Math 27800 / CS 27800, Winter 2024: Assignment 6
Due Friday, February 23rd

Problem 2 is a "take-home exam" problem, to be worked out individually.

1. Let T be a theory in a language that includes 0 and S. Suppose that the unary relation R is representable in T, and so are (the graphs of the) unary functions g and h. Show that the graph of

$$f(a) = \begin{cases} g(a) & \text{if } a \in R \\ h(a) & \text{if } a \notin R \end{cases}$$

is also representable in T.

2 (take-home exam problem). Show that the Halting Problem is definable in the natural numbers $\mathfrak{N}$ but not representable in any axiomatizable subtheory of $\mathrm{Th}(\mathfrak{N})$. Explain how this shows that $\mathrm{Th}(\mathfrak{N})$ is not axiomatizable.

3. Let W be a noncomputable, computably enumerable set (such as the Halting Problem). Then W is the domain of some partial computable function, so there is a Turing machine M such that $n \in W$ iff M halts on input n. Let

$$R = \{(n, s) : M \text{ halts on input } n \text{ in at most } s \text{ many steps}\}.$$

Let $W_s = \{n : (n, s) \in R\}$. Notice that the W_s's are uniformly computable, $W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots$, and $W = \bigcup_s W_s$.

Let $f(n)$ be the least s such that for every $k < n$, if $k \in W$ then $k \in W_s$. Let G be the graph of f, i.e. $\{(n, f(n)) : n \in \mathbb{N}\}$.

a. Show that G does not contain any infinite computably enumerable subsets.

b. Show that G is Π^0_1-definable but not Σ^0_1-definable in $\mathfrak{N}$.

c. Let $\theta(x, y)$ define G in $\mathfrak{N}$. Let $\Sigma \subset \mathrm{Th}(\mathfrak{N})$ be axiomatizable. Show that there are infinitely many pairs (n, s) such that $\mathfrak{N} \vDash \theta(S^n 0, S^s 0)$, but only finitely many such that $\Sigma \vdash \theta(S^n 0, S^s 0)$.

4. We will show in class that no consistent axiomatizable extension of Q is complete. Assuming this result, show that Q has continuum many completions (i.e., there are continuum many complete theories in the language of arithmetic that contain Q).