The Reverse Mathematics of Noether’s Decomposition Lemma

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Definition

A *computable ring* is a computable subset $A \subseteq \mathbb{N}$ equipped with two computable binary operations $+$ and $\cdot$ on $A$, together with elements $0, 1 \in A$ such that $R = (A, 0, 1, +, \cdot)$ is a ring.
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All rings will be *countable* and *commutative*, unless we say otherwise.
Primary Decomposition Lemma

If $R$ is Noetherian, then $R$ contains only finitely many minimal prime ideals.
Noether’s Primary Decomposition Lemma

Primary Decomposition Lemma

If \( R \) is Noetherian, then \( R \) contains only finitely many minimal prime ideals.

Primary Decomposition Lemma

If \( R \) contains infinitely many minimal prime ideals, then \( R \) is not Noetherian, i.e. \( R \) contains an infinite strictly ascending chain of ideals

\[
I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \subset R, \quad n \in \mathbb{N}.
\]
Assume that $R$ contains infinitely many distinct minimal primes.
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$$I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \subset R.$$ 

Let $I_0 = \langle 0 \rangle_R \subset R$. Since $R$ contains infinitely many minimal primes, $\langle 0 \rangle_R \subset R$ is not a prime ideal. Therefore there exist $a_1, b_1 \in R$ such that $a_1, b_1 \notin I_0$ but $a_1 b_1 = 0 \in I_0$. Now, either $a_1$ or $b_1$ is contained in infinitely many minimal primes; add it to $I_0$ to get $I_1 \supset I_0$. 

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$$I_k = \langle c_1, c_2, \cdots, c_k \rangle_R \subset R, \ k \in \mathbb{N},$$

is contained in infinitely many minimal primes, and therefore is not prime itself. Uses $\emptyset''$. 

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The Reverse Mathematics of Noether’s Decomposition Lemma
Reverse Mathematics

The “Big Five:”

- $\text{RCA}_0$: Recursive Comprehension Axiom
- $\text{WKL}_0$: Weak König’s Lemma
- $\text{ACA}_0$: Arithmetic Comprehension Axiom
- $\text{ATR}_0$: Arithmetic Transfinite Recursion
- $\Pi^1_1-\text{CA}_0$: $\Pi^1_1$–Comprehension Axiom
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- $\text{ADS}$ : Ascending-Descending Chain Principle
The “Big Five:”

- RCA\(_0\) : Recursive Comprehension Axiom
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- ACA\(_0\) : Arithmetic Comprehension Axiom
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- \(\Pi^1_1\)-CA\(_0\) : \(\Pi^1_1\)-Comprehension Axiom

- ADS : Ascending-Descending Chain Principle
- 2 – MLR : Existence of 2-Random sets
Reverse Mathematics

The “Big Five:”

- RCA₀ : Recursive Comprehension Axiom
- WKL₀ : Weak König’s Lemma
- ACA₀ : Arithmetic Comprehension Axiom
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- \( \Pi^1_1 - CA_0 \) : \( \Pi^1_1 \) – Comprehension Axiom

- ADS : Ascending-Descending Chain Principle
- 2 – MLR : Existence of 2-Random sets
- COH : Cohesive set principle

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Reverse Mathematics

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- ADS: Ascending-Descending Chain Principle
- 2 – MLR: Existence of 2-Random sets
- COH: Cohesive set principle
- AMT: Atomic Model Theorem
The Tree Antichain Theorem

Definition

Let $T \subseteq 2^{<\mathbb{N}}$ be a tree. We say that $T$ is completely branching if for all $\sigma \in T$, $\sigma^+ = \{\sigma_0, \sigma_1\} \subset 2^{<\mathbb{N}}$, either

$$\sigma^+ \subset T \quad \text{or} \quad \sigma^+ \cap T = \emptyset.$$
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TAC (Tree Antichain Theorem)

Every infinite completely branching computably enumerable tree $T \subseteq 2^{\mathbb{N}}$ contains an infinite antichain.
The Tree Antichain Theorem

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TAC (Tree Antichain Theorem)

Every infinite completely branching computably enumerable tree $T \subseteq 2^{\mathbb{N}}$ contains an infinite antichain.

TAC (Tree Antichain Theorem–Equivalent Version)

Every infinite tree $T \subseteq 2^{\mathbb{N}}$ with no terminal nodes and infinitely many splittings has an infinite antichain.
Two Paths to TAC

Fact (RCA₀)

*TAC follows from each of 2-MLR and ADS (individually).*

Fact (RCA₀)

*TAC is restricted \( \Pi^1_2 \).*

Fact (RCA₀)

*TAC does not follow from WKL*

Corollary

*TAC is not equivalent to any other “known” subsystem of Second-Order Arithmetic.*
Definition

Let $R$ be a ring with multiplicative identity $1_R$.

- We say that ideals $I, J \subseteq R$ are coprime whenever $I + J = R$, i.e. $1_R \in I + J$. 

Theorem A

If $R$ has infinitely many coprime minimal primes, then $R$ is not Noetherian.

Theorem B

If $R$ has infinitely many uniformly coprime minimal primes, then $R$ is not Noetherian.
Primary Decomposition for Restricted Classes of Rings

**Definition**

Let $R$ be a ring with multiplicative identity $1_R$.

- We say that ideals $I, J \subseteq R$ are **coprime** whenever $I + J = R$, i.e. $1_R \in I + J$.

- We say that ideals $I, J \subseteq R$ are **uniformly coprime** if for all $x \in I \cap J$ there exist $y \in I$, $z \in J$, and $a, b \in R$ such that $x = yz$ and $ay + bz = 1_R$. 

**Theorem A**

If $R$ has infinitely many coprime minimal primes, then $R$ is not Noetherian.

**Theorem B**

If $R$ has infinitely many uniformly coprime minimal primes, then $R$ is not Noetherian.

The Reverse Mathematics of Noether’s Decomposition Lemma
### Definition

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Algebraic Characterizations of TAC

Theorem (RCA_0 + BΣ_2)

*Theorem B is equivalent to TAC.*
Algebraic Characterizations of TAC

Theorem (RCA$_0 + B\Sigma_2$)

Theorem B is equivalent to TAC.

Conjecture (RCA$_0 + B\Sigma_2$)

Theorem A is equivalent to TAC.
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1. every $\sigma \in T$ corresponds to some (zero-divisor) $x_\sigma \in R$;
2. $\prod_{\sigma \in S} x_\sigma = 0_R$ whenever $S$ covers $2^{\mathbb{N}}$;
3. paths in $T$ correspond to annihilator ideals;
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- paths in \( T \) correspond to annihilator ideals;
- maximal paths correspond to maximal annihilator (hence minimal prime) ideals.

If \( \{\alpha_i : i \in \mathbb{N}\} \) is an infinite \( T \)—antichain, and

\[
I_N = \text{Ann}(\prod_{i=1}^{N} x_{\alpha_i}),
\]

then

\[
l_0 \subset l_1 \subset l_2 \cdots \subset l_N \subset \cdots.
\]
Given infinite $\Sigma^0_1$ completely branching $T \subseteq 2^{<\mathbb{N}}$. Construct $R$ via:

- $R$ is a quotient of $\mathbb{Q}[X_\sigma : \sigma \in T]$ such that
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  - $X_\emptyset = 0 \in R$,
  - $X_{\sigma_0}X_{\sigma_1} = X_\sigma$, and
  - inverses for all polynomials such that the intersection of the partial $2^{<\mathbb{N}}$ coverings yielded by the monomials is empty.

$R$ is a PIR; every ideal $I \subset R$ is generated by a monomial.

Given an infinite strictly ascending $R$−chain, one can effectively find a principle generator for each ideal in the chain and use $B\Sigma^2$ along with the sequence of exponents of these generators to build an infinite antichain of $T$ in the context.
Given infinite $\Sigma_1^0$ completely branching $T \subseteq 2^{\mathbb{N}}$.

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Over RCA$_0$ we have that TAC $\rightarrow$ Theorem B.
Over $\text{RCA}_0$ we have that $\text{TAC} \rightarrow \text{Theorem B}$. The converse follows from $\text{RCA}_0 + \text{B}\Sigma_2$. 

**Definition ($\text{RCA}_0$)**

For each $n \in \mathbb{N}$, let $n^{-\text{TAC}}$ be the principle that says “for every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with infinitely many splittings, there is a (path-)nonincreasing $f_T: T \rightarrow \mathbb{N}$ such that:

- $f_\emptyset = n$;
- there exist infinitely many $\sigma \in T$ and $i_\sigma \in \{0, 1\}$ such that:
  
  $f(\sigma) > f(\sigma i_\sigma)$.
First-Order Considerations

Over RCA$_0$ we have that TAC $\rightarrow$ Theorem B. The converse follows from RCA$_0 + B\Sigma_2$.

**Definition (RCA$_0$)**

For each $n \in \mathbb{N}$, let $n$–TAC be the principle that says “for every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with infinitely many splittings, there is a (path-)nonincreasing $f_T : T \rightarrow \mathbb{N}$ such that:

- $f_T(\emptyset) = n$;
- there exist infinitely many $\sigma \in T$ and $i_{\sigma} \in \{0, 1\}$ such that:
  \[ f_T(\sigma) > f_T(\sigma_{i_{\sigma}}). \]
First-Order Considerations

Over RCA$_0$ we have that TAC $\rightarrow$ Theorem B. The converse follows from RCA$_0$+$\Sigma_2$.

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For each $n \in \mathbb{N}$, let $n$–TAC be the principle that says “for every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with infinitely many splittings, there is a (path-)nonincreasing $f_T : T \rightarrow \mathbb{N}$ such that:

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  \[
  f(\sigma) > f(\sigma i_\sigma).
  \]
Over RCA₀ we have that TAC $\rightarrow$ Theorem B. The converse follows from RCA₀+$B\Sigma_2$.

**Definition (RCA₀)**

For each $n \in \mathbb{N}$, let $n$–TAC be the principle that says “for every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with infinitely many splittings, there is a (path-)nonincreasing $f_T : T \rightarrow \mathbb{N}$ such that:

- $f(\emptyset) = n$;
- there exist infinitely many $\sigma \in T$ and $i_\sigma \in \{0, 1\}$ such that:
  \[
  f(\sigma) > f(\sigma i_\sigma).
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TAC is equivalent to 1-TAC.
Over RCA\(_0\) we have that TAC $\rightarrow$ Theorem B.
The converse follows from RCA\(_0\)+B\(\Sigma_2\).

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For each $n \in \mathbb{N}$, let $n$–TAC be the principle that says “for every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with infinitely many splittings, there is a (path-)nonincreasing $f_T : T \rightarrow \mathbb{N}$ such that:

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TAC is equivalent to 1-TAC. Let WTAC be $n$–TAC without the $n$. 
Over RCA$_0$ we have that TAC $\rightarrow$ Theorem B. The converse follows from RCA$_0 + \text{B}\Sigma_2$.

**Definition (RCA$_0$)**

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TAC is equivalent to 1-TAC. Let WTAC be $n$–TAC without the $n$.

TAC $\rightarrow$ Theorem B $\rightarrow$ WTAC, over RCA$_0$. 
Over $\text{RCA}_0$ we have that $\text{TAC} \rightarrow \text{Theorem B}$. The converse follows from $\text{RCA}_0 + \text{B}\Sigma_2$.

**Definition ($\text{RCA}_0$)**

For each $n \in \mathbb{N}$, let $n-\text{TAC}$ be the principle that says “for every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with infinitely many splittings, there is a (path-)nonincreasing $f_T : T \rightarrow \mathbb{N}$ such that:

- $f(\emptyset) = n$;
- there exist infinitely many $\sigma \in T$ and $i_\sigma \in \{0, 1\}$ such that:
  
  $$f(\sigma) > f(\sigma i_\sigma).$$

$\text{TAC}$ is equivalent to $1-\text{TAC}$. Let $\text{WTAC}$ be $n-\text{TAC}$ without the $n$.

$\text{TAC} \rightarrow \text{Theorem B} \rightarrow \text{WTAC}$, over $\text{RCA}_0$.

$\text{TAC} \leftrightarrow \text{Theorem A/B} \leftrightarrow \text{WTAC}$, over $\text{RCA}_0 + \text{B}\Sigma$. 
Over RCA\(_0\) we have that TAC \(\rightarrow\) Theorem B. The converse follows from RCA\(_0\)+B\(\Sigma_2\).

**Definition (RCA\(_0\))**

For each \(n \in \mathbb{N}\), let \(n-\text{TAC}\) be the principle that says “for every infinite tree \(T \subseteq 2^{<\mathbb{N}}\) with infinitely many splittings, there is a (path-)nonincreasing \(f_T : T \rightarrow \mathbb{N}\) such that:

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\[
    f(\sigma) > f(\sigma i_\sigma).
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TAC is equivalent to 1-TAC. Let WTAC be \(n-\text{TAC}\) without the \(n\).

\[
    \text{TAC} \rightarrow \text{Theorem B} \rightarrow \text{WTAC, over RCA}_0.
\]

\[
    \text{TAC} \leftrightarrow \text{Theorem A/B} \leftrightarrow \text{WTAC, over RCA}_0 + B\Sigma.
\]

Q: What is the first order part of \(n-\text{TAC}, \text{WTAC}\)?
Consequences of the Hilbert Basis Theorem: 
The Krull Intersection Theorem

Theorem (Krull Intersection Theorem; KIT)

If $R$ is an integral domain, $I \subset R$ an ideal, then

$$\bigcap_{n \in \mathbb{N}} I^n = 0_R.$$  

Theorem (RCA$_0$, Conidis (2021))

KIT implies WKL$_0$.  

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The Primary Decomposition Lemma

We need to use infinite combinatorial structures (graphs) that are more general than trees and include (undirected) cycles.
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Theorem

*The Primary Decomposition Lemma follows from $CAC + WKL_0$.*
The Primary Decomposition Lemma

We need to use infinite combinatorial structures (graphs) that are more general than trees and include (undirected) cycles.

**Theorem**

*The Primary Decomposition Lemma follows from $CAC + WKL_0$.***

**Lemma ($\text{RCA}_0$)**

- *If $R$ is Noetherian, then the nilradical $N \subset R$ exists and $N^n = 0_R$, for some $n \in \mathbb{N}$.***
We need to use infinite combinatorial structures (graphs) that are more general than trees and include (undirected) cycles.

**Theorem**

*The Primary Decomposition Lemma follows from CAC + WKL$_0$.*

**Lemma (RCA$_0$)**

- *If $R$ is Noetherian, then the nilradical $N \subset R$ exists and $N^n = 0_R$, for some $n \in \mathbb{N}$.*
- *PDL implies KIT (and thus WKL$_0$).*
The Primary Decomposition Lemma

We need to use infinite combinatorial structures (graphs) that are more general than trees and include (undirected) cycles.

**Theorem**

*The Primary Decomposition Lemma follows from CAC$\vdash$WKL$^0$.***

**Lemma (RCA$_0$)**

- *If $R$ is Noetherian, then the nilradical $N \subset R$ exists and $N^n = 0_R$, for some $n \in \mathbb{N}$.***
- *PDL implies KIT (and thus WKL$^0$).***

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The Reverse Mathematics of Noether’s Decomposition Lemma
The Primary Decomposition Lemma

We need to use infinite combinatorial structures (graphs) that are more general than trees and include (undirected) cycles.

**Theorem**

*The Primary Decomposition Lemma follows from CAC + WKL₀.*

**Lemma (RCA₀)**

- If \( R \) is Noetherian, then the nilradical \( N \subset R \) exists and \( N^n = 0_R \), for some \( n \in \mathbb{N} \).
- \( PDL \) implies \( KIT \) (and thus \( WKL₀ \)).

**Conjecture (RCA₀)**

*The Primary Decomposition Lemma implies:*

- \( KIT \); *(Milne’s Lecture Notes; online)*
- \( WKL₀ \);
- \( TAC + WKL₀ \).*
Thank You!