1. Show that the Axiom of Choice and the Well-Ordering Principle are equivalent over ZF. (If you need a reminder of how transfinite recursion works, see Kunen, p. 25.)

2. Give an example of a complete theory \( T \) in a countable language such that there are exactly two models of \( T \) (up to isomorphism) of cardinality \( \aleph_1 \). Make sure to prove that \( T \) is complete.

3. This problem gives a technique for showing that the Ehrenfeucht theory with exactly three countable models given in class is complete. Recall that two structures in the same language are elementarily equivalent if they satisfy the same sentences.
   
a. Let \( T \) be a theory in a countable language such that \( T \) has no finite models and any two countable models of \( T \) are elementarily equivalent. Show that \( T \) is complete.

b. Show that for any language \( L \), two \( L \)-structures \( \mathcal{M} \) and \( \mathcal{N} \) are elementarily equivalent iff for every finite sublanguage \( L' \subseteq L \), the restrictions of \( \mathcal{M} \) and \( \mathcal{N} \) to \( L' \) are elementarily equivalent.

c. Recall the Ehrenfeucht theory: In the language with a binary relation symbol \( < \) and constant symbols \( c_0, c_1, \ldots \), let \( \Sigma \) consist of the theory of dense linear orders without endpoints together with sentences asserting that \( c_0 < c_1 < \cdots \), and let \( T = Cn(\Sigma) \). Show that \( T \) is complete.

4. Recall that a substructure \( \mathcal{M} \) of a structure \( \mathcal{N} \) is an elementary substructure, written \( \mathcal{M} \preceq \mathcal{N} \), if for all first-order formulas \( \varphi(x_1, \ldots, x_n) \) and all \( a_1, \ldots, a_n \in \mathcal{M} \), we have \( \mathcal{M} \models \varphi(a_1, \ldots, a_n) \) iff \( \mathcal{N} \models \varphi(a_1, \ldots, a_n) \).

Let \( \kappa \leq \lambda \) be infinite cardinals. Let \( L \) be a language of cardinality at most \( \kappa \) and \( \mathcal{M} \) be an \( L \)-structure of cardinality at most \( \kappa \). Let \( \varphi(x) \) be an \( L \)-formula such that \( \{a \in \mathcal{M} : \mathcal{M} \models \varphi(a)\} \) is infinite. Show that there is an \( L \)-structure \( \mathcal{N} \) such that \( \mathcal{M} \preceq \mathcal{N} \) and \( |\{b \in \mathcal{N} : \mathcal{N} \models \varphi(b)\}| = \lambda \).