The following problem uses the ideas in the proof of Lindström’s Theorem. For reference, the relevant section of Ebbinghaus, Flum, and Thomas is on the course website.

Let $\psi$ be a first-order sentence such that $\text{Mod}(\psi)$ is closed under substructures. (I.e., if $\mathcal{A} \models \psi$ and $\mathcal{B}$ is a substructure of $\mathcal{A}$, then $\mathcal{B} \models \psi$.) Show that $\psi$ is logically equivalent to a universal sentence. [It is easy to see that the converse is also true.]

Hint: We can assume that the language of $\psi$ is finite and relational. First define a notion of “$m$-embeddability” of a structure $\mathcal{B}$ in another structure $\mathcal{A}$, akin to the notion of $m$-isomorphism used in the proof of Lindström’s Theorem. Then show that for each $m$ and $\mathcal{A}$, this notion can be captured by a universal formula $\theta^m_{\mathcal{A}}$. (I.e., $\mathcal{B}$ is $m$-embeddable in $\mathcal{A}$ iff $\mathcal{B} \models \theta^m_{\mathcal{A}}$.) Next, show that there is a universal first order formula expressing the disjunction of $\theta^m_{\mathcal{A}}$ over all models $\mathcal{A}$ of $\psi$. Assuming that $\psi$ is not logically equivalent to a universal sentence, proceed as in the proof of Lindström’s Theorem to find $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \models \psi$ and $\mathcal{B} \models \neg \psi$, but $\mathcal{B}$ is embeddable in $\mathcal{A}$. 