True Stages and Descriptive Set Theory

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We work in Baire space $\omega^\omega$ (or Cantor space $2^\omega$).
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This is a (Polish) topological space with basic clopen sets

$$[\sigma] = \{ \tau \in \omega^{<\omega} : \tau \geq \sigma \}.$$

Closed sets correspond to paths through trees.
The Borel Hierarchy

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- $\Pi_1^0$: Closed sets.
- $\Sigma_0^0$: Countable unions of $\Pi_0^\beta$ sets for $\beta < \alpha$.
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- $\Sigma^0_1$: Open sets.
- $\Pi^0_1$: Closed sets.
- $\Sigma^0_\alpha$: Countable unions of $\Pi^0_\beta$ sets for $\beta < \alpha$.
- $\Pi^0_\alpha$: Countable intersections of $\Sigma^0_\beta$ sets for $\beta < \alpha$. 
The Difference Hierarchy

We will need two more types of sets as well:

- A set is $\Delta^0_\alpha$ if it is both $\Sigma^0_\alpha$ and $\Pi^0_\alpha$.
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- A set is $\Delta^0_\alpha$ if it is both $\Sigma^0_\alpha$ and $\Pi^0_\alpha$.
- A set is $D_\eta(\Sigma^0_\alpha)$ if it is a difference of $\eta$-many $\Sigma^0_\alpha$ sets. E.g., if $\eta$ even, of the form

$$\bigcup_{\gamma < \eta \text{ odd}} \left( U_\gamma - \bigcup_{\gamma' < \gamma} U_{\gamma'} \right)$$

where each $U_\gamma$ is $\Sigma^0_\alpha$. 
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- A set is $\Delta^0_\alpha$ if it is both $\Sigma^0_\alpha$ and $\Pi^0_\alpha$.

- A set is $D_\eta(\Sigma^0_\alpha)$ if it is a difference of $\eta$-many $\Sigma^0_\alpha$ sets. E.g., if $\eta$ even, of the form

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where each $U_\gamma$ is $\Sigma^0_\alpha$.

For example, a $D_2(\Sigma^0_\alpha)$ set is of the form

$$U_1 - U_0$$

and a $D_3(\Sigma^0_\alpha)$ set is of the form

$$U_2 - (U_1 - U_0).$$
The Hausdorff-Kuratowski Theorem

**Theorem (Hausdorff-Kuratowski)**

\[ \Delta_2^0 = \bigcup_{\eta} D_{\eta}(\Sigma_1^0). \]

**Proof.**

See blackboard.
The Hausdorff-Kuratowski Theorem

Theorem (Hausdorff-Kuratowski)

$$\Delta^0_{\alpha+1} = \bigcup_{\eta} D_\eta(\Sigma^0_\alpha).$$

If you look in Kechris, the proof is essentially:

Proof.

Let $A$ be $\Delta^0_{\alpha+1}$. 
The Hausdorff-Kuratowski Theorem

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\[ \Delta^0_{\alpha+1} = \bigcup_{\eta} D_\eta(\Sigma^0_\alpha). \]

If you look in Kechris, the proof is essentially:

Proof.

Let \( A \) be \( \Delta^0_{\alpha+1} \).

Change the topology so that \( A \) is \( \Delta^0_2 \).
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**Theorem (Hausdorff-Kuratowski)**

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Let \( A \) be \( \Delta_{\alpha+1}^0 \).

Change the topology so that \( A \) is \( \Delta_2^0 \).

By the \( \alpha = 1 \) case, \( A \) is \( D_\eta(\Sigma_1^0) \) in the new topology.
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By the \( \alpha = 1 \) case, \( A \) is \( D_{\eta}(\Sigma^0_{1}) \) in the new topology.

Each \( \Sigma^0_{1} \) sets in the new topology is \( \Sigma^0_{\alpha} \) in the old topology.
Change-of-Topology

Change-of-topology is a useful tool in descriptive set theory.

**Theorem**

Let \((X, \mathcal{T})\) be a Polish space with topology \(\mathcal{T}\).
Let \(B_1, B_2, \ldots\) be any countable collection of Borel sets in \((X, \mathcal{T})\).
There is a finer Polish topology \(\mathcal{T}' \supseteq \mathcal{T}\) such that \(B_1, B_2, \ldots\) are open.
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There is a finer Polish topology \(\mathcal{T}' \supseteq \mathcal{T}\) such that \(B_1, B_2, \ldots\) are open.

Often you can also say something about the open sets in the new topology. Before, we needed that the open sets in the new topology are \(\Sigma^0_\alpha\) in the old topology.
What is this talk about?

This talk will be about a way of understanding change-of-topology in descriptive set theory using iterated true stages from computability theory.
The Effective Borel Hierarchy

The effective Borel hierarchy allows only computable unions and intersections. For $\alpha$ a computable ordinal:

- $\Sigma^0_1$: Effectively open sets, i.e., sets of the form $\bigcup_{\sigma \in W} [\sigma]$ for $W$ c.e.
- $\Pi^0_1$: Effectively closed sets, i.e., paths through a computable tree.
- $\Sigma^0_\alpha$: Unions of c.e. collections of (names for) $\Pi^0_\beta$ sets for $\beta < \alpha$.
- $\Pi^0_\alpha$: Intersections of c.e. collections of (names for) $\Sigma^0_\beta$ sets for $\beta < \alpha$.

We can also define $\Delta^0_\alpha$, $D_\eta(\Sigma^0_\alpha)$, etc.

These hierarchies also relativize to an oracle.
Effective Descriptive Set Theory

Any $\Sigma^0_\alpha$ set is $\Sigma^0_\alpha(X)$ (relative to $X$) for some set $X$. Thus it can be useful to apply effective methods even if we are not initially interested in computability.

Theorem (Hausdorff-Kuratowski, Selivanov)

$$\Delta^0_2 = \bigcup_{\eta<\omega^CK_1} D_\eta(\Sigma^0_1).$$
The key connection is that there is a way of thinking about $\Sigma^0_{\alpha+1}$ sets using the $\alpha$th jump.

**Fact**

A set $A \subseteq \omega^\omega$ is $\Sigma^0_{\alpha+1}$ if and only if there is a $\Sigma^0_1$ set $V \subseteq \omega^\omega$ such that $A = \{x : x^{(\alpha)} \in V\}$.

We will use true stage constructions to approximate the jumps.
Iterated True Stage Constructions

The idea is to think of $\emptyset^{(\alpha)}$ as an iteration of the limit lemma. Each jump is a simple step, and we just need a good way to organize how they fit together.
Iterated True Stage Constructions

The idea is to think of $\varnothing^{(\alpha)}$ as an iteration of the limit lemma. Each jump is a simple step, and we just need a good way to organize how they fit together.

Many computability-theoretic frameworks have been introduced to help organize this:

- Harrington worker arguments
- Lempp and Lerman's tree of strategies
- Ash and Knight's $\alpha$-systems
- Montalbán's $\eta$-systems
- Greenberg and Turetsky's variation on the $\eta$-systems
Approximating the First Jump

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We can computably approximate $K$ by

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where $K_s$ is the finite set containing $e < s$ if the $e$th program has halted at stage $s$. 
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where \( K_s \) is the finite set containing \( e < s \) if the \( e \)th program has halted at stage \( s \).

We could think of approximating the infinite binary string \( K \) by the finite binary strings \( K_s \upharpoonright s \). But it might be that every \( K_s \upharpoonright s \) makes some incorrect guess.
Approximating the First Jump

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Say that $s$ is a Dekker non-deficiency stage if for all $t > s$, $n_t > n_s$. There are infinitely many non-deficiency stages.
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Suppose that at stage $s$, we guess that $K_s \restriction n_s$ is an initial segment of $K$. At non-deficiency stages, our guess is correct.
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Suppose that at stage $s$, we guess that $K_s \upharpoonright n_s$ is an initial segment of $K$. At non-deficiency stages, our guess is correct.

A stage $s$ is $1$-true if $K_s \upharpoonright n_s < K$. 

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- There are infinitely many 1-true stages.
- If $s$ is a 1-true stage, then it appears 1-true at every stage $t > s$. 
A stage $s$ is **1-true** if $K_s \uparrow n_s < K$.

- There are infinitely many 1-true stages.
- If $s$ is a 1-true stage, then it appears 1-true at every stage $t > s$.
- If $s$ is not 1-true, then there might be stages $t > s$ which do not have enough information to see this, i.e.,

$$K_s \uparrow n_s < K_t \uparrow n_t.$$  

We say that $s$ **appears 1-true** at stage $t$. Such $t$ are also not 1-true.
Approximating More Jumps

Montalbán: Iterate this through the hyperarithmetic hierarchy:

- Having approximated \( \emptyset^{(\alpha)} \) at stage \( s \) by a finite string \( \nabla_s^\alpha \), use this finite string as an oracle to approximate \( \emptyset^{(\alpha+1)} \) by a finite string \( \nabla_{s+1}^\alpha \).
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- At limits, take joins.

Disclaimer: This is all morally correct, but needs some adjustment for technical reasons.
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- At limits, take joins.
- Use non-deficiency stages to ensure that there are infinitely many $\alpha$-true stages $s$ with $\nabla_s^\beta < \varnothing^{(\beta)}$ for $\beta \leq \alpha$.
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▸ At limits, take joins.

▸ Use non-deficiency stages to ensure that there are infinitely many $\alpha$-true stages $s$ with $\nabla^\beta_s \prec \emptyset^{(\beta)}$ for $\beta \leq \alpha$.

▸ Say that $s$ appears $\alpha$-true at stage $t$, and write $s \leq_\alpha t$, if $\nabla^\beta_s \leq \nabla^\beta_t$ for $\beta \leq \alpha$. 
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▸ Use non-deficiency stages to ensure that there are infinitely many $\alpha$-true stages $s$ with $\nabla_s^{\beta} < \emptyset^{(\beta)}$ for $\beta \leq \alpha$.

▸ Say that $s$ appears $\alpha$-true at stage $t$, and write $s \leq_\alpha t$, if $\nabla_s^{\beta} \leq \nabla_t^{\beta}$ for $\beta \leq \alpha$.

▸ The $\nabla_s^{\alpha}$ and the relations $\leq_\alpha$ are all computable.
Montalbán: Iterate this through the hyperarithmetic hierarchy:

- Having approximated $\emptyset^{(\alpha)}$ at stage $s$ by a finite string $\nabla^\alpha_s$, use this finite string as an oracle to approximate $\emptyset^{(\alpha+1)}$ by a finite string $\nabla^\alpha_{s+1}$.
- At limits, take joins.
- Use non-deficiency stages to ensure that there are infinitely many $\alpha$-true stages $s$ with $\nabla^\beta_s < \emptyset^{(\beta)}$ for $\beta \leq \alpha$.
- Say that $s$ appears $\alpha$-true at stage $t$, and write $s \leq_{\alpha} t$, if $\nabla^\beta_s \leq \nabla^\beta_t$ for $\beta \leq \alpha$.
- The $\nabla^\alpha_s$ and the relations $\leq_{\alpha}$ are all computable.

Disclaimer: This is all morally correct, but needs some adjustment for technical reasons.
References

For the technical details, see:

- Ash and Knight’s book *Computable Structures and the Hyperarithmetical Hierarchy*
- Montalban, $\eta$-systems, in *Priority Arguments via True Stages* and *Computable Structure Theory: Beyond the arithmetic*
- Day, Greenberg, HT, Turetsky, *An effective classification of Borel Wadge classes* and *Iterated priority arguments in descriptive set theory*
Relativizing True Stages

In the true stage constructions before, we approximated $\emptyset, \emptyset', \emptyset'', \ldots$. We can also relativise this to any $x$, approximating $x, x', x'', \ldots$. 

Note that being true is now relative to the extension $x$. 
Relativizing True Stages

In the true stage constructions before, we approximated $\emptyset, \emptyset', \emptyset'', \ldots$. We can also relativise this to any $x$, approximating $x, x', x'', \ldots$. In fact, given $x \in \omega^\omega$, we can make it so that the approximation to $x^{(\alpha)}$ at stage $s$ only depends on $x \upharpoonright s$:

- For each finite string $\sigma$ and computable ordinal $\alpha$, define $\sigma^{(\alpha)}$, the approximation to $x^{(\alpha)}$ for $x$ extending $\sigma$ at stage $|\sigma|$. 

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- For each finite string $\sigma$ and computable ordinal $\alpha$, define $\sigma^{(\alpha)}$, the approximation to $x^{(\alpha)}$ for $x$ extending $\sigma$ at stage $|\sigma|$.
- Define $\sigma \leq_{\alpha} \tau$ if $\sigma^{(\beta)} \leq \tau^{(\beta)}$ for $\beta \leq \alpha$. We say $\sigma$ appears $\alpha$-true at $\tau$. 

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In fact, given $x \in \omega^\omega$, we can make it so that the approximation to $x^{(\alpha)}$ at stage $s$ only depends on $x \upharpoonright s$:

- For each finite string $\sigma$ and computable ordinal $\alpha$, define $\sigma^{(\alpha)}$, the approximation to $x^{(\alpha)}$ for $x$ extending $\sigma$ at stage $|\sigma|$.

- Define $\sigma \leq_\alpha \tau$ if $\sigma^{(\beta)} \leq \tau^{(\beta)}$ for $\beta \leq \alpha$. We say $\sigma$ appears $\alpha$-true at $\tau$.

- Say that $\sigma$ is $\alpha$-true for $x \in 2^\omega$, and write $\sigma \leq_\alpha x$, if $\sigma^{(\beta)} \leq x^{(\beta)}$ for $\beta \leq \alpha$.

Note that being true is now relative to the extension $x$. 
The Structure of the Approximations

These orderings $\leq_\alpha$ on $\omega^< \cup \omega^\omega$ have lots of nice properties:

- The relations $\leq_\alpha$, when restricted to finite strings $\omega^<$, are computable.
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The Structure of the Approximations

These orderings \( \leq_\alpha \) on \( \omega^\omega \cup \omega \) have lots of nice properties:

- The relations \( \leq_\alpha \), when restricted to finite strings \( \omega^{<\omega} \), are computable.
- \( \sigma \leq_0 \tau \iff \sigma \leq \tau \).
- \( \sigma \leq_\alpha \tau \Rightarrow \sigma \leq_\beta \tau \) for \( \beta < \alpha \).
The Structure of the Approximations

These orderings $\leq_\alpha$ on $\omega^\omega \cup \omega^\omega$ have lots of nice properties:

- The relations $\leq_\alpha$, when restricted to finite strings $\omega^\omega$, are computable.
- $\sigma \leq_0 \tau \iff \sigma \leq \tau$.
- $\sigma \leq_\alpha \tau \implies \sigma \leq_\beta \tau$ for $\beta < \alpha$.
- for each $x \in \omega^\omega$, there infinitely many strings which are $\alpha$-true for $x$:
  \[ \sigma_0 \leq_\alpha \sigma_1 \leq_\alpha \sigma_2 \leq_\alpha \cdots \leq_\alpha x. \]
- $\left(\omega^\omega, \leq_\alpha\right)$ is a tree/forest.
True Stages and Topology

Some additional properties of our true stages:

▸ \([\sigma]_\alpha = \{\bar{x} : \sigma \leq_\alpha \bar{x}\} \text{ is } \Sigma^0_\alpha.\)
True Stages and Topology

Some additional properties of our true stages:

- $[\sigma]_\alpha = \{\bar{x} : \sigma \leq_\alpha \bar{x}\}$ is $\Sigma^0_\alpha$.
- Each $\Sigma^0_\alpha$ set is of the form

$$\bigcup_{\sigma \in W} [\sigma]_\alpha = \bigcup_{\sigma \in W} \{\bar{x} : \sigma \leq_\alpha \bar{x}\}$$

for some c.e. set $W$. 
True Stages and Topology

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\bigcup_{\sigma \in W} [\sigma]_{\alpha} = \bigcup_{\sigma \in W} \{ \bar{x} : \sigma \leq \alpha \bar{x} \}
$$

for some c.e. set $W$.

Taking $[\sigma]_{\alpha} = \{ \bar{x} : \sigma \leq \alpha \bar{x} \}$ as a basis, we get a Polish topology $T'$ extending the standard topology where the open sets are exactly those generated by the $\Sigma^0_{\alpha}$ sets.
Hausdorff-Kuratowski

This way of constructing the change of topology is particularly nice because it looks like the standard topology on $\omega^\omega$ in the sense that it comes from a tree.

Theorem (Hausdorff-Kuratowski, Selivanov)

For all computable $\alpha$, $\Delta_0^{\alpha+1} = \bigcup_{\eta < \omega} \ck_{\eta}(\Sigma_0^\alpha)$.

Proof. See blackboard.
Hausdorff-Kuratowski

This way of constructing the change of topology is particularly nice because it looks like the standard topology on $\omega^\omega$ in the sense that it comes from a tree.

We can adjust our proof of the Hausdorff-Kuratowski theorem to get a proof for $\Delta^0_{\alpha+1}$ by replacing the standard tree $(\omega^\omega, \preceq)$ by the tree $(\omega^\omega, \leq_\alpha)$.

**Theorem (Hausdorff-Kuratowski, Selivanov)**

For all computable $\alpha$,

$$\Delta^0_{\alpha+1} = \bigcup_{\eta<\omega_1^{ck}} D_\eta(\Sigma^0_\alpha).$$

**Proof.**

See blackboard.
This sounds great, but can we do anything new?
Definition (Wadge)

Let $A$ and $B$ be subsets of Baire space $\omega^\omega$.

We say that $A$ is \textit{Wadge reducible} to $B$, and write $A \leq_W B$, if there is a continuous function $f$ on $\omega^\omega$ with $A = f^{-1}[B]$, i.e.

$$x \in A \iff f(x) \in B.$$
Structure of Wadge Degrees

Theorem (Martin and Monk, AD)

The Wadge order is well-founded.

Theorem (Wadge’s Lemma, AD)

Given $A, B \subseteq \omega^\omega$, either $A \leq_W B$ or $B \leq_W \omega^\omega - A$.

These theorems are proved by playing a game. For Borel sets, we have Borel Determinacy without having to assume AD, and so these are always true for Borel sets.
Wadge Degrees in Second-order Arithmetic

Borel determinacy requires iterations of power-set.

**Theorem (Friedman)**

*Borel determinacy requires $\omega_1$ iterations of the Power Set Axiom.*

Martin showed that $\Sigma^0_4$ Determinacy is not provable in second-order arithmetic.

One the other hand, one can prove that Borel Wadge games are determined in second-order arithmetic.

**Theorem (Louveau and Saint-Raymond)**

*Borel Wadge determinacy is provable in second-order arithmetic.*
Description of Wadge Degrees

There are also many comprehensive descriptions of the Borel Wadge classes:

▸ Louveau (1983)
▸ Duparc (2001)
▸ Selivanov, for $k$-partitions (2007, 2017)
▸ Kihara and Montalbán, for functions into a countable BQO (2019)

We use our true stage machinery to give a new description of the Borel Wadge classes, and use them to prove Borel Wadge determinacy in a reasonable fragment of second-order arithmetic.
Theorem (Day, Greenberg, HT, Turetsky)

Borel Wadge determinacy is provable in $\text{ATR}_0 + \Pi^1_1 - \text{Ind}$, and there is a complete description of the Borel Wadge classes.

Thus the Borel Wadge degrees are semilinearly ordered and well-founded.

This simplifies Louveau and Saint-Raymond’s proof in second-order arithmetic and uses a weaker subsystem. Our descriptions of the classes are inherently dynamic, and naturally lightface.
▸ Make a list of described classes. These are non-self-dual. Our descriptions are dynamic in nature.

▸ Prove a Louveau-Saint Raymond separation result for each described class $\Gamma$, which implies that if $A$ is universal for $\Gamma$, and $B$ is Borel, then either $A \leq W B$ or $B \in \tilde{\Gamma}$, in which case $B \leq W A^c$.

▸ Prove that the intersection of a described class and its dual is either a union of described classes of lower Wadge degree, like

$$\Delta^0_{\xi+1} = \bigcup_{\eta} D_\eta(\Sigma^0_\xi),$$

or is a Wadge class in its own right like $\Delta^0_1$.

▸ Given a Borel set, take the least described class (or dual of a described class, or $\Delta(\Gamma)$) containing it. Prove that it is complete for that class.
Theorem (Louveau, Saint Raymond)

Suppose that $\Gamma$ is a described class. Let $A \in \Gamma$. Let $B_0$ and $B_1$ be two disjoint $\Sigma^1_1$ sets. Then either:

- There is a continuous reduction of $(A, A^c)$ into $(B_0, B_1)$, or
- There is a $\tilde{\Gamma}$ separator of $B_0$ from $B_1$.

If $A$ is universal for $\Gamma$, and $B$ is Borel, then either $A \leq_W B$ or $B \in \tilde{\Gamma}$, in which case $B \leq_W A^c$.

The direct way to prove this would be to use Borel determinacy for a naturally associated game.

Louveau and Saint Raymond show by an unravelling process that there is an associated closed game.

Using true stages, we get a relatively simple description of such a game.
Theorem (Louveau, Saint Raymond)

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- There is a continuous reduction of $(A, A^c)$ into $(B_0, B_1)$, or
- There is a $\check{\Gamma}$ separator of $B_0$ from $B_1$.

Take $\Gamma = \Sigma^0_\xi$. Let $T_i$ be a tree whose projection is $B_i$.

- Player 1 plays $x$ in $A$ or $A^c$.
- Player 2 attempts to play $y$ in $B_0$ (if $x \in A$) or $B_1$ (if $x \notin A$), with a corresponding witness $f$ in $[T_0]$ or $[T_1]$.
- Player 2 guesses, using the true stage machinery, at whether $x$ is in $A$ or not. At each stage, they play an attempt at extending $y$ and $f$. But they are only committed to which $f$ they play at true stages.
Theorem (Day, Greenberg, HT, Turetsky)

*Borel Wadge determinacy is provable in* \( \text{ATR}_0 + \Pi^1_1 - \text{Ind} \), *and there is a complete description of the Borel Wadge classes.*

*Thus the Borel Wadge degrees are semilinearly ordered and well-founded.*

This simplifies Louveau and Saint-Raymond’s proof in second-order arithmetic and uses a weaker subsystem. Our descriptions of the classes are inherently dynamic, and naturally lightface.
References

Day, Greenberg, Harrison-Trainor, Turetsky:

- *Iterated priority arguments in descriptive set theory*
- *An effective classification of Borel Wadge classes*
Thanks!