1. Show that there is a representable relation \( S_{bl} \) such that for any formula \( \alpha \), variable \( x \), and term \( t \), we have \( S_{bl}(\downarrow \alpha \downarrow, \downarrow x \downarrow, \downarrow t \downarrow) \) if and only if \( t \) is substitutable for \( x \) in \( \alpha \). (I have included below the page in Enderton where substitutability is defined.) In this and the next problem, do not use the fact that every computable relation is representable.

2. Show that there is a representable relation \( Tr \) such that for any formula \( \alpha \) and any \( v \) that encodes a truth assignment for \( \alpha \), we have \( Tr(\downarrow \alpha \downarrow, v) \) if and only if this truth assignment satisfies \( \alpha \).

3. Give an example of a representable unary relation \( R \) such that \( D = \{ n : n \text{ divides some element of } R \} \) is not representable. (Justify your answer.) Show that \( D \) is nevertheless definable in \( \mathcal{R} = (\mathbb{N}; 0, S, <, +, \cdot, E) \).

4. Let \( T = \downarrow \text{Cn}(A_E) \downarrow = \{ \downarrow \varphi \downarrow \mid A_E \vdash \varphi \} \) and \( F = \{ \downarrow \varphi \downarrow \mid A_E \vdash \neg \varphi \} \). For parts a-c, suppose that we could compute a set \( A \) such that \( T \subseteq A \) and \( F \cap A = \emptyset \).

   a. Show that then, for each pair of disjoint computably enumerable sets \( B \) and \( C \), we could compute a set \( D \) such that \( B \subseteq D \) and \( C \cap D = \emptyset \).

   b. Show that then, for every computable set of axioms \( S \) in a computable language we could compute a set \( E \) such that \( \{ \downarrow \varphi \downarrow \mid S \vdash \varphi \} \subseteq E \) and \( \{ \downarrow \varphi \downarrow \mid S \vdash \neg \varphi \} \cap E = \emptyset \).

   c. Show that then, for every computable infinite binary tree \( V \), we could compute an infinite path of \( V \).

   d. Show that there is a computable infinite binary tree \( V \) such that if we could compute an infinite path on \( V \), then we could compute an \( A \) such that \( T \subseteq A \) and \( F \cap A = \emptyset \).
obtained. Although it is tempting (and in some ways more elegant) to define a deduction to be such a tree, it will be simpler to take deductions to be the linear sequences obtained by squashing such trees into straight lines.

Now at last we give the set $\Lambda$ of logical axioms. These are arranged in six groups. Say that a wff $\varphi$ is a generalization of $\psi$ if for some $n \geq 0$ and some variables $x_1, \ldots, x_n$,

$$\varphi = \forall x_1 \cdots \forall x_n \psi.$$  

We include the case $n = 0$; any wff is a generalization of itself. The logical axioms are then all generalizations of wffs of the following forms, where $x$ and $y$ are variables and $\alpha$ and $\beta$ are wffs:

1. Tautologies;
2. $\forall x \alpha \rightarrow \alpha^x_\gamma$, where $t$ is substitutable for $x$ in $\alpha$;
3. $\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x \alpha \rightarrow \forall x \beta)$;
4. $\alpha \rightarrow \forall x \alpha$, where $x$ does not occur free in $\alpha$.

And if the language includes equality, then we add

5. $x = x$;
6. $x = y \rightarrow (\alpha \rightarrow \alpha')$, where $\alpha$ is atomic and $\alpha'$ is obtained from $\alpha$ by replacing $x$ in zero or more (but not necessarily all) places by $y$.

For the most part groups 3–6 are self-explanatory; we will see various examples later. Groups 1 and 2 require explanation. But first we should admit that the above list of logical axioms may not appear very natural. Later it will be possible to see where each of the six groups originated.

**Substitution**

In axiom group 2 we find

$$\forall x \alpha \rightarrow \alpha^x_\gamma.$$  

Here $\alpha^x_\gamma$ is the expression obtained from the formula $\alpha$ by replacing the variable $x$, wherever it occurs free in $\alpha$, by the term $t$. This concept can also be (and for us officially is) defined by recursion:

1. For atomic $\alpha$, $\alpha^x_\gamma$ is the expression obtained from $\alpha$ by replacing the variable $x$ by $t$. (This is elaborated upon in Exercise 1. Note that $\alpha^x_\gamma$ is itself a formula.)
2. $(-\alpha)^x_\gamma = (-\alpha^x_\gamma)$.
3. $(\alpha \rightarrow \beta)^x_\gamma = (\alpha^x_\gamma \rightarrow \beta^x_\gamma)$.
4. $$(\forall x \alpha)^x_\gamma = \begin{cases} \forall y \alpha & \text{if } x = y, \\ \forall y (\alpha^x_\gamma) & \text{if } x \neq y. \end{cases}$$