1. Show that the $<$ relation is not definable in $\mathcal{R}_S = (\mathbb{N}; 0, S)$.

2. Show that if $T$ is a consistent, $\omega$-complete theory (as defined in the previous assignment) in the language of $\mathcal{R} = (\mathbb{N}; 0, S, <, +, \cdot, E)$ and $A_E \subseteq T$, then $T = \text{Th} \mathcal{R}$.

3. Show that the set of sequence numbers is representable.

4. Let $R$ be a representable unary relation and let $g$ and $h$ be representable unary functions. Show that

$$ f(a) = \begin{cases} g(a) & \text{if } a \in R \\ h(a) & \text{if } a \notin R \end{cases} $$

is representable.

5. Show that the theory of the natural numbers has $2^{\aleph_0}$ many nonisomorphic countable models. [Hint: For each set of primes $A$, construct a model containing an element whose prime divisors are exactly the elements of $A$.]