Lecture 17
Second Fundamental Theorem of Calculus

Math 13200

If you were paying close attention last lecture, you may have been wondering to yourself if there was a slicker way to evaluate definite integrals. Specifically, is there a formulaic way to find \( C \)?

If you were paying any amount of attention, you will have noticed that the previous theorem we talked about had ‘First’ in the name, so there had better be a ‘Second.’ Indeed, the Second Fundamental Theorem gives an explicit formula for a definite integral (if you have an antiderivative at hand) by giving you the value \( C \) in terms of the antiderivative \( F \).

Statement of Second Fundamental Theorem of Calculus

**Theorem:** Suppose \( f(x) \) is integrable on \([a, b]\), and that \( F(x) \) is an antiderivative of \( f \). Then

\[
\int_a^b f(t) \, dt = F(b) - F(a)
\]

To save space, we will sometimes write this as

\[
\int_a^b f(t) \, dt = F(x)\bigg|_a^b
\]

The content of SFTC is that the constant \( C \) we determined is equal to \(-F(a)\). Let me explain why \( C = -F(a) \). Indeed, since both \( F(x) \) and

\[
G(x) = \int_a^x f(t) \, dt
\]

are antiderivatives of \( f(x) \), we conclude that

\[
\int_a^x f(t) \, dt = F(x) + C
\]

for some constant \( C \). Setting \( x = a \), we conclude that

\[
0 = \int_a^a f(t) \, dt = F(a) + C
\]
which implies $C = -F(a)$, as desired.

Remark: The second fundamental theorem is the most frequently used of the two theorems. It is insanely helpful at calculating integrals. However, the heart of the content is in the first fundamental theorem. I don’t expect that you would be able to rigorously prove FTFC - but I just gave the proof of STFC using the first.

Examples

Example: Evaluate the integral

$$\int_{0}^{\frac{3\pi}{4}} \cos 2x \, dx$$

Solution: Since $\frac{1}{2} \sin 2x$ is an antiderivative of $\cos 2x$, applying the SFTC gives us

$$\int_{0}^{\frac{3\pi}{4}} \cos 2x \, dx = \left[ \frac{1}{2} \sin 2x \right]_{0}^{\frac{3\pi}{4}} = -\frac{1}{2} - 0 = -\frac{1}{2}$$

Example: Evaluate the integral

$$\int_{1}^{2} (2x^3 + 4x) \, dx$$

Notation for Antiderivatives

I will now succumb to the book’s demands. Indeed, if any of you ever see a general antiderivative of $f(x)$ after this course, it will look like this:

$$\int f(x) \, dx$$

This actually sort of makes sense - if you already understand the fundamental theorems of calculus. Indeed, the general antiderivative of $f$ is the one that doesn’t care about the exact value of $C$ (i.e., the general antiderivative ‘includes’ all values of $C$). But a definite integral $\int_{a}^{b} f$ pins down that $C$ value - specifically, $-F(a)$. So getting rid of the limits of integration is analogous to letting $C$ be anything you want.

Summary:

$$A_x(f(x)) = \int f(x) \, dx = F(x) + C \quad \text{General antiderivatives/Indefinite integral}$$

$$\int_{a}^{b} f(x) \, dx \quad \text{Area cut out by } f \text{/ Definite integral}$$
where the definite integral happens to be a specific antiderivative (because of the fundamental theorems).

The definition of indefinite integral is unrelated to the area cut out by \( f \) (which is why this notation and name is unfortunate when you first learn it).

**Method of \( u \)-substitution (or as I like to call it, \( u \)-transformation)**

Let’s recall the chain rule, and see what it means for integration.

\[
D_x(F(g(x)) = F'(g(x)) \cdot g'(x)
\]

Thus, if \( F \) is an antiderivative of \( f \), then we have

\[
A_x(f(g(x)) \cdot g'(x)) = \int f(g(x)) \cdot g'(x) \, dx = F(g(x)) + C
\]

Now let’s use this for (definite) integrals:

**Proposition:** Suppose \( g \) has continuous derivative and \( f \) is continuous. Then

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

Here, we are thinking of \( u \) as \( g(x) \). (Of course \( u \) is a dummy variable, so it doesn’t matter what we think - but it gives good intuition.)

**Explanation:** Let \( F \) be an antiderivative of \( f \). Then we have

\[
\int_{g(a)}^{g(b)} f(u) \, du = F(u) \bigg|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))
\]

by SFTC. Further, since \( F(g(x)) \) is an antiderivative of \( f(g(x))g'(x) \), we have that

\[
\int_a^b f(g(x))g'(x) \, dx = \left[F(g(x))\right]_a^b = F(g(b)) - F(g(a))
\]

**How I think of \( u \)-substitution**

You should think of \( u \)-substitution as \( u \)-transformation. If \( u = g(x) \), we imagine that by inserting \( g(x) \) for \( u \), we’ve ‘transformed’ the \( u \)-axis into the \( x \)-axis. This transformation can be stretch/flip/double itself.

When we do this transformation, our Riemann sums get transformed too. Specifically, the heights changes from \( f(u) \) to \( f(g(x)) \), and the widths change from the change \( \Delta u \) in \( u \) to \( \Delta g(x) \). But the change in \( g \) is given by derivative! So the \( du \) turns into a \( g'(x) \, dx \).
Finally, the transformation also transforms the limits of integration - from $a$ to $b$ to $g(a)$ to $g(b)$.

**Example:** Evaluate

$$\int_{0}^{1} \frac{x + 1}{(x^2 + 2x + 6)^2} \, dx$$

**Solution:** Must identify $u = g(x)$. In this case, it is

$$g(x) = x^2 + 2x + 6$$

**Example:** Evaluate

$$\int_{\pi/2}^{\pi} \cos \sqrt{x} \, dx$$

**Solution:** Must identify $u = g(x)$. In this case, it is

$$g(x) = \sqrt{x}$$