1. Show that a theory is computably enumerable iff it is axiomatizable. [Hint: Any formula $\varphi$ is equivalent to $\varphi \land \varphi \land \cdots \land \varphi$.]

2. A theory $T$ in a language that includes 0 and $S$ is $\omega$-complete if for any formula $\varphi(x)$, if $\varphi(S^n0) \in T$ for every $n \in \mathbb{N}$, then $\forall x \varphi(x) \in T$. (Here the notation $S^n0$ means $S$ applied to 0 $n$ many times.) Let $L$ be a language whose symbols include the constant symbol 0, the unary function symbol $S$, and the ternary relation symbol $H$. Let $T$ be a theory in this language such that $T \vdash H(S^e0, S^x0, S^t0)$ iff the $e$th Turing machine on input $x$ runs for exactly $t$ many steps and then halts. Suppose that $T$ is axiomatizable and $\omega$-complete. Show that $T$ is not complete.

3. Working in a language with equality, countably many unary relation symbols $R_0, R_1, \ldots$, and no function or constant symbols, let $\Gamma$ consist of the following axioms:

$$\exists x (R_i x \land \forall y (R_i y \rightarrow y = x)) \quad \text{for each } i \in \omega$$

and

$$\forall x (R_i x \rightarrow \neg R_j x) \quad \text{for each } i, j \in \omega \text{ such that } i \neq j.$$ 

Let $T = Cn(\Gamma)$. Prove that $T$ admits elimination of quantifiers.

4. Let $\mathcal{A}$ be the directed graph consisting of exactly one $n$-cycle for each $n > 1$ (with all of the cycles disjoint, and no other vertices or edges than the ones in these cycles). Let $T$ be the theory of $\mathcal{A}$.

   a. Show that $T$ does not admit elimination of quantifiers.

   b. Show that $T$ is axiomatizable but not finitely axiomatizable. [You may use the following fact (Theorem 26H in Chapter 2 of Enderton): If $Cn \Sigma$ is finitely axiomatizable then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $Cn \Sigma_0 = Cn \Sigma$.]