Contrasting the halves of an Ahmad pair

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Midwest Computability Seminar XXXV
University of Chicago
26 September 2025

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The Enumeration Degrees

Definition

We say $X \leq_e Y$ if every enumeration of Y computes an enumeration of X.

A more useful characterization for constructions is the following: $X \leq_e Y$ if there is a c.e. operator Γ such that $x \in X \iff \exists \langle x, D \rangle \in \Gamma$ with $D \subseteq Y$.

1. This is a positive analog of Turing reducibility. One can think of $X \leq_T Y$ as X is $\Delta^0_1(Y)$. Here we can think of $X \leq_e Y$ as X is $\Sigma^+_1(Y)$, so we can only ask positive questions about Y.
2. $\leq_e$ is a preorder on $\mathcal{P}(\omega)$ and the corresponding degree structure induced on $\mathcal{P}(\omega)$ is called the enumeration degrees denoted by $(\mathcal{D}_e, \leq_e)$.
3. The enumeration jump of a set X is defined by $X' = K_X \oplus \overline{K}_X$ where $K_X := \bigoplus_{n\in\omega} \Gamma_n(X)$. The skip of a set is defined as $X^\diamond = \overline{K_X}$. While these two operations do not agree in general, they will agree for the degrees we care about in this talk.

The Enumeration Degrees

4. Observe : $X \leq_T Y$ iff $X \oplus \overline{X} \leq_e Y \oplus \overline{Y}$. So we can think of Turing reducibility in terms of enumeration reducibility.

5. The Turing degrees $(\mathcal{D}_T, \leq_T)$ embed into the enumeration degrees as a partial order under the map $\iota : A \to A \oplus \overline{A}$ and this embedding respects join and jump. The image of this mapping are the total degrees: degrees containing a set X for which $\overline{X} \leq_e X$.

6. $\mathcal{D}_T$ and $\mathcal{D}_e$ are both upper semilattices with a least element. The least element in D_T is the family of computable sets while in $\mathcal{D}_e$ the least element 0_e is made up of the c.e. sets.

7. Monotonicity is a salient property of enumeration operators : If $A \subseteq B$ then $\Gamma(A) \subseteq \Gamma(B)$ for any enumeration operator Γ. Note that no such simple relation holds in the Turing world.

Local Structure

Let $\mathcal{D}_T(\leq 0'_T)$ be the Turing degrees below $0'_T$ while $\mathcal{D}_e(\leq 0'_e)$ denote the enumeration degrees below $0'_e$. Both of these sets form a countable ideal in their respective degree structures. The standard representative of $0'_e$ is the Π^0_1 complete set $\overline{K} = \{e : e \notin \Gamma_e(\emptyset)\}$.

1. $X \leq_T 0'_T$ iff X is Δ^0_2 while $X \leq_e 0'_e$ iff X is Σ^0_2 iff X is c.e. relative to $0'$.
2. We have the relation : $\iota(\mathcal{D}_T(\leq 0'_T)) \subset D_e(\leq 0'_e)$.
3. These two structures are not elementarily equivalent as partial orders: For example, Sacks constructed a minimal Turing degree below $0'_T$ while Guttridge showed that the enumeration degrees are downward dense (globally).
4. In fact Cooper [Coo84] was able to show that given $X <_e Y \leq 0'_e$, there is a Z such that $X <_e Z <_e Y$. So $\mathcal{D}_e(\leq 0'_e)$ is actually dense.
5. This is far from true in the global structure: Calhoun and Slaman [CS96] construct an empty interval in the Π^0_2 enumeration degrees.

Ahmad pairs

Definition

Let A, B be Σ^0_2 sets. Then (A, B) is an Ahmad pair if $A \not\leq_e B$ and $\forall Z <_e A(Z <_e B)$.

This is almost like a strong minimal cover, except $A \not\geq_e B$.

Theorem (Ahmad [AL98])

There is an Ahmad pair (A, B) with A, B - Δ^0_2. Moreover if (A, B) is an Ahmad pair, then (B, A) is not an Ahmad pair.

This phenomenon cannot occur in the c.e. Turing degrees due to Sacks splitting, and hence the c.e. Turing degrees are not elementarily equivalent to $\mathcal{D}_e(\leq 0'_e)$. The following work is more recent.

Theorem (Goh, Lempp, Ng, M. Soskova[Goh+22])

If (A, B) is an Ahmad pair, then (B, C) is not an Ahmad pair for any C.

Theorem (Kalimullin, Lempp, Ng, Yamaleev[Kal+24])

If (A, B) is an Ahmad pair, then $A \oplus B <_e 0'_e$.

Importance of Ahmad pairs

1. Slaman and Soare [SS01] solved the extension of embeddings problem in the c.e. Turing degrees: For finite partial orders $\mathcal{P} \subset \mathcal{Q}$, when can every embedding of $\mathcal{P}$ be extended to an embedding of $\mathcal{Q}$. They provide two obstructions barring which an extension is always possible.
2. Lempp, Slaman and Sorbi [LSS05] solve the extension of embeddings problem for the Σ^0_2 enumeration degrees, where the only added obstruction to extension is the phenomenon of Ahmad pairs.

Recently several researchers have been focusing on solving the $\forall\exists$ theory of the Σ^0_2 enumeration degrees which is in general a strictly harder problem than the extensions of embeddings problem. This has resulted in renewed interest in Ahmad pairs with the two recent results above.
Kent [Ken06] showed that the $\forall\exists\forall$ theory of $\mathcal{D}_e(\leq 0'_e)$ is undecidable, so this is the best we can hope for.

New Work

Previous work separating the right and left half [Goh+22] was quite involved, using a $0'''$ priority construction. Moreover, it did not explain why the two halves are separate. We present a simpler proof of a more general result, which gives a natural separation between the two halves in terms of jump classes.

Definition

The tuple $(A, B_0, ..., B_{n-1})$ is an Ahmad n-pair if $A \not\leq_e B_i$ for every $i < n$ and any $Z <_e A$ is below at least one of the B_i's.

Theorem

For every $n > 1$, there is an Ahmad n-pair whose left half is not part of an Ahmad $(n-1)$ pair.

Theorem

If $(A, B_0, B_1, ..., B_{n-1})$ is a proper Ahmad n-pair, then A is low_3 and B_i is not low_3 for any $i < n$. Therefore the possible left and right halves are disjoint.

The left half

Note that if (A, B) is an Ahmad pair, then the set $\{Z : Z <_e A\}$ is an ideal. In particular A is join irreducible. Observe that (A, B) form an exact pair for this ideal.

Theorem

Let f, g be computable. There is a computable function h such that $\{\Gamma_{h(n)}(0'_e)\}_n = \{\Gamma_{f(n)}(0'_e)\}_n \cap \{\Gamma_{g(n)}(0'_e)\}_n$

Proof.

Recall that the Σ^0_2 sets are precisely those which are $c.e.$ relative to $0'$. So let $\{W^{0'}_{f(n)}\}_n$ and $\{W^{0'}_{g(n)}\}$ be the uniform families of Σ^0_2 sets. Then we define $h(n)$ as follows:

$$x \in W^{0'}_{h(\langle e,i\rangle),s} \iff x \in W^{0'}_{f(e),s} \cap W^{0'}_{g(i),s} \text{ and } W^{0'}_{f(e),s} \upharpoonright x = W^{0'}_{g(i),s} \upharpoonright x.$$

So we successfully copy $W^{0'}_{f(e)}$ into $W^{0'}_{h(\langle e,i\rangle)}$ precisely when $W^{0'}_{f(e)} = W^{0'}_{g(i)}$, while $W^{0'}_{h(\langle e,i\rangle)}$ is finite otherwise. □

The left half

Theorem

If $(A, B_0, ..., B_{n-1})$ form an Ahmad n pair, the ideal $\{Z : Z <_e A\}$ has a uniform enumeration.

Proof.

By the parameter theorem the ideals $\{Z : Z \leq_e A\}$ and $\{Z : Z \leq_e B_i\}$ have uniform enumerations. Therefore so does their intersection $\mathcal{F}_i := \{Z : Z \leq_e A, B_i\}$. Then $\mathcal{F} = \bigcup_{i<n} \mathcal{F}_i$ has a uniform enumeration as well and $\mathcal{F} = \{Z : Z <_e A\}$. □

We call a uniform enumeration of $\{Z : Z <_e A\}$ via indices relative to $0'_e$, an Ahmad sequence for A.

Definition

A set A is called low_3 if $A''' \leq_e 0'''$. (We are dealing with enumeration jumps here!)

The results below are stated in terms of jump, although strictly speaking they are statements about the skip. For Σ_2 (in fact cototal) degrees, these two notions coincide (upto degree).

The left half

Theorem

The following are equivalent:

1. A *has an Ahmad sequence* $\{\Gamma_{f(n)}(0'_e)\}$.
2. $X = \{n : \Gamma_n(A) <_e A\}$ *is* Σ^0_4.
3. A *is* low_3.

Proof.

$(1 \implies 2)$ $X \leq_1 A'''$ and so is always Π^0_4. Using the Ahmad sequence it also has a Σ^0_4 definition.
$(2 \implies 3)$ We will show that $A''' \leq_e X$ below. Then $A''' \in \Sigma^0_4$ so $A''' \leq_e 0'''$.
$(3 \implies 1)$ Note that $X \leq_e A''' \leq (0'_e)''$ and so X has a $\Sigma^0_3(0'_e)$ definition. Let

$$e \in X \iff \exists n \forall m \exists i R(e, n, m, i)$$

with $R \leq_1 0'_e$. Then using this we can define an Ahmad sequence $\{A_{e,n}\}$ for A such that if $e \in X$ then $\exists n(A_{e,n} = \Gamma_e(X))$ while if $e \notin X$ then $A_{e,n}$ is finite for every n. □

The left half

Lemma

For any A, there is a computable function g :

1. $e \in A^{\langle 3 \rangle} \iff \Gamma^{[i]}_{g(e)}(A)$ *is finite for every i.*
2. $e \notin A^{\langle 3 \rangle} \iff \Gamma^{[i]}_{g(e)}(A)$ *is infinite for some i .*

Lemma

let $A >_e 0'_e$. For any set $X \leq_e A$, can uniformly build an enumeration operator Θ, such that:

1. $\Theta(A) <_e A \iff X^{[i]}$ *is finite for every i*
2. $\Theta(A) \geq_e A \iff X^{[i]}$ *is infinite for some i.*

Corollary

$A^{\langle 3 \rangle} \leq_e \{e : \Gamma_e(A) <_e A\}$.

The left half

Definition

1. A good approximation to X is a computable sequence $\{X_s\}_s$ of finite sets with infinitely many good stages $G_X := \{s : X_s \subseteq X\}$ such that $\lim_{s \in G_X} X_s(n) = X(n)$ for every n.
2. A correct approximation $\{Y_s\}$ to Y with respect to $\{X_s\}$ satisfies $G_X \subseteq G_Y$ and $\lim_{s \in G_X} Y_s(n) = Y(n)$.

Proof.

Let $X = \Delta(A)$ and let $\{A_s\}_s$ be a good approximation to A. We shall build the enumeration operator Θ to meet the requirements:

$$\mathcal{R}_e : \Gamma_e(\Theta(A)) \neq A \iff X^{[\leq e]} \text{ is finite}$$

i) Let $l_{e,s} = l(\Gamma_{e,s}(\Theta(A_s)), A_s)$. Then $\forall x \in \Gamma_{e,s}(\Theta(A_s)^{[\leq e]} \cup \mathbb{N}^{[>e]}$, via axiom $\langle x, D \rangle$, add the axiom $\langle y, A_s \rangle$ into Θ for $y \in D^{[>e]}$.
ii) Copy $A \upharpoonright n$ where $n = |\Delta_s^{[t]}(A_s)|$ into the t^{th} column by enumerating axioms $\langle \langle t, x \rangle, \{x\} \cup A_s \rangle$ into Θ for every $x \leq n$. □

Characterizing the left half

Definition

A is n join irreducible if for every $A_0, \ldots, A_n <_e A$ there is an $i, j \leq n$ with $A_i \oplus A_j <_e A$.

Theorem

A is the left half of an Ahmad n-pair $\iff$ A is low_3 and n join irreducible.

Proof.

If A is the left half of an Ahmad n pair it has an Ahmad sequence and is therefore low_3. By the pigeonhole principle, A is n join irreducible.
For the converse, suppose A is n join irreducible and $(n-1)$ join reducible. Let $X_1, \ldots, X_n$ satisfy $X_i \oplus X_j \equiv_e A$ for $i \neq j$. Let $\mathcal{F}_i = \{Z : Z \oplus X_i <_e A\}$. □

Lemma

Let f be computable and $\mathcal{F} = \{\Gamma_{f(n)}(A)\}_n$ be an ideal such that $\forall n(A \not\leq_e \Gamma_{f(n)}(A))$. Then there is a Σ^0_2 set B with $A \not\leq_e B$ and $\forall X \in \mathcal{F}(X \leq_e B)$.

Proof.

We will build a B by coding $\Gamma_{f(n)}(A)$ into the n^{th} column of B while ensuring that $A \not\leq_e B$. Let $\{A_s\}_s, \{B_{n,s}\}_s$ be correct approximations to $A, \Gamma_{f(n)}(A)$ respectively with respect to a good approximation K_s to $\overline{K}$. We will build an enumeration operator Θ so that $B = \Theta(\overline{K})$ will meet the requirements:

$$\mathcal{N}_e : A \neq \Gamma_e(B)$$

$$\mathcal{P}_n : \Gamma_{f(n)}(A) \leq_e B^{[n]}$$

At stage $s = 0$, let $\Theta = \emptyset$.
At stage $s + 1$:

1. For $e \leq s$ let $l_{e,s} = l(A_s, \Gamma_{e,s}(B_s))$. Then $\forall x < l_{e,s}$ with $x \in \Gamma_{e,s}(B_s^{[\leq e]} \cup \mathbb{N}^{[>e]})$, pick the least axiom $\langle x, D\rangle \in \Gamma_e$ which witnesses this. Now for all $y \in D^{[>e]}$ enumerate the axioms $\langle y, K_s\rangle$ into Θ.
2. For $n \leq s$ if $x \in \Gamma_{f(n),s+1}(0'_e)$ then for every new axiom $\langle x, D\rangle \in \Gamma_{f(n),s+1} - \Gamma_{f(n),s}$ enumerate the axiom $\langle\langle n, x\rangle, D\rangle$ into Θ.

□

On the right half

Lemma

Suppose (A, B) is an Ahmad pair. Then $A' \leq_e B'$.

Theorem

Suppose B is low_3 and $A \not\leq_e B$. We can build an enumeration operator Θ such that $\Theta(A) <_e A$ and $\Theta(A)|_e B$.

Proof.

Suppose $\{\Theta_n\}_{n\in\omega}$ are a family of enumeration operators. Consider the statement $\forall n\ \Theta_n(A) \not\leq_e B$:

$$\forall n, m \exists x (x \in \Theta_n(A) \wedge x \notin \Gamma_m(B)) \text{or} (x \in \Gamma_m(B) \wedge x \notin \Theta_n(A))$$

This statement is $\leq_e B'''$ and is Σ_4 if B is low_3. Let $\exists n S_n$ where S_n is Π^0_3 be a Σ_4 definition of the statement above. We construct a Θ such that its columns $\Theta^{[n]}$ correspond to Θ_n. □

On the right half

Proof.

By the recursion theorem, we may assume we know an index for Θ and so while constructing Θ we can reason about the statement $\exists n S_n$. We will ensure that the following holds:

1. $S_n \implies \Theta_n(A) <_e A$
2. $\neg S_n \implies \Theta_n(A) \equiv_e A$

Consider the following cases:

1. The statement is false: $\exists n\, \Theta_n(A) \leq_e B$. Then $\forall n \neg S_n$ and so by construction $\Theta_n(A) \equiv_e A$ for every n, a contradiction to $A \not\leq_e B$.
2. The statement is true: $\forall n\, \Theta_n(A) \not\leq_e B$. Then we also have $\exists n S_n$ and so $\Theta_n(A) <_e A$ for some n and we are done!

To construct the Θ_n's with the property that $S_n \iff \Theta_n(A) <_e A$ we just need to use the lemma above! □

Unanswered questions

We know the following about right halves:

1. For every left half A, there is a high right half B.
2. There is a right half B which is high_2 but not high.
3. If (A, B) is an Ahmad pair, then $A \oplus B$ is high_2.
4. There is no maximal right half: If (A, B) is an Ahmad pair, then any degree C with $B <_e C <_e A \oplus B$ is such that (A, C) is an Ahmad pair.
5. There is a high degree which does not bound any right half.

We end with some questions.

Question

Is there a characterization of the right halves of an Ahmad pair? Do the right halves have to be high_2*?*

Question

If $(A, B_0, \ldots, B_{n-1})$ *is an Ahmad* n*-pair, then is* $A \oplus B_i <_e 0'_e$*? Is there a simpler explanation in this framework for the non cupping of Ahmad pairs?*

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