Coarse Computability and Generic Reduction

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Outline

1. Generic Computability and Coarse Computability
2. Generic, Coarse, and Other Reducibilities
3. Density-1 Generic Degrees and Hyperarithmeticity

Background

In complexity theory, it has been observed that problems can be difficult in theory while being quite easy to solve in practice.

1986: Levin introduces "average-case complexity."

2003: Kapovich, Miasnikov, Schupp and Shpilrain introduce "generic-case complexity."

Generic computablity

In 2012, Jockusch, and Schupp introduce and analyze the notion of generic computability. A real is genericaly computable if it is possible to compute the *majority* of the bits of the real, in the following sense:

Definition

A real A is density-1 if the limit of the densities of its initial segments is 1, or in other words, if $\lim_{n\to\infty} \frac{|A\cap n|}{n} = 1$.

Generic and coarse computablity

Definition

A real A is generically computable if there exists a partial computable function φ whose domain is density-1 such that $\varphi(n) = \chi_A(n)$ for all $n \in \text{dom}(\varphi)$.

This is distinct from the following related notion.

Definition

A real A is coarsely computable if there exists a computable real B such that $\{n \mid \chi_A(n) = \chi_B(n)\}$ is density-1.

So a generic computation is a computation that usually halts, always correctly, while a coarse computation is a computation that always halts, usually correctly.

Examples

Any subset of the powers of 2 is both generically computable and coarsely computable.

Any density-1 real is coarsely computable, but a density-1 real is generically computable if and only if it has a density-1 subset that is c.e.

In 2012, Jockusch and Schupp showed that neither generic computability nor coarse computability implies the other.

Theorem (Downey, Jockusch, Schupp, 2013)

Every nonzero c.e. degree contains a real that is generically computable but not coarsely computable.

Theorem (Hirschfeld, Jockusch, McNicholl, Schupp)

If A is 1-generic, or weakly 2-random, then A does not compute any sets that are generically computable but not coarsely computable.

Theorem (I, 2013)

Every nonzero Turing degree contains a real that is coarsely computable but not generically computable.

Note, however, that the real content of the theorem is that every nonzero Turing degree computes a real that is coarsely computable but not generically undecidable. The reverse direction is achieved by coding into a density-0 set.

This is slightly unsatisfying – We can make coarsely computable reals that are not generically computable, and that are arbitrarily difficult to compute, but can we make coarsely computable reals that are arbitrarily difficult to generically compute?

To address this question, we introduce generic reducibility, and the generic degrees.

Generic oracles (informal)

Generic computations do not necessarily halt everywhere, and so, if we wish to make generic reduction transitive, we need a good notion of partial oracles

If we tried to use a generic computation of A as an oracle for A, we would be ensured that if we ever asked it a question, and it responded, it would respond truthfully. We would also be ensured that, for the majority of questions we could ask it it would respond.

On the other hand, the generic computation would not necessarily always tell us *whether* it was going to respond – it could always respond to a question a long time after we asked it!

Partial Oracles

Definition

Let A be a real. Then a (time-dependent) partial oracle, (A), for A is a set of ordered triples $\langle n, x, s\rangle$ such that:
$\exists s(\langle n, 0, s\rangle \in (A)) \implies n \notin A$,
$\exists s(\langle n, 1, s\rangle \in (A)) \implies n \in A$.

We think of (A) as a partial function, sending n to x. We think of s as the number of steps it takes (A) to converge.

The domain of (A) is the set of n for which there exists such an x, s.

Generic reductions

Definition

Let A be a real. Then a generic oracle for A is a partial oracle whose domain is density-1.

Note that generically computing A is equivalent to computing a generic oracle for A.

Definition

Let A, B be reals. We say A is generically reducible to B (or $A \leq_g B$) if there is a Turing functional φ such that for every generic oracle (B), for B, $\varphi^{(B)}$ is a generic computation of A.

Embedding the Turing degrees in the generic degrees

There is a natural embedding of the Turing degrees into the generic degrees:

Definition

For any real X, let $\mathcal{R}(X)$ be defined as follows.
$\mathcal{R}(X) = \{2^n(2k+1) : n \in X\}$.

So we have "stretched" every bit of X into a positive density "column" of $\mathcal{R}(X)$.

Since every generic computation of $\mathcal{R}(X)$ must include at least one bit from every column, it must be able to compute X.

As a result, generically computing $\mathcal{R}(X)$ is the same as computing X, and working with $\mathcal{R}(X)$ as a generic oracle is the same as working with X as an oracle in the usual sense.

Incorrect Embedding

To illustrate the effect of the uniformity assumption, consider the following flawed embedding.

Definition

For any real X, let $\widetilde{\mathcal{R}}(X)$ be defined as follows.
$\widetilde{\mathcal{R}}(X) = \{2^n + k : n \in X, k < 2^n\}$.

So the bits of X are coded into larger and larger finite initial segments of $\widetilde{\mathcal{R}}(X)$.

A generic computation of $\widetilde{\mathcal{R}}(X)$ computes cofinitely many of the bits of X.

For nonuniform generic reduction, $\mathcal{R}(X) \equiv \widetilde{\mathcal{R}}(X)$. However, in the uniform generic degrees, if X is not autoreducible, then $\mathcal{R}(X) \not\equiv_g \widetilde{\mathcal{R}}(X)$.

In fact, if X is 1-generic, or weakly 2-random, then $\widetilde{\mathcal{R}}(X)$ is quasi-minimal in the generic degrees. (Cholak, I.)

To discuss this distinction directly, we make the following definition.

Definition

Let A, B be reals. We say A is cofinitely reducible to B (or $A \leq_{\text{cf}} B$) if there is a Turing functional φ such that for every partial oracle (B), for B, with cofinite domain, $\varphi^{(B)}$ is a partial computation of A with cofinite domain.

Cofinite Reducibility

Observation (Dzhafarov, I.)

The map $X \mapsto \widetilde{\mathcal{R}}(X)$ induces an embedding of the cofinite degrees into the generic degrees.

Also, the map $X \mapsto \mathcal{R}(X)$ induces an embedding of the Turing degrees into the cofinite degrees.

The previously mentioned embedding of the Turing degrees into the generic degrees is (degreewise) the composition of the two.

Definition

Let A, B be reals. We say A is coarsely reducible to B (or $A \leq_{\text{cor}} B$) if there is a Turing functional φ such that for every C,
if $\{n : C(n) = B(n)\}$ is density-1,
then $\{n : \varphi^C(n) = A(n)$ is density-1.

Definition

Let A, B be reals. We say A is mod-finitely reducible to B (or $A \leq_{\text{mf}} B$) if there is a Turing functional φ such that for every C,
if $\{n : C(n) = B(n)\}$ is cofinite,
then $\{n : \varphi^C(n) = A(n)$ is cofinite.

Note, the map $X \mapsto \mathcal{R}(X)$ does not induce an embedding of the Turing degrees into the coarse degrees:

If X is Δ^0_2, then $\mathcal{R}(X)$ is coarsely computable.

However,

Theorem (Dzhafarov, I.)

The map $X \mapsto \widetilde{\mathcal{R}}(X)$ induces an embedding of the mod-finite degrees into the coarse degrees.

Also, the map $X \mapsto \mathcal{R}(X)$ induces an embedding of the Turing degrees into the mod-finite degrees.

So, the Turing degrees embed in the coarse degrees.

Cofinite and mod-finite reduction allow us to directly consider the difference between computations that are incomplete and computations that are incorrect without the added baggage introduced by considerations of density.

In this simplified setting, we get the following implication.

Theorem (Dzhafarov, I.)

If $A \leq_{mf} B$, then $A \leq_{cf} B$. The converse does not hold.

The implication hinges on the fact that a cofinite oracle for B is able to effectively emulate a mod-finite oracle for B.

Density-1 Generic Degrees

We say that a generic degree is density-1 if it is the generic degree of a density-1 real.

Lemma

The density-1 generic degrees are precisely the generic degrees of the coarsely computable reals.

Lemmas

Recall that a generic computation of a density-1 real is basically an enumeration of a density-1 subset of that real. This gives us a number of lemmas.

- $A \supseteq B \rightarrow A \leq_g B$
- $A \cap B \equiv_g A \oplus B$
- $A \leq_g B \leftrightarrow \exists \tilde{B}(\tilde{B} \equiv_g B)\&(A \supseteq \tilde{B})$

(All reals here are density-1)

How low?

We consider the question of how far down the density-1 degrees go in the generic degrees.

Theorem (I. 2013)

Given any finite set of noncomputable reals $\{A_0, \dots A_n\}$, there is a density-1 real B such that $\forall i \leq n\, (B \leq_g \mathcal{R}(A_i))$.

Proposition (I. 2013)

Given any two density-1 degrees **a** *and* **b**, *if* **a** $>_g$ **b**, *then there exists a density-1 degree* **c**, *such that* **a** $>_g$ **c** $>_g$ **b**.

Theorem (Solovay)

Let A be a real. Then A is hyperarithmetic if and only if there is a function f, and a Turing functional φ such that for every function g dominating f, φ^g is a computation of A. In this case, we say that f is a modulus of computation for A.

(Here, g dominates f if and only if $\forall n$, $g(n) \geq f(n)$. In this case, we sometimes write $g \geq f$.)

Theorem (I.)

Let A be a real. Then A is hyperarithmetic if and only if there is a density-1 real B, such that $B \geq_g \mathcal{R}(A)$.

($\Rightarrow$) This direction is easy:
Make the density of B approach 1 very slowly. Then any generic oracle will have density that also approaches 1 at least as slowly.

($\Leftarrow$) This direction is harder:

- Start with $B \geq_g \mathcal{R}(A)$
- Choose f so that for any $g \gg f$, g can generate a tree of density-1 oracles that includes B.
- Those oracles then repeatedly attempt to elect a "leader" who can cause them to vote unanimously.
- B is such a leader, so eventually they will find one.
- B always votes correctly, so when they find a leader, the vote will be correct.

Nonequivalence of density-1 degrees and moduli of computation

The forward direction was easy because we could capture the entire computation power of a fast growing function within a density-1 real.
For the reverse direction, this is not possible.

Theorem (I.)

There exists a density-1 real, A, such that for every $f : \mathbb{N} \to \mathbb{N}$, and every φ, there is a $g \geq f$ such that φ^g is not a generic computation of A.

In fact, *A* can be built to be quasi-minimal.

Minimal degrees and pairs

Now, we turn our attention to two open questions.

Question

Are there any minimal generic degrees?

Question

Are there minimal pairs in the generic degrees?

In this consideration, we ask one additional question:

Question

Given a nonzero generic degree **a***, is there always a density-1 degree* **b** *such that is* $\mathbf{a} \geq_g \mathbf{b}$*?*

If the answer to the third question is "yes," then there cannot be any minimal generic degrees, because the density-1 degrees are dense.

On the other hand, if the answer to the question is "no," then the counterexample is half of a minimal pair for generic reduction.

From these two observations, we get a free result:

Corollary

If there are minimal generic degrees, then there are minimal pairs of generic degrees.

End

Thank you

References

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