A non-trivial 3-REA Set Not Computing a Weak 3-generic

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Midwest Computability Seminar, 2023
1 Notation & Definitions

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   - Weak 1-genericity
   - R.E. Sets and 1-genericity
   - 2-genericity
   - 3-genericity

3 3-REA Sets
   - Differences From $\Delta^0_3$ Escaping Functions
   - Main Result
   - Naive Strategies
   - Complications
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Notation

- $\sigma, \tau, \nu, \delta$ range over $\{0, 1, \uparrow\}^{<\omega}$ (partial binary valued functions with finite domain).
- We write $\sigma < \tau$ if $\tau$ extends $\sigma$ and $\sigma < X$ if $\sigma$ is extended by the characteristic function of $X$.
- $\theta$ meets $\Gamma \subset \{0, 1, \uparrow\}^{<\omega}$ ($\theta \models \Gamma$) if $(\exists \sigma \in \Gamma)(\theta > \sigma)$ and $\theta$ strongly avoids $\Gamma$ ($\theta \models \neg \Gamma$) if some $(\exists \tau < \theta)(\forall \sigma \in \Gamma)(\tau \not\in \gamma)$.
- $f \in \omega^\omega$ dominates $g \in \omega^\omega$ ($f \gg g$) if $(\forall^* x \in \omega)(f(x) \geq g(x))$.
- $f$ is $\Delta^0_{n+1}$ escaping if $f$ isn't dominated by any $g \leq_T 0^{(n)}$.
The $i$-th hop is $\mathcal{H}_i(A) \overset{\text{def}}{=} A \oplus W_i^A$.

- REA sets are the result of iterating the Hop operation on $\emptyset$.
- The 1-REA sets are just the r.e. sets.
- The 2-REA sets are sets of the form $W_i \oplus W_j^W_i$

Components as Columns

- For this talk we only care about $n$-REA sets up to Turing degree.
- Useful to identify the components of $n$-REA sets with their columns.

\[
\begin{array}{c}
\vdots \\
W_{i_0} & W_{i_1} & W_{i_2} & \cdots
\end{array}
\]
In this talk we only consider the (standard) forcing relation on $2^{<\omega}$

- $G$ is $n$-generic ($n > 0$) if $G \vDash \phi$ or $G \vDash \neg \phi$ for all $\Sigma^0_n$ sentences.
- Equivalently, $G$ is $n$-generic if $G$ meets or strongly avoids every $\Sigma^0_n$ subset of $2^{<\omega}$ (equivalently $\{0, 1, \uparrow\}^{<\omega}$)
- $\Gamma \subset 2^{<\omega}$ is dense if $(\forall \tau \in 2^{<\omega})(\exists \sigma \in \Gamma) \tau < \sigma$
- $G$ is weakly $n$-generic if $G$ meets every dense $\Sigma^0_n$ subset of $2^{<\omega}$
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Computing Weak 1-Generics

Theorem

If $f \in \omega^\omega$ is $\Delta^0_1$ escaping then $f$ computes a weak 1-generic

- WLOG $f$ is monotonically increasing and let $U_i$ be $i$-th r.e. subset of $2^{<\omega}$.
- Build $G = \lim_{n \to \infty} \tau_n$, $\tau_0 = \langle \rangle$, $\tau_{n+1} > \tau_n$.
- Let $\tau_{n+1} > \tau_n$ be in $U_{i,f(n+1)}$ for least $i \leq n$ or $\tau_n$ if no such $i$ exists.
Suppose $U_i$ is dense but $G$ doesn’t meet $U_i$.

Let $n > 0$ large enough that $\tau_n$ meets every $U_j, j < i G$ will ever meet.

Suppose we can compute a bound $l_m > |\tau_m|$ for $m > n$.

Let $h(m)$ be the least stage $s$ such that $U_{i,h(m)}$ includes an extension of every string of length $l_m$.

If $f(m) \geq h(m), m > n$ then $\tau_m$ meets $U_i$.

We compute $l_m$ by assuming $f(x) < h(x)$ for $n < x < m$.

Can’t extend to 1-generics because we can’t guarantee number of stages needed to find an extension in a non-dense $U_i$ is computably bounded.
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R.E. Sets Compute 1-generics

**Theorem**

If $A \not\leq_T 0$ is r.e. then $A$ computes a 1-generic

- The modulus for $A$ ($m(n) \equiv \mu t \left( A_t \upharpoonright_{n+1} = A \upharpoonright_{n+1} \right)$) is $\Delta^0_1$ escaping.
- But we can compute full 1-generic by using the computable approximation to $A$.
- Same construction as before but we use stagewise approximations and allow restraint.
- Now, if we extend $\tau_{n,s}$ to $\tau_{n+1,s}$ to meet $U_i$ then we preserve $\tau_{n+1,s}$ from changes trying to meet $U_j, j > i$
Constructing 1-generic Below R.E.

\[ m_s(n) \equiv \mu t \left( A_t \upharpoonright n+1 = A_s \upharpoonright n+1 \right) \]
\[ r_s(i) \equiv \max \left\{ n \mid n \leq s \land \left( \exists \sigma > \tau_{n,s} \right) \left( \tau_{n,s} \not \models U_{i,s-1} \land \sigma \models U_{i,s-1} \right) \right\} \]
\[ \bar{r}_s(i) \equiv \max_{j < i} r_s(i) \]
\[ i^*_{n+1,s} \equiv \min_{i \leq n} \neg (\tau_{n,s} \models U_{i,m_s(n)}) \land \left( \exists \sigma > \tau_{n,s} \right) \left( \sigma \models U_{i,m_s(n+1)} \right) \]
\[ \tau_{n,s} \begin{cases} \langle \rangle & \text{if } s \leq n \lor s = 0 \lor n = 0 \\ \tau_{n,s-1} & \text{unless } m_s(n+1) > m_{s-1}(n) \\ \tau_{n,s-1} & \text{if } \bar{r}_s(i^*_{n,s}) \geq n \\ \sigma & \text{o.w. where } \sigma \text{ is least witness for } i^*_{n,s} \end{cases} \]
\[ G \equiv \lim_{n \to \infty} \lim_{s \to \infty} \tau_{n,s} \]

Note that \( \tau_{n,\infty} = \tau_{n,m(n)} \) so \( G \leq_T A \).
Suppose $i$ is least s.t. $G \vdash U_i \land G \nvdash \neg U_i$. We show that $A$ is computable.

Let $n$ large enough that $n > \bar{r}_\infty(i)$ (exists by fact $i$ least) and for all $j < i \tau_n \vdash U_j \lor \tau_n \vdash \neg U_j$ and $t$ large enough that $\tau_{n,t} = \tau_n$.

If there are $n' \geq n, s \geq \max(t, n'), \sigma > \tau_{n',s}, \sigma \vdash U_{i,s}$ then $m(n') < s$.

Otherwise we'd preserve $\tau_{n',s}$ and have $\tau_{n',m(n')} \vdash U_i$.

But, by assumption, there must be infinitely many such $m, s$ showing $m \leq_T 0$

Contradiction.
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Computing Weak 1-Generics

**Theorem (Andrews, Gerdes and Miller)**

If \( f \in \omega^\omega \) is \( \Delta^0_2 \) escaping then \( f \) computes a weak \( 2 \)-generic

- Proved in [1]. Won’t prove it here.
- Idea is to try and extend to meet \( \Sigma^0_2 \) sets \( \mathcal{U}_i \) by favoring those \( \sigma \) for which \((\exists x)(\forall y)\phi(\sigma, x, y)\) appears true with least \( \max(|\sigma|, x) \).

**Hypothesis**

If \( A \nless T 0' \) is 2-REA then \( A \) computes a 2-generic
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Theorem (Andrews, Gerdes and Miller)

There is a (pruned) perfect \( \omega \)-branching tree \( T \subset \omega^{<\omega}, T \leq_T 0'' \) such that if \( f \in [T] \) then \( f \) doesn't compute a weak 3-generic.

vertex Node with multiple successors (\( \sigma \langle i \rangle, \sigma \langle j \rangle \in T, i \neq j \)).

\( \omega \)-branching Every vertex has infinitely many immediate successors.

pruned No terminal nodes (all nodes extend to paths)

perfect Every node is extended by a vertex.
Theorem (Andrews, Gerdes and Miller)

There is a (pruned) perfect $\omega$-branching tree $T \subset \omega^{<\omega}, T \leq_T 0''$ such that if $f \in [T]$ then $f$ doesn’t compute a weak 3-generic.

- No amount of (countable) non-domination suffices to compute a weak 3-generic, e.g., $g_j \gg f$, $j \in \omega$.
- View $T$ as function on $\omega^\omega$ by defining $T[h]$ to be the path taking the $h(n)$-th option at the $n$-th vertex.
- Let $f = T[h]$ with $h(k)$ picked large enough that $T[h](n_k) > g_j(n_k)$, $j \leq k$ where $T[h]|_{n_k}$ is the $k$-th vertex along $T[h]$.
- Note that if $f$ is monotonic and $\Delta^0_{n+3}, n \geq 0$ escaping then $T[f] \leq_T f \oplus 0''$ is as well.
- If $g \gg T[f]$ then $g^*(k) = g(n_k)$ satisfies $g^* \gg f$, $g^* \leq_T g \oplus 0''$. 
Intuition Behind Failure

Question

What prevents the pattern from continuing indefinitely?

- Pattern worked because more non-domination strength gave us more computational power (guessing at membership in $\Sigma^0_1$ sets then $\Sigma^0_2$ sets).
- But, a computable reduction can’t hope to always distinguish $0^{(n)}$ big and $0^{(n+k)}$ big.
- Given finitely many potential values of $\Phi_e(\sigma \langle n \rangle)$, $0''$ can figure out which value is compatible with infinitely many $n$.
- Allows us to limit $\Phi_e(f)$ to a narrow range of options (while allowing $f$ to take arbitrarily large values).
- Can build $U_e \subset 2^{<\omega}$ a dense $\Sigma^0_3$ set $\Phi_e(f)$ can’t meet by enumerating strings outside that narrow range.
Utility Lemma

Lemma

Suppose for infinitely many $l \in \omega$, $0''$ can enumerate $k > 0$, $\eta_i \in 2^{<\omega}$, $i < 2^k - 1$, $|\eta_i| \geq l + k$. If $f \in [T] \land \Phi_e(f) \downarrow \implies \Phi_e(f) > \eta_i$ then $\Phi_e(f)$ isn’t weakly 3-generic for any $f \in [T]$.

Proof.

For each $\sigma$ with $|\sigma| = l$ there are $2^k$ strings $\tau > \sigma$ of length $l + k$. At least one of those strings $\tau_\sigma$ must be incompatible with $\eta_i$, $i < 2^k - 1$.

For each such $l > 0$ and $\sigma$ with $|\sigma| = l$ enumerate $\tau_\sigma$ into $U_e$. $U_e$ is a dense $\Sigma^0_3$ set that isn’t met by $\Phi_e(f)$ for any $f \in [T]$.  \[\square\]
Building $T$

**Conditions**

- A finite set $V_s$ of vertexes ($\sigma$)
- For each $\sigma \in V_s$ an infinite r.e. set of strings
  \[ \Sigma_s(\sigma) \subset \{ \sigma \hat{\langle n \rangle} \hat{\tau} \mid n \in \omega, \tau \in 2^{<\omega} \} \]
- $\theta^e_s : 2^{<\omega} \mapsto 2^{<\omega} \cup \{ \uparrow \}, e \in \omega$ such that if $\sigma \in V_s, \tau \in \Sigma_s(\sigma)$ then $\Phi_e(\tau) > \theta^e_s(\sigma)$ (where that means $\Phi_e(f) \uparrow$ if $f \succ \tau$ if $\theta^e_s(\sigma) = \uparrow$)

$V_s$: Nodes we commit to making $\omega$-branching vertexes in $T$.

$\Sigma_s(\sigma)$: Possible (i.e. not in $V_s$) branches extending $\sigma$.

$\theta^e_s(\sigma)$: Specifies initial segment of $\Phi_e(\tau)$ agreed on by all $\tau \in \Sigma_s(\sigma)$ (or that all such $\tau$ force partiality)
Building $T$

**Conditions**

- A finite set $V_s$ of vertexes ($\emptyset$)
- For each $\sigma \in V_s$ an infinite r.e. set of strings
  
  $$\Sigma_s(\sigma) \subset \{ \sigma \hat{\langle n \rangle} \hat{\tau} \mid n \in \omega, \tau \in 2^{<\omega} \}$$

- $\theta^e_s : 2^{<\omega} \mapsto 2^{<\omega} \cup \{ \uparrow \}, e \in \omega$ such that if $\sigma \in V_s, \tau \in \Sigma_s(\sigma)$ then
  
  $$\Phi^e(\tau) > \theta^e_s(\sigma) \text{ (where that means } \Phi^e(f) \uparrow \text{ if } f > \tau \text{ if } \theta^e_s(\sigma) = \uparrow \text{)}$$

- $V_0 = \{ \langle \rangle \}$ if $s = 0 \lor \sigma \notin V_s \lor e \geq s$ then $\Sigma_s(\sigma) = \{ \sigma \hat{\langle n \rangle} \}$ and $\theta^e_s(\sigma) = \langle \rangle$.

- $V_{s+1} = V_s \cup \{ \tau_\sigma \mid \sigma \in V_s \}$ where $\tau_\sigma \in \Sigma_s(\sigma)$ with $\tau_\sigma(|\sigma|)$ large. (Hence $|V_s| = 2^s$).

- $\Sigma_{s+1}(\sigma) \subset \Sigma_s(\sigma)$ and $\theta^e_{s+1}(\sigma) > \theta^e_s(\sigma)$ (where $\uparrow$ is considered $>$ maximal).

- We ensure that if $e < s, \sigma \in V_s$ then $|\theta^e_s(\sigma)| > 2s + 1$
Every $\sigma_i \in \Sigma_0(\langle \rangle)$ has $\Phi_e(\sigma_i) > \theta^e_0(\langle \rangle)$.
Add new vertex in $\Sigma_s(\tau)$ for each $\tau \in V_s$. 
Prune and extend (e.g. replace $\sigma_i$ with an extension) so

$$\sigma_i \in \Sigma_1(\langle \rangle) \implies \Phi_e(\sigma_i) > \theta_1^e(\langle \rangle)$$ (now longer) and $$\Phi_e(\sigma_{0i}) > \theta_1^e(\sigma_0)$$
If $f \in [T]$ then $\Phi_e(f) > \theta_1^e(\langle \rangle)$ or $\Phi_e(f) > \theta_1^e(\sigma_0)$
Visualizing $T$ Construction

- Extend each vertex with a node from allowed branches.
If $f \in [T]$ then $\Phi_e(f) > \theta^e_2(\langle \rangle)$ or $\Phi_e(f) > \theta^e_2(\sigma_0)$ or $\Phi_e(f) > \theta^e_2(\sigma_2)$ or $\Phi_e(f) > \theta^e_2(\sigma_{00})$.
Verifying Construction

- To complete proof we must only show that we can always construct $\Sigma_{s+1}(\tau)$ from $\Sigma_s(\tau)$ that makes $\theta^e_{s+1}(\tau)$ sufficiently long.
- But given the length $0''$ can ask if there are infinitely many elements $\sigma \in \Sigma_s(\tau)$ that can be extended to $\sigma'$ with $\Phi_e(\sigma')$ of sufficient length.
- If not remove the finitely many elements that allow convergence.
- If so $0''$ can determine which of the finitely many options for $\Sigma_{s+1}(\tau)$ permits $\Sigma_{s+1}(\tau)$ to be infinite.
- Repeat for each $e < s + 1$ and $\tau \in V_{s+1}$. 
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Genericity From 3-REA Sets

Question

If $A \not\leq_T 0''$ is 3-REA does $A$ compute a (weak) 3-generic?

- $A$ computes a $\Delta^0_3$ escaping function $m^{[3]}(x)$ (where $m^{[n+1]}(x)$ is modulus of $A^{[n+1]}$ over $A^{[n]}$) but that’s not enough.
- But several reasons to think that 3-REA sets have extra power to compute generics.
  - We get $m^{[3]}, m^{[2]}, m^{[1]}$ with $m^{[n]} \Delta^0_n, 1 \leq n \leq 3$ escaping. Modifications even ensure all three functions simultaneously escape a tuple $h^1 \leq_T 0, h^2 \leq_T 0', h^3 \leq_T 0''$
  - Our ability to effectively approximate $A$ offers additional power (remember non-trivial r.e. sets compute 1-generics not just weak 1-generics).
  - Approach used to build $T$ doesn’t directly translate.
Isolating Large Values

- When we built $T$ functionals $\Phi_e(f)$ had to meet $\mathcal{U}_e$ using only one large value.
  - If $\sigma \in V_s$, $e < s$, $x \in \omega$ we could wait until we found $\tau > \sigma^\langle n \rangle$ with $\Phi_e(\tau; x)$ converging before choosing the next large value.
- Given $A \not\leq_T 0''$, 3-REA, $k > 1$ and $h \leq_T 0''$ there are infinitely many tuples $x_0 < x_1, <, \ldots, < x_k < m^{[3]}(x_0)$ such that $m^{[3]}(x_i) > h(x_i), i \leq k$.
- So, infinitely often, $\Phi_e(A; x)$ can consult $k$ large values before trying to meet $\mathcal{U}_e$. 
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There is a 3-REA set $A \nleq_T 0''$ that doesn't compute a weak 3-generic.

- We know $A$ computes a weak 2-generic.
- By result in [1] every $\Delta^0_3$ escaping function computes a 2-generic.
- Thus, result is sharp.
Requirements

\( \mathcal{P}_i: \quad A^{[3]}(c^i) \neq \lim_{s \to \infty} \lim_{t \to \infty} p_i(c^i, s, t) \)

\( \mathcal{Q}_{e,\sigma}: \quad X_e \downarrow \implies [\exists \tau > \sigma](\tau \in \mathcal{U}_e \land \tau \not\in X_e) \)

\[
X_e \overset{\text{def}}{=} \Phi_e(A) \overset{\text{def}}{=} \Phi_e(A) \quad \mathcal{U}_e : \Sigma_1^0(0^{\prime\prime}) \text{ subset of } 2^{<\omega}
\]

\( \mathcal{P}_i \) Ensures that \( A \nsubseteq T \ 0^{\prime\prime} \)

\( \mathcal{Q}_{e,\sigma} \) Builds dense \( \mathcal{U}_e \) avoiding \( X_e \) (no other additions)

We’ll want to break these requirements up into \( \Pi_2^0 \) subrequirements (to use tree method and let \( 0^{\prime\prime} \) see outcome).
### Requirements

\( P_\alpha: \quad A^{[3]}(c^\alpha) \neq \lim_{s \to \infty} \lim_{t \to \infty} p_\alpha(c^\alpha, s, t) \)

\( Q_{\alpha, \sigma}: \quad X_\alpha \downarrow \implies [\exists \tau > \sigma](\tau \in U_\alpha \land \tau \not\in X_\alpha) \)

\[
X_\alpha \overset{\text{def}}{=} \Phi_\alpha(A) \overset{\text{def}}{=} \Phi_e_\alpha(A) \quad U_\alpha : \Sigma_1^0(0^{"}) \text{ subset of } 2^{<\omega}
\]

\( P_\alpha \) Ensures that \( A \not\equiv_T 0^{"} \)

\( Q_{\alpha, \sigma} \) Builds dense \( U_\alpha \) avoiding \( X_e \) (no other additions)

- We’ll want to break these requirements up into \( \Pi_2^0 \) subrequirements (to use tree method and let \( 0^{"} \) see outcome).
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Strategy for $\mathcal{P}_\alpha$

### Requirement

$$\mathcal{P}_\alpha: A^{[3]}(c^\alpha) \neq \lim_{s \to \infty} p'_\alpha(c^\alpha, s)$$

where

$$p'_\alpha(c^\alpha, s) \overset{\text{def}}{=} \lim_{t \to \infty} p_\alpha(c^\alpha, s, t)$$

### Sub-requirements

$$\mathcal{P}^k_\alpha:$$

$$b^\alpha_k \in A^{[2]} \iff |\{t \mid p'_\alpha(c^\alpha, t)\} = 1| > k$$

- Place $c^\alpha \in A^{[3]}$ iff $(\exists k)(b^\alpha_k \notin A^{[2]})$
- At stage $s$ place $b_k$ into $A^{[2]}$ if it’s not currently in and $|\{t \mid p_\alpha(c^\alpha, t, s)\} = 1| > k$.
- We remove $b_k$ at $s_1 > s$ (by enumerating into $A^{[1]}$) if $|\{t \mid (\forall s' \in [s, s_1])(p_\alpha(c^\alpha, t, s') = 1)\}| \leq k$
- $c^\alpha \notin A^{[3]}$ if $\lim_{s \to \infty} p'_\alpha(c^\alpha, s)$ is 1 or DNE
Let’s try same approach as constructing $T$, ensure that all ‘options’ for $A$ agree on ‘alot’ of $\Phi_e(A)$.

But $0''$ can’t determine if $c^\alpha \in A^{[3]}$. But we can accomodate both options by agreeing on sufficently long initial segments.

Harder problem is ensuring that $\Phi_e(A)$ takes the same value no matter what value we get for $\bar{k}^\alpha \overset{\text{def}}{=} \mu k \left( b^\alpha_k \notin A^{[3]} \right)$.

This is analog of allowing $f(x)$ to take on infinitely many values in construction of $T$.

- (Up to $0''$ equivalence) $\bar{k}^\alpha$ measures stage at $c^\alpha$ enters $A^{[3]}$
- Effectively, we need to accomodate infinitely many options for $m^{[3]}(c^\alpha)$. 

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Ensuring $\Phi_e(A) > \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing $0^{''}$ to determine $\tau$, $|\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$

- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$

- $0^{''}$ would find some other long $\tau$ if $c^\alpha \notin A^{[3]}$. Easy (can only happen one way).

- Remember, elements can be removed from $A^{[2]}$ by enumeration into $A^{[1]}$

- Like a $\Delta^0_2$ construction for $A^{[2]}$ but stays out if removed infinitely many times.

- For simplicity assume totality ($0^{''}$ will be able to check)
Ensuring $\Phi_e(A) > \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing 0" to determine \( \tau, |\tau| = 2 \) with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$
- Try \( \tau = \langle 00 \rangle \) with highest priority, then \( \langle 01 \rangle, \langle 10 \rangle \) and then \( \langle 11 \rangle \)
- $\Phi_e(A_s) > \langle 11 \rangle$. 
Ensuring $\Phi_e(A) > \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing $0''$ to determine $\tau$, $|\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$

- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$

- Enumerate $b_1$.

- $\Phi_e(A_s) > \langle 00 \rangle$. 

\[\begin{array}{c}
\vdots \\
A^{[2]} \\
A^{[3]} \\
\vdots \\
\end{array}\]

\[b_1 \quad x\]
Ensuring $\Phi_e(A) > \tau$

- Satisfy $P_\alpha$ allowing $0^\prime$ to determine $\tau$, $|\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$

- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$

- Enumerate $b_1$.

- $\Phi_e(A_s) > \langle 00 \rangle$.

- Preserve higher priority string.

- Cancelation can only happen at $b_k$ removing $b_k$ and all larger enumerations.
Ensuring $\Phi_e(A) > \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing $0^\prime$ to determine $\tau$, $|\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate $b_1$.
- $\Phi_e(A_s) > \langle 10 \rangle$. 

\[ \begin{array}{c c c c c c c}
\vdots \\

\hline
\hline
x \\

\hline
x \\

\hline
x \\

\hline
A^{[2]} \\

\hline
b_1 \\
\end{array} \]
Ensuring $\Phi_e(A) > \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing $0''$ to determine $\tau$, $|\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate $b_2$.
- $\Phi_e(A_s) > \langle 01 \rangle$.
Ensuring $\Phi_e(A) > \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing $0^\prime$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$

- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$

- Enumerate $b_2$.

- $\Phi_e(A_s) > \langle 01 \rangle$.

- Preserve higher priority string.

- But don’t restrain/move $b_1$ because that belongs to higher priority string $\langle 00 \rangle$. 
Ensuring $\Phi_e(A) > \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing $0''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate $b_2$.
- $\Phi_e(A_s) > \langle 00 \rangle$. 
Ensuring $\Phi_e(A) > \tau$

- Satisfy $P_\alpha$ allowing $0''$ to determine $\tau$, $|\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate $b_2$.
- $\Phi_e(A_s) > \langle 00 \rangle$.
- Preserve higher priority string.
- Don’t restrain/move $b_1$ because it belongs to same string $\langle 00 \rangle$. 

\[
\begin{array}{cccc}
 b_2 & b_1 \\
 & x & x & x \\
 & x & x & x \\
 & b_2 & b_1 \\
 & x & x & x \\
 & x & x & x \\
 A^{[2]} & A^{[2]} & A^{[2]} & A^{[2]}
\end{array}
\]
Ensuring $\Phi_e(A) \succ \tau$

- Satisfy $\mathcal{P}_\alpha$ allowing $0''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Later we may need to cancel $b_1$
- But this restores state we had at earlier $\langle 00 \rangle$ stage so $\Phi_e(A_s) > \langle 00 \rangle$. 
Ensuring $\Phi_e(A) > \tau$

1. Satisfy $P_\alpha$ allowing $0''$ to determine $\tau$, $|\tau| = 2$ with $\Phi_e(A) > \tau$ assuming $c^\alpha \in A^{[3]}$
2. Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
3. If $c^\alpha \in A^{[3]}$ then $\Phi_e(A)$ extends highest priority $\tau$, $|\tau| = 2$ seen infinitely.
4. Critically $0''$ can determine what $\tau$ would be if $c^\alpha \in A^{[3]}$.
5. Doesn’t affect whether (eventually) all $b_k$ stay in $A^{[3]}$.
1 Notation & Definitions

2 Background
   - Weak 1-genericity
   - R.E. Sets and 1-genericity
   - 2-genericity
   - 3-genericity

3 3-REA Sets
   - Differences From $\Delta_3^0$ Escaping Functions
   - Main Result
   - Naive Strategies
   - Complications
Fortunately (for me), the method derived from $T$ isn’t enough.

If the limit DNE then $0''$ never gets confirmation that $c^\alpha \notin A^{[3]}$

So, unlike $T$, we can’t wait to see how $\mathcal{P}_\alpha$ is met before starting on $\mathcal{P}_\beta$.

- Requirements guessing that $\bar{k}^\alpha = n$ (i.e. each way $c^\alpha \in A^{[3]}$) can execute on cancelation of $b_n$ (e.g. they get to know how $\mathcal{P}_\alpha$ is met)
- But $\mathcal{P}_\beta$ - which guesses that $c^\alpha \notin A^{[3]}$ - can’t wait.

If guess $c^\alpha \notin A^{[3]}$ we do know how $\mathcal{P}_\alpha$ is met but must work on $\mathcal{P}_\beta$ allowing for possibility $c^\alpha \in A^{[3]}$ with really large $\bar{k}^\alpha$

This is the concrete instantiation of fact that $\Phi_e(A)$ can wait to see multiple large values before committing.
Interference Finding $\tau < \Phi_e(A)$

- Trick to let $0''$ determine common $\tau < \Phi_e(A)$ above can’t respect both $P_\alpha$ and $P_\beta$ simultaneously.

- $P_\beta$ is guessing $c^\alpha \in A^{[3]}$ so even if $b^\beta_m$ is cancelled infinitely often that must not cancel any $b^\alpha_k$ infinitely many times.

- Has consequence that we can’t ensure that cancelling $b^\beta_m$ doesn’t return us to a lower priority option for $\tau$. 
Instead of ensuring that if $b_i^\alpha$ gets cancelled we restore $\Phi_e(A) > \tau$ instead ensure that if $b_i^\alpha$ cancelled we restore $\Phi_e(A) > \sigma \langle 00 \cdots 0 \rangle$ where $|\langle 00 \cdots 0 \rangle| = i$.

0″ can tell if we eventually succeed at this for infinitely many $i$.

If this succeeds we can (at stages we see progress) then go ahead and try to meet $\mathcal{P}_{\beta'}$ (where $\beta'$ guesses this succeeds) certain that when 0″ finds out that $b_i^\alpha \in A^{[2]}$ we can conclude $\Phi_e(A) > \sigma \langle 00 \cdots 0 \rangle$.

This means that even if 0″ never sees exactly how $\mathcal{P}_\alpha$ is satisfied we can enumerate a dense set of strings that $\Phi_e(A)$ avoids if $c^\alpha \in A^{[3]}$.

OTOH, if this fails we 0″ discovers a string $\sigma \langle 00 \cdots 0 \rangle$ that $\Phi_e(A)$ avoids.

We can try this again and again for different $\sigma$ and interleave (in priority) with $\mathcal{P}_\beta^k$ meaning each $\mathcal{P}_\beta^k$ is only injured finitely many times.