Math 27800 / CS 27800, Winter 2024: Assignment 2

Due Friday, January 26th

Problems 3.c and 4.c are "take-home exam" problems, to be worked out individually, without consultation with other students or anyone else. It is fine to ask me clarifying questions about these problems. For all other problems, collaboration is encouraged, but each person should write their own solutions independently.

1. Use transfinite induction to prove that the Axiom of Choice implies the Well-Ordering Principle over ZF. (For the Axiom of Choice, you can use any of the versions given in Problem 5 of Assignment 1.)

2. Give an example of a complete theory T in a countable language such that there are exactly two models of T (up to isomorphism) of cardinality $\aleph_1$. Make sure to prove that T is complete.

3. This problem gives a technique for showing that the Ehrenfeucht theory with exactly three countable models given in class is complete. Recall that two structures in the same language are elementarily equivalent if they satisfy the same sentences.

- **a.** Let T be a theory in a countable language such that T has no finite models and any two countable models of T are elementarily equivalent. Show that T is complete.
- **b.** Show that for any language L, two L-structures $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent iff for every finite sublanguage $L' \subseteq L$, the restrictions of $\mathfrak{M}$ and $\mathfrak{N}$ to L' are elementarily equivalent.
- **c.** (take-home exam problem) Recall the Ehrenfeucht theory: In the language with a binary relation symbol $<$ and constant symbols $c_0, c_1, \ldots$, let Σ consist of the theory of dense linear orders without endpoints together with sentences asserting that $c_0 < c_1 < \cdots$, and let $T = \mathrm{Cn}(\Sigma)$. Show that T is complete.

4. For this problem, it is enough to give informal (but precise) descriptions of your algorithms. Let $\mathcal{L}$ be a language with finitely many relation symbols and constant symbols, and no function symbols.

a. Let Γ be a set of $\mathcal{L}$-sentences such that for every $\sigma \in \Gamma$, if σ is satisfiable then σ has a finite model. Give an algorithm for deciding, given $\sigma \in \Gamma$, whether σ is satisfiable.

b. A sentence is Σ_2 if it is of the form

$$\exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_n \theta,$$

where θ is quantifier-free. Give an algorithm for deciding, given a Σ_2 $\mathcal{L}$-sentence σ, whether σ is satisfiable.

c. (take-home exam problem) A sentence is Π_2 if it is of the form

$$\forall x_1 \ldots \forall x_m \exists y_1 \ldots \exists y_n \theta,$$

where θ is quantifier-free. Let T be a theory that has a finite axiomatization Γ such that every sentence in Γ is Σ_2. Give an algorithm for deciding, given a Π_2 $\mathcal{L}$-sentence σ, whether $T \vdash \sigma$.