$_\alpha$ degrees.

$$\left(\bigoplus_{\text{parity}(\alpha)\neq\text{parity}(\gamma)} \hat{M}_\alpha, \bigoplus_{\text{parity}(\alpha)=\text{parity}(\gamma)} \hat{M}_\alpha\right) \qquad \square$$

A different reducibility

Definition

Suppose K and L are linear orders. Define $K \leq_{woce} L$ if there are ordinals β_0, β_1 and linear orders $(K_\alpha)_{\alpha<\beta_0}$ and $(L_\alpha)_{\alpha<\beta_1}$ satisfying:

- $K \cong \sum_{\alpha<\beta_0} K_\alpha$;
- $L \cong \sum_{\alpha<\beta_1} L_\alpha$; and
- $(K_\alpha)_{\alpha<\beta_0}$ occurs as a subsequence of $(L_\alpha)_{\alpha<\beta_1}$.

Question

What is the complexity of $\{(K, L) : K \leq_{woce} L\}$?

Conjecture: $\neg\mathcal{A}\Pi^1_1$.

The βs

Note that we can assume $\beta_1 = 2\beta_0 + 1$.

Example: Fix H the Harrison order. Let

$$K = \sum_{x \in H} \left((H \restriction x) \cdot \omega + \mathbb{Z} \right)$$

and

$$L = (H + \mathbb{Z}) \cdot H.$$

Then $K \leq_{woce} L$, but only with $\beta_0 \geq \omega_1^{ck} + 1$.

A lower bound

Theorem (Aguilera, T)

$\leq_{woce}$ reducibility is hard for $\neg\mathcal{A}\Pi^1_1$.

Proof.

Suppose $B \subseteq \mathbb{Q}$ is Π^1_1. Build N_q such that

- $N_q \cong H$ if $q \notin B$; and
- N_q well-founded if $q \in B$.

Let

$$K = \sum_{q \in \mathbb{Q}} N_q$$

and

$$L = H \cdot \mathbb{Q}.$$

□

Back to the βs

How large does β_0 need to be?

Suppose there were a $\delta < \omega_2^{ck}$ such that any computable ordinals $K \leq_{woce} L$ could be witnessed with $\beta_0 < \delta$. Fix $\delta = \Phi_e(\mathcal{O})$.

Game for $K \leq_{woce} L$:

1. Player I builds $\mathcal{O}$ and witnesses for any claims that $n \notin \mathcal{O}$.
2. Player I also builds witnessing partitions and isomorphisms using $\Phi_e(\mathcal{O})$.
3. Player II can challenge any claim $n \in \mathcal{O}$ by building a descending sequence.

This would show $\leq_{woce}$ is $\mathcal{A}\Pi^1_1$, a contradiction.
So the necessary βs aren't bounded below ω_2^{ck}.

Thank you.