1. Give an example of a representable $R \subseteq \mathbb{N}$ such that

$$D = \{ n : n \text{ divides some element of } R \}$$

is not representable. (Justify your answer.) Show that $D$ is nevertheless definable in $\mathfrak{N} = (\mathbb{N}; 0, S, <, +, \cdot, E)$.

2. Let $T$ be a decidable consistent theory (in an effective language). Show that there is a complete decidable consistent theory $C \supseteq T$. [Hint: Recall that theories are deductively closed. List the sentences $\sigma_0, \sigma_1, \ldots$ in the language and for each sentence in turn, add it or its negation to $C$, in such a way as to maintain consistency and decidability.]

3. How many complete extensions does $Cn A_E$ have? Justify your answer.

4. Let $A$ be a computable set of sentences in a computable language that includes 0 and $S$. Suppose that every computable relation is representable in $Cn A$, that every computable function if functionally representable in $Cn A$, and that $A$ is $\omega$-consistent, which means that if $A \vDash \neg \varphi(S^n0)$ for all $n$ then $A \not\vDash \exists x \varphi(x)$. Show how to obtain a sentence $\sigma$ such that $A \not\vDash \sigma$ and $A \not\vDash \neg \sigma$. (This $\sigma$ should “say” that it is not a theorem of $A$. Note that $A$ does not necessarily include $A_E$, nor is there any requirement that the elements of $A$ be true in the natural numbers.)