A non-trivial 3-REA Set Not Computing a Weak 3-generic

Peter M. Gerdes

Midwest Computability Seminar, 2023

Outline

Table of Contents

Notation

- $\sigma, \tau, \nu, \delta$ range over $\{0, 1, \uparrow\}^{<\omega}$ (partial binary valued functions with finite domain).
- We write $\sigma \prec \tau$ if τ extends σ and $\sigma \prec X$ if σ is extended by the characteristic function of X.
- θ meets $\Gamma \subset \{0, 1, \uparrow\}^{<\omega}$ $(\theta \Vdash \Gamma)$ if $(\exists \sigma \in \Gamma)(\theta \succ \sigma)$ and θ strongly avoids Γ $(\theta \Vdash \neg\Gamma)$ if some $(\exists \tau \prec \theta)(\forall \sigma \in \Gamma)(\tau \nprec \gamma)$.
- $f \in \omega^\omega$ dominates $g \in \omega^\omega$ $(f \gg g)$ if $\left(\forall^* x \in \omega\right)(f(x) \geq g(x))$.
- f is Δ^0_{n+1} escaping if f isn't dominated by any $g \leq_{\mathbf{T}} \mathbf{0}^{(n)}$

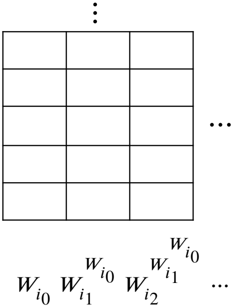
α-REA Sets

- The i-th hop is $\mathcal{H}_i(A) \stackrel{\text{def}}{=} A \oplus W_i^A$.
- REA sets are the result of iterating the Hop operation on $\emptyset$.
- The 1-REA sets are just the r.e. sets.
- The 2-REA sets are sets of the form $W_i \oplus W_j^{W_i}$

See Jockusch and Shore [2] for a more explicit definition.

Components as Columns

- For this talk we only care about n-REA sets up to Turing degree.
- Useful to identify the components of n-REA sets with their columns.



Genericity

- In this talk we only consider the (standard) forcing relation on $2^{<\omega}$
- G is n-generic $(n > 0\ )$ if $G \Vdash \phi$ or $G \Vdash \neg\phi$ for all $\Sigma^{0,G}_n$ sentences.
- Equivalently, G is n-generic if G meets or strongly avoids every Σ^0_n subset of $2^{<\omega}$ (equivalently $\{0, 1, \uparrow\}^{<\omega}$)
- $\Gamma \subset 2^{<\omega}$ is dense if $\left(\forall\tau \in 2^{<\omega}\right)(\exists\sigma \in \Gamma)\tau \prec \sigma$
- G is weakly n-generic if G meets every dense Σ^0_n subset of $2^{<\omega}$

Table of Contents

Table of Contents

Computing Weak 1-Generics

Theorem

If $f \in \omega^\omega$ is Δ^0_1 escaping then f computes a weak 1-generic

- WLOG f is monotonicly increasing and let U_i be i-th r.e. subset of $2^{<\omega}$.
- Build $G = \lim_{n\to\infty} \tau_n$, $\tau_0 = \langle\rangle, \tau_{n+1} \succ \tau_n$.
- Let $\tau_{n+1} \succ \tau_n$ be in $U_{i,f(n+1)}$ for least $i \leq n$ or τ_n if no such i exists.

Verifying Weak 1-Generic

- Suppose U_i is dense but G doesn't meet U_i.
- Let $n > 0$ large enough that τ_n meets every $U_j, j < i$ G will ever meet.
- Suppose we can compute a bound $l_m > |\tau_m|$ for $m > n$.
- Let $h(m)$ be the least stage s such that $U_{i,h(m)}$ includes an extension of every string of length l_m.
- If $f(m) \geq h(m), m > n$ then τ_m meets U_i.
- We compute l_m by assuming $f(x) < h(x)$ for $n < x < m$.

Can't extend to 1-generics because we can't guarantee number of stages needed to find an extension in a non-dense U_i is computably bounded.

Table of Contents

R.E. Sets Compute 1-generics

Theorem

If $A \not\leq_{\mathbf{T}} \mathbf{0}$ is r.e. then A computes a 1-generic

- The modulus for A $\left(m(n) \overset{\text{def}}{=} \mu t \left(A_t\restriction_{n+1} = A\restriction_{n+1}\right)\right)$ is Δ^0_1 escaping.
- But we can compute full 1-generic by using the computable approximation to A.
- Same construction as before but we use stagewise approximations and allow restraint.
- Now, if we extend $\tau_{n,s}$ to $\tau_{n+1,s}$ to meet U_i then we preserve $\tau_{n+1,s}$ from changes trying to meet $U_j, j > i$

Constructing 1-generic Below R.E.

$$
\begin{aligned}
m_s(n) &\stackrel{\text{def}}{=} \mu t \left(A_t\restriction_{n+1} = A_s\restriction_{n+1} \right) \\
r_s(i) &\stackrel{\text{def}}{=} \max \left\{ n \mid n \leq s \wedge \left(\exists \sigma \succ \tau_{n,s}\right)\left(\tau_{n,s} \neg \Vdash U_{i,s-1} \wedge \sigma \Vdash U_{i,s-1}\right)\right\} \\
\bar{r}_s(i) &\stackrel{\text{def}}{=} \max_{j<i} r_s(i) \\
i^*_{n+1,s} &\stackrel{\text{def}}{=} \min_{i \leq n} \neg(\tau_{n,s} \Vdash U_{i,m_s(n)}) \wedge \left(\exists \sigma \succ \tau_{n,s}\right)\left(\sigma \Vdash U_{i,m_s(n+1)}\right) \\
\tau_{n,s} &\stackrel{\text{def}}{=} \begin{cases} \langle\rangle & \text{if } s \leq n \vee s = 0 \vee n = 0 \\ \tau_{n,s-1} & \text{unless } m_s(n+1) > m_{s-1}(n) \\ \tau_{n,s-1} & \text{if } \bar{r}_s(i^*_{n,s}) \geq n \\ \sigma & \text{o.w. where } \sigma \text{ is least witness for } i^*_{n,s} \end{cases} \\
G &\stackrel{\text{def}}{=} \lim_{n\to\infty} \lim_{s\to\infty} \tau_{n,s}
\end{aligned}
$$

Note that $\tau_{n,\infty} = \tau_{n,m(n)}$ so $G \leq_{\mathbf{T}} A$.

Verifying R.E. Sets Compute 1-generics

- Suppose i is least s.t. $G\neg \Vdash U_i \wedge G\neg \Vdash \neg U_i$. We show that A is computable.
- Let n large enough that $n > \bar{r}_\infty(i)$ (exists by fact i least) and for all $j < i$ $\tau_n \Vdash U_j \vee \tau_n \Vdash \neg U_j$ and t large enough that $\tau_{n,t} = \tau_n$.
- If there are $n' \geq n, s \geq \max(t, n'), \sigma \succ \tau_{n',s}, \sigma \Vdash U_{i,s}$ then $m(n') < s$.
- Otherwise we'd preserve $\tau_{n',s}$ and have $\tau_{n',m(n')} \Vdash U_i$.
- But, by assumption, there must be infinitetly many such m, s showing $m \leq_{\mathbf{T}} \mathbf{0}$
- Contradiction.

Table of Contents

Computing Weak 1-Generics

Theorem (Andrews, Gerdes and Miller)

If $f \in \omega^\omega$ is Δ^0_2 escaping then f computes a weak 2-generic

- Proved in [1]. Won't prove it here.
- Idea is to try and extend to meet Σ^0_2 sets $\mathfrak{U}_i$ by favoring those σ for which $(\exists x)(\forall y)\phi(\sigma, x, y)$ appears true with least $\max(|\sigma|, x)$.

Hypothesis

If $A \not\leq_{\mathbf{T}} \mathbf{0}'$ is 2-REA then A computes a 2-generic

Table of Contents

Pattern Ends at $n = 3$

Theorem (Andrews, Gerdes and Miller)

There is a (pruned) perfect ω-branching tree $T \subset \omega^{<\omega}, T \leq_{\mathbf{T}} \mathbf{0}''$ such that if $f \in [T]$ then f doesn't compute a weak 3-generic.

vertex Node with multiple successors ($\sigma\widehat{\ }\langle i \rangle, \sigma\widehat{\ }\langle j \rangle \in T, i \neq j$).

ω-branching Every vertex has infinitely many immediate successors.

pruned No terminal nodes (all nodes extend to paths)

perfect Every node is extended by a vertex.

Pattern Ends at $n = 3$

Theorem (Andrews, Gerdes and Miller)

There is a (pruned) perfect ω-branching tree $T \subset \omega^{<\omega}, T \leq_{\mathbf{T}} \mathbf{0}''$ such that if $f \in [T]$ then f doesn't compute a weak 3-generic.

- No amount of (countable) non-domination suffices to compute a weak 3-generic, e.g., $g_j \not\gg f, j \in \omega$.
  - View T as function on ω^ω by defining $T[h]$ to be the path taking the $h(n)$-th option at the n-th vertex.
  - Let $f = T[h]$ with $h(k)$ picked large enough that $T[h](n_k) > g_j(n_k), j \leq k$ where $T[h]\upharpoonright_{n_k}$ is the k-th vertex along $T[h]$
- Note that if f is monotonic and $\Delta^0_{n+3}, n \geq 0$ escaping then $T[f] \leq_{\mathbf{T}} f \oplus \mathbf{0}''$ is as well .
  - If $g \gg T[f]$ then $g^*(k) = g(n_k)$ satisfies $g^* \gg f, g^* \leq_{\mathbf{T}} g \oplus \mathbf{0}''$

Intuition Behind Failure

Question

What prevents the pattern from continuing indefinitely?

- Pattern worked because more non-domination strength gave us more computational power (guessing at membership in Σ^0_1 sets then Σ^0_2 sets).
- But, a computable reduction can't hope to always distinguish $\mathbf{0}^{(n)}$ big and $\mathbf{0}^{(n+k)}$ big.
- Given finitely many potential values of $\Phi_e\left(\sigma\widehat{\ }\langle n\rangle\right)$, $\mathbf{0}''$ can figure out which value is compatible with infinitely many n.
- Allows us to limit $\Phi_e(f)$ to a narrow range of options (while allowing f to take arbitrarily large values).
- Can build $\mathfrak{U}_e \subset 2^{<\omega}$ a dense Σ^0_3 set $\Phi_e(f)$ can't meet by enumerating strings outside that narrow range.

Utility Lemma

Lemma

Suppose for infinitely many $l \in \omega$, $\mathbf{0}''$ can enumerate $k > 0$, $\eta_i \in 2^{<\omega}, i < 2^k - 1, |\eta_i| \geq l + k$. If $f \in [T] \wedge \Phi_e(f)\downarrow \implies \Phi_e(f) \succ \eta_i$ then $\Phi_e(f)$ isn't weakly 3-generic for any $f \in [T]$.

Proof.

For each σ with $|\sigma| = l$ there are 2^k strings $\tau \succ \sigma$ of length $l + k$. At least one of those strings τ_σ must be incompatible with $\eta_i, i < 2^k - 1$.

For each such $l > 0$ and σ with $|\sigma| = l$ enumerate τ_σ into $\mathfrak{U}_e$. $\mathfrak{U}_e$ is a dense Σ^0_3 set that isn't met by $\Phi_e(f)$ for any $f \in [T]$. □

Building T

Conditions

- *A finite set V_s of vertexes ()*
- *For each $\sigma \in V_s$ an infinite r.e. set of strings* $\Sigma_s(\sigma) \subset \left\{ \sigma\widehat{\ }\langle n \rangle\widehat{\ }\tau \mid n \in \omega, \tau \in 2^{<\omega} \right\}$
- $\theta^e_s : 2^{<\omega} \mapsto 2^{<\omega} \cup \{\uparrow\}, e \in \omega$ *such that if* $\sigma \in V_s, \tau \in \Sigma_s(\sigma)$ *then* $\Phi_e(\tau) \succ \theta^e_s(\sigma)$ *(where that means* $\Phi_e(f)\uparrow$ *if* $f \succ \tau$ *if* $\theta^e_s(\sigma) = \uparrow$ *)*

V_s: Nodes we commit to making ω-branching vertexes in T.

$\Sigma_s(\sigma)$: Possible (i.e. not in V_s) branches extending σ.

$\theta^e_s(\sigma)$: Specifies initial segment of $\Phi_e(\tau)$ agreed on by all $\tau \in \Sigma_s(\sigma)$ (or that all such τ force partiality)

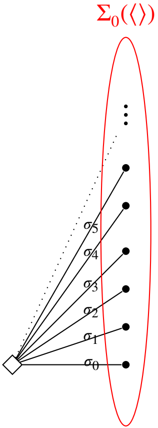
Building T

Conditions

- *A finite set V_s of vertexes ()*
- *For each $\sigma \in V_s$ an infinite r.e. set of strings* $\Sigma_s(\sigma) \subset \left\{\sigma^\frown\langle n\rangle^\frown\tau \mid n \in \omega, \tau \in 2^{<\omega}\right\}$
- $\theta^e_s : 2^{<\omega} \mapsto 2^{<\omega} \cup \{\uparrow\}, e \in \omega$ *such that if* $\sigma \in V_s, \tau \in \Sigma_s(\sigma)$ *then* $\Phi_e(\tau) \succ \theta^e_s(\sigma)$ *(where that means* $\Phi_e(f)\uparrow$ *if* $f \succ \tau$ *if* $\theta^e_s(\sigma) = \uparrow$ *)*

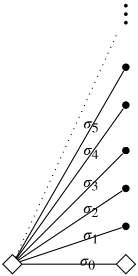
- $V_0 = \{\langle\rangle\}$ if $s = 0 \vee \sigma \notin V_s \vee e \geq s$ then $\Sigma_s(\sigma) = \left\{\sigma^\frown\langle n\rangle\right\}$ and $\theta^e_s(\sigma) = \langle\rangle$.
- $V_{s+1} = V_s \bigcup \left\{\tau_\sigma \mid \sigma \in V_s\right\}$ where $\tau_\sigma \in \Sigma_s(\sigma)$ with $\tau_\sigma(|\sigma|)$ large. (Hence $|V_s| = 2^s$).
- $\Sigma_{s+1}(\sigma) \subset \Sigma_s(\sigma)$ and $\theta^e_{s+1}(\sigma) \succ \theta^e_s(\sigma)$ (where $\uparrow$ is considered $\succ$ maximal).
- We ensure that if $e < s, \sigma \in V_s$ then $|\theta^e_s(\sigma)| > 2s + 1$

Visualizing T Construction



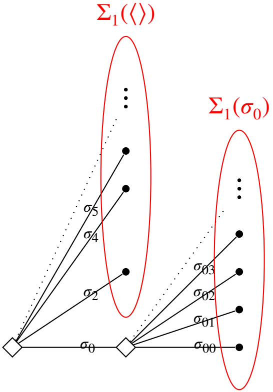
- Every $\sigma_i \in \Sigma_0(\langle\rangle)$ has $\Phi_e(\sigma_i) \succ \theta^e_0(\langle\rangle)$

Visualizing T Construction



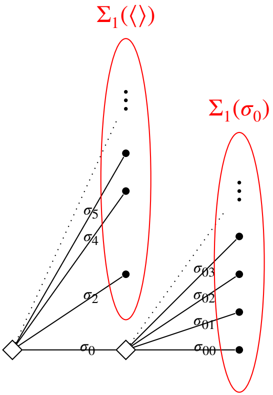
- Add new vertex in $\Sigma_s(\tau)$ for each $\tau \in V_s$.

Visualizing T Construction



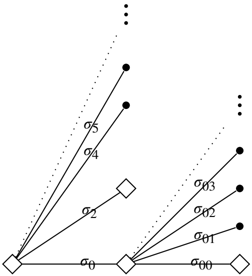
- Prune and extend (e.g. replace σ_i with an extension) so $\sigma_i \in \Sigma_1(\langle\rangle) \implies \Phi_e(\sigma_i) \succ \theta_1^e(\langle\rangle)$ (now longer) and $\Phi_e(\sigma_{0i}) \succ \theta_1^e(\sigma_0)$

Visualizing T Construction



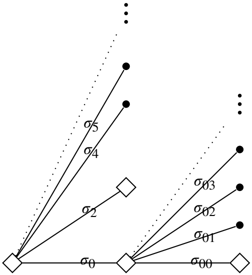
- If $f \in [T]$ then $\Phi_e(f) \succ \theta_1^e(\langle\rangle)$ or $\Phi_e(f) \succ \theta_1^e\left(\sigma_0\right)$

Visualizing T Construction



- Extend each vertex with a node from allowed branches.

Visualizing T Construction



- If If $f \in [T]$ then $\Phi_e(f) \succ \theta_2^e(\langle\rangle)$ or $\Phi_e(f) \succ \theta_2^e\left(\sigma_0\right)$ or $\Phi_e(f) \succ \theta_2^e\left(\sigma_2\right)$ or $\Phi_e(f) \succ \theta_2^e\left(\sigma_{00}\right)$

Verifying Construction

- To complete proof we must only show that we can always construct $\Sigma_{s+1}(\tau)$ from $\Sigma_s(\tau)$ that makes $\theta^e_{s+1}(\tau)$ sufficently long.
- But given the length $\mathbf{0}''$ can ask if there are infinitely many elements $\sigma \in \Sigma_s(\tau)$ that can be extended to σ' with $\Phi_e(\sigma')$ of sufficent length.
- If not remove the finitely many elements that allow convergence.
- If so $\mathbf{0}''$ can determine which of the finitely many options for $\Sigma_{s+1}(\tau)$ permits $\Sigma_{s+1}(\tau)$ to be infinite.
- Repeat for each $e < s+1$ and $\tau \in V_{s+1}$.

Table of Contents

Table of Contents

Genericity From 3-REA Sets

Question

If $A \not\leq_{\mathbf{T}} \mathbf{0}''$ is 3-REA does A compute a (weak) 3-generic?

- A computes a Δ^0_3 escaping function $m^{[3]}(x)$ (where $m^{[n+1]}(x)$ is modulus of $A^{[n+1]}$ over $A^{[n]}$) but that's not enough.
- But several reasons to think that 3-REA sets have extra power to compute generics.
  - We get $m^{[3]}, m^{[2]}, m^{[1]}$ with $m^{[n]}$ $\Delta^0_n, 1 \leq n \leq 3$ escaping. Modifications even ensure all three functions simultaneously escape a tuple $h^1 \leq_{\mathbf{T}} \mathbf{0}, h^2 \leq_{\mathbf{T}} \mathbf{0}', h^3 \leq_{\mathbf{T}} \mathbf{0}''$
  - Our ability to effectively approximate A offers additional power (remember non-trivial r.e. sets compute 1-generics not just weak 1-generics).
  - Approach used to build T doesn't directly translate.

Isolating Large Values

- When we built T functionals $\Phi_e(f)$ had to meet $\mathfrak{U}_e$ using only one large value.
  - If $\sigma \in V_s, e < s, x \in \omega$ we could wait until we found $\tau \succ \sigma^\frown\langle n\rangle$ with $\Phi_e(\tau; x)$ converging before choosing the next large value.
- Given $A \not\leq_{\mathbf{T}} \mathbf{0}''$, 3-REA, $k > 1$ and $h \leq_{\mathbf{T}} \mathbf{0}''$ there are infinitely many tuples $x_0 < x_1, <, \ldots, < x_k < m^{[3]}(x_0)$ such that $m^{[3]}(x_i) > h(x_i), i \leq k$.
- So, infinitely often, $\Phi_e(A; x)$ can consult k large values before trying to meet $\mathfrak{U}_e$.

Table of Contents

Ultimately Insufficent

Theorem

There is a 3-REA set $A \not\leq_{\mathbf{T}} \mathbf{0}''$ that doesn't compute a weak 3-generic.

- We know A computes a weak 2-generic
- By result in [1] every Δ^0_3 escaping function computes a 2-generic.
- Thus, result is sharp.

Requirements

$\mathscr{P}_i$: $A^{[3]}(c^i) \neq \lim_{s\to\infty} \lim_{t\to\infty} p_i(c^i, s, t)$

$\mathscr{Q}_{e,\sigma}$: $X_e\downarrow \implies [\exists \tau \succ \sigma]\big(\tau \in \mathfrak{U}_e \land \tau \not\prec X_e\big)$

$$X_e \stackrel{\text{def}}{=} \Phi_e(A) \stackrel{\text{def}}{=} \Phi_e(A) \qquad \mathfrak{U}_e : \Sigma^0_1\left(\mathbf{0}''\right) \text{ subset of } 2^{<\omega}$$

$\mathscr{P}_i$ Ensures that $A \not\leq_{\mathbf{T}} \mathbf{0}''$

$\mathscr{Q}_{e,\sigma}$ Builds dense $\mathfrak{U}_e$ avoiding X_e (no other additions)

- We'll want to break these requirements up into Π^0_2 subrequirements (to use tree method and let $\mathbf{0}''$ see outcome).

(Alt) Requirements

Requirements

$\mathscr{P}_\alpha$: $A^{[3]}(c^\alpha) \neq \lim_{s\to\infty} \lim_{t\to\infty} p_\alpha(c^\alpha, s, t)$

$\mathscr{Q}_{\alpha,\sigma}$: $X_\alpha\downarrow \implies [\exists \tau \succ \sigma]\big(\tau \in \mathfrak{U}_\alpha \wedge \tau \nprec X_\alpha\big)$

$$X_\alpha \stackrel{\text{def}}{=} \Phi_\alpha(A) \stackrel{\text{def}}{=} \Phi_{e_\alpha}(A) \qquad \mathfrak{U}_\alpha : \Sigma^0_1\left(\mathbf{0}''\right) \text{ subset of } 2^{<\omega}$$

$\mathscr{P}_\alpha$ Ensures that $A \not\leq_{\mathbf{T}} \mathbf{0}''$

$\mathscr{Q}_{\alpha,\sigma}$ Builds dense $\mathfrak{U}_\alpha$ avoiding X_e (no other additions)

- We'll want to break these requirements up into Π^0_2 subrequirements (to use tree method and let $\mathbf{0}''$ see outcome).

Table of Contents

Strategy for $\mathscr{P}_\alpha$

Requirement

$\mathscr{P}_\alpha$: $A^{[3]}(c^\alpha) \neq \lim_{s\to\infty} p'_\alpha(c^\alpha, s)$ where $p'_\alpha(c^\alpha, s) \overset{\text{def}}{=} \lim_{t\to\infty} p_\alpha(c^\alpha, s, t)$

Sub-requirements

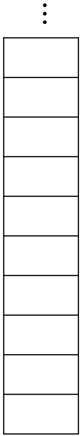
$\mathscr{P}^k_\alpha$: $b^\alpha_k \in A^{[2]} \iff |\{t \mid p'_\alpha(c^\alpha, t)\} = 1| > k$

- Place $c^\alpha \in A^{[3]}$ iff $(\exists k)\big(b^\alpha_k \notin A^{[2]}\big)$
- At stage s place b_k into $A^{[2]}$ if it's not currently in and $|\{t \mid p_\alpha(c^\alpha, t, s)\} = 1| > k$.
- We remove b_k at $s_1 > s$ (by enumerating into $A^{[1]}$) if $|\{t \mid (\forall s' \in [s, s_1])\big(p_\alpha(c^\alpha, t, s') = 1\big)\}| \leq k$
- $c^\alpha \notin A^{[3]}$ if $\lim_{s\to\infty} p'_\alpha(c^\alpha, s)$ is 1 or DNE

First Attempt At $\mathcal{Q}_{\alpha,\sigma}$

- Let's try same approach as constructing T, ensure that all 'options' for A agree on 'alot' of $\Phi_e(A)$.
- But $\mathbf{0}''$ can't determine if $c^\alpha \in A^{[3]}$. But we can accomodate both options by agreeing on sufficently long initial segments.
- Harder problem is ensuring that $\Phi_e(A)$ takes the same value no matter what value we get for $\bar{k}^\alpha \stackrel{\text{def}}{=} \mu k \left(b^\alpha_k \notin A^{[3]} \right)$.
- This is analog of allowing $f(x)$ to take on infinitely many values in construction of T.
  - (Up to $\mathbf{0}''$ equivalence) $\bar{k}^\alpha$ measures stage at c^α enters $A^{[3]}$
  - Effectively, we need to accomodate infinitely many options for $m^{[3]}(c^\alpha)$.

Ensuring $\Phi_e(A) \succ \tau$



$A^{[2]}$

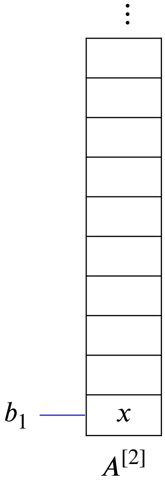
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- $\mathbf{0}''$ would find some other long τ if $c^\alpha \notin A^{[3]}$. Easy (can only happen one way).
- Remember, elements can be removed from $A^{[2]}$ by enumeration into $A^{[1]}$
- Like a Δ^0_2 construction for $A^{[2]}$ but stays out if removed infinitely many times.
- For simplicity assume totality ($\mathbf{0}''$ will be able to check)

Ensuring $\Phi_e(A) \succ \tau$

$A^{[2]}$

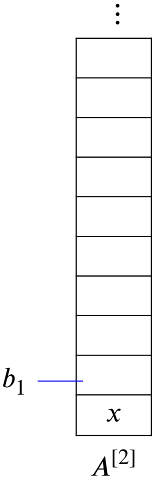
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- $\Phi_e\left(A_s\right) \succ \langle 11 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



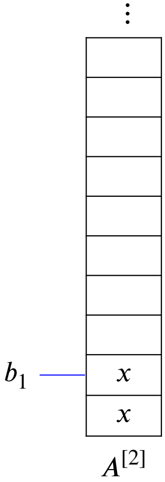
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate b_1.
- $\Phi_e\left(A_s\right) \succ \langle 00 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



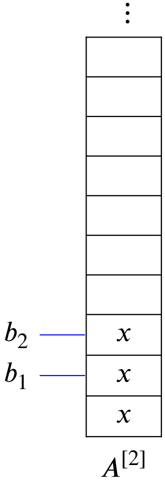
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate b_1.
- $\Phi_e\left(A_s\right) \succ \langle 00 \rangle$.
- Preserve higher priority string.
- Cancelation can only happen at b_k removing b_k and all larger enumerations.

Ensuring $\Phi_e(A) \succ \tau$



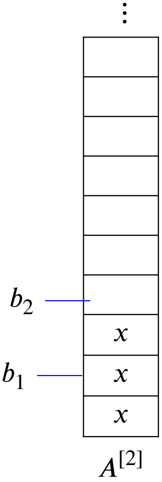
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate b_1.
- $\Phi_e\left(A_s\right) \succ \langle 10 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



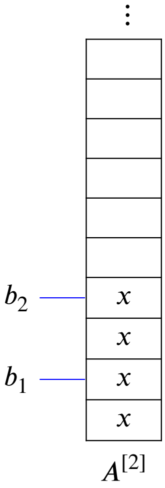
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate b_2.
- $\Phi_e\left(A_s\right) \succ \langle 01 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



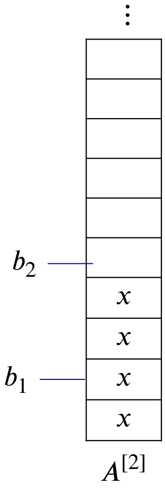
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate b_2.
- $\Phi_e\left(A_s\right) \succ \langle 01 \rangle$.
- Preserve higher priority string.
- But don't restrain/move b_1 because that belongs to higher priority string $\langle 00 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



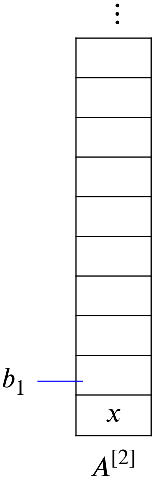
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate b_2.
- $\Phi_e\left(A_s\right) \succ \langle 00 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



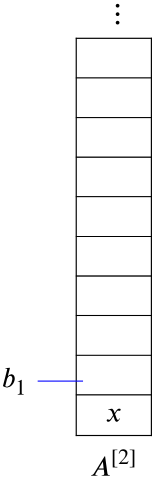
- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Enumerate b_2.
- $\Phi_e\left(A_s\right) \succ \langle 00 \rangle$.
- Preserve higher priority string.
- Don't restrain/move b_1 because it belongs to same string $\langle 00 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- Later we may need to cancel b_1
- But this restores state we had at earlier $\langle 00 \rangle$ stage so $\Phi_e\left(A_s\right) \succ \langle 00 \rangle$.

Ensuring $\Phi_e(A) \succ \tau$



- Satisfy $\mathscr{P}_\alpha$ allowing $\mathbf{0}''$ to determine $\tau, |\tau| = 2$ with $\Phi_e(A) \succ \tau$ assuming $c^\alpha \in A^{[3]}$
- Try $\tau = \langle 00 \rangle$ with highest priority, then $\langle 01 \rangle, \langle 10 \rangle$ and then $\langle 11 \rangle$
- If $c^\alpha \in A^{[3]}$ then $\Phi_e(A)$ extends highest priority $\tau, |\tau| = 2$ seen infinitely.
- **Critically** $\mathbf{0}''$ can determine what τ would be *if* $c^\alpha \in A^{[3]}$.
- Doesn't affect whether (eventually) all b_k stay in $A^{[3]}$

Table of Contents

Limit May Not Exist

- Fortunately (for me), the method derived from T isn't enough.
- If the limit DNE then $\mathbf{0}''$ never gets confirmation that $c^\alpha \notin A^{[3]}$
- So, unlike T, we can't wait to see how $\mathscr{P}_\alpha$ is met before starting on $\mathscr{P}_\beta$.
  - Requirements guessing that $\bar{k}^\alpha = n$ (i.e. each way $c^\alpha \in A^{[3]}$) can execute on cancelation of b_n (e.g. they get to know how $\mathscr{P}_\alpha$ is met)
  - But $\mathscr{P}_\beta$ - which guesses that $c^\alpha \notin A^{[3]}$ - can't wait.
- If guess $c^\alpha \notin A^{[3]}$ we do know how $\mathscr{P}_\alpha$ is met but must work on $\mathscr{P}_\beta$ allowing for possibility $c^\alpha \in A^{[3]}$ with really large $\bar{k}^\alpha$
- This is the concrete instantiation of fact that $\Phi_e(A)$ can wait to see multiple large values before commiting.

Interference Finding $\tau \prec \Phi_e(A)$

- Trick to let $\mathbf{0}''$ determine common $\tau \prec \Phi_e(A)$ above can't respect both $\mathscr{P}_\alpha$ and $\mathscr{P}_\beta$ simultaneously.
- $\mathscr{P}_\beta$ is guessing $c^\alpha \in A^{[3]}$ so even if b^β_m is cancelled infinitely often that must not cancel any b^α_k infinitely many times.
- Has consequence that we can't ensure that cancelling b^β_m doesn't return us to a lower priority option for τ.

Final Trick

- Instead of ensuring that if b_i^α gets cancelled we restore $\Phi_e(A) \succ \tau$ instead ensure that if b_i^α cancelled we restore $\Phi_e(A) \succ \sigma\hat{}\langle 00 \cdots 0\rangle$ where $|\langle 00 \cdots 0\rangle| = i$.
- $\mathbf{0}''$ can tell if we eventually succeed at this for infinitely many i.
- If this succeeds we can (at stages we see progress) then go ahead and try to meet $\mathscr{P}_{\beta'}$ (where β' guesses this succeeds) certain that when $\mathbf{0}''$ finds out that $b_i^\alpha \in A^{[2]}$ we can conclude $\Phi_e(A) \succ \sigma\hat{}\langle 00 \cdots 0\rangle$.
  - This means that even if $\mathbf{0}''$ never sees exactly how $\mathscr{P}_\alpha$ is satisfied we can enumerate a dense set of strings that $\Phi_e(A)$ avoids *if* $c^\alpha \in A^{[3]}$.
- OTOH, if this fails we $\mathbf{0}''$ discovers a string $\sigma\hat{}\langle 00 \cdots 0\rangle$ that $\Phi_e(A)$ avoids.
- We can try this again and again for different σ and interleave (in priority) with $\mathscr{P}_\beta^k$ meaning each $\mathscr{P}_\beta^k$ is only injured finitely many times.

References I

[1] Uri Andrews, Peter Gerdes, and Joseph S. Miller. "The degrees of bi-hyperhyperimmune sets". en. In: *Annals of Pure and Applied Logic* 165.3 (Mar. 2014), pp. 803–811. ISSN: 0168-0072. DOI: 10.1016/j.apal.2013.10.004. URL: https://www.sciencedirect.com/science/article/pii/S0168007213001528 (visited on 10/15/2021).

[2] Carl G Jockusch and Richard A Shore. "PSEUDO JUMP OPERATORS. I: THE R. E. CASE". en. In: *Transactions of the American Mathematical Society* 275.2 (Feb. 1983), p. 11. DOI: 10/fdstv2.