1 Introduction

Random graphs have become an indispensable tool in combinatorics, appearing in a variety of proofs and applications throughout the field. In the typical random graph model, a graph on \( n \) vertices is constructed by independently including each of the possible \( \binom{n}{2} \) edges in the graph with some probability \( 0 < p < 1 \). This creates a probability distribution \( G(n, p) \) on the space of \( n \)-vertex graphs. In this paper, we will primarily be concerned with the case where \( p \) is constant, but \( p \) can be a function of \( n \) in general.

Instead of constructing random graphs in this fashion, one might be inclined to go in the other direction, looking at a deterministic graph \( G \) and asking whether it is “random”. Of course, this is an ill-formed question: any \( n \)-vertex graph has positive probability in the distribution \( G(n, p) \). To make this notion more precise, we can instead look at sequences of graphs \( (G_n) \), where the graph \( G_n \) has \( n \) vertices and \( n \to \infty \). Then we can ask whether the sequence \( (G_n) \) has certain properties that a random graph has almost surely, such as having an edge density that approaches \( p \). (We say that a random graph has a property almost surely if the probability that the property holds for a graph in \( G(n, p) \) approaches 1 as \( n \to \infty \).) It turns out that a number of these properties with sufficient strength are all equivalent; a few of them are described in section 3. A sequence of graphs satisfying any of them is called quasirandom, as it captures a certain amount of the behavior expected of random graphs while being fully deterministic.\(^1\)

The concept of quasirandomness can be extended beyond graphs to other combinatorial objects. The most obvious generalization is to hypergraphs, for which the situation is somewhat more complicated. As we will see in section 4, quasirandomness for graphs corresponds to one level of a more general hierarchy of quasirandomness properties for hypergraphs. In section 5, we will briefly describe the form quasirandomness takes in several other contexts, forming comparisons to graphs and hypergraphs along the way.

2 Notation

Given a graph \( G \), let \( V(G) \) and \( E(G) \) denote the vertex and edge sets of \( G \), respectively, and let \( |G| \) and \( e(G) \) denote their cardinalities. For a given vertex \( v \in V(G) \), the neighborhood \( N(v) \) of \( v \) is the set of vertices adjacent to \( v \), and \( \deg(v) = |N(v)| \) is the degree of \( v \). The graph \( \bar{G} \) is the complement of \( G \), whose set of edges is the complement of \( E(G) \).

Oftentimes, we will be interested in the number of occurrences of a smaller graph \( H \) in a graph \( G \). Let \( N_G(H) \) be the number of labeled, not necessarily induced copies of \( H \) in the graph \( G \). Equivalently, \( N_G(H) \) is the number of injective graph homomorphisms from \( H \) to \( G \). The quantity

\(^1\)In earlier literature, the terms quasirandom and pseudorandom are both used. More recently, quasirandom has become preferred to avoid ambiguity.
$N_G^*(H)$ is defined in the same way, but we require that the copies of $H$ are induced by the vertices of $G$.

3 Quasirandom Graphs

3.1 Definitions

The defining characteristic of a random graph $G$ in $G(n,p)$ is its expected edge distribution: we would anticipate that the edges are evenly distributed throughout the graph, in some sense. That is, for large $n$ the entire graph $G$ should have about $p$ fraction of its possible edges, and the same thing should hold if we restrict $G$ to a large subset of its vertices. This intuition is made concrete by the following theorem [Bol01,KS06]:

**Theorem 3.1.** Let $p \leq 0.99$ and $G \in G(n,p)$ be a random graph. Almost surely, any induced subgraph $H$ of $G$ satisfies

$$\left| e(H) - p \left( \binom{|H|}{2} \right) \right| = O \left( |H|^{3/2} p^{1/2} \log \left( \frac{2n}{|H|} \right)^{1/2} \right).$$

Thomason introduced the first definition to quantify how random a graph appears to be, the $(p,\alpha)$-jumbled graph, based on this property [Tho87]:

**Definition 3.2.** Let $0 < p < 1$ and $\alpha \geq 1$. A graph $G$ is $(p,\alpha)$-jumbled if for every induced subgraph $H$ of $G$,

$$\left| e(H) - p \left( \binom{|H|}{2} \right) \right| \leq \alpha |H|$$

Comparing Theorem 3.1 and Definition 3.2, we see that a random graph $G(n,p)$ is almost surely $(p,O(\sqrt{n}))$-jumbled. Note that since $H$ contains at most $\binom{|H|}{2}$ edges, any graph is $(p,O(n))$-jumbled for any $p$. Therefore, the nontrivial case of a graph being $(p,o(n))$-jumbled seems like a reasonable criterion for quasirandomness, and it would be illuminating to investigate what other graph properties follow from it.

In [CGW89], Chung, Graham, and Wilson did just that. They defined the following seven conditions for a sequence of graphs to be quasirandom, all of which hold almost surely for a random graph in $G(n,p)$, and showed that they are equivalent. Condition $P_4$ says that $G_n$ is $(p,o(n))$-jumbled.

**Theorem 3.3.** Let $(G_n)$ be a sequence of graphs and $0 < p < 1$. The following are equivalent:

- **$P_1(s)$**: For all graphs $L$ on $s \geq 4$ vertices,
  $$N_{G_n}^s(L) = (1 + o(1))p^{|E(L)|} \left( 1 - p \right)^{\binom{s}{2} - |E(L)|} n^s$$

---

2The little-o error terms in each property can be explicitly computed in terms of each other. If we know that some sequence $(G_n)$ satisfies property $P_i$ with error term $f(n)$, then it also satisfies property $P_j$ with an error term $g(n)$ that depends only on $f$. 

2
• $P_2(t)$: For an even $t \geq 4$,

$$e(G_n) = \frac{pn^2}{2} + o(n^2)$$

and $N_{G_n}(C_t) \leq p^t n^t + o(n^t)$.

• $P_3$: Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of $G_n$. Then $e(G_n) \geq \frac{pn^2}{2} + o(n^2)$, $\lambda_1 = (1 + o(1))pn$, and $\lambda_2 = o(n)$.

• $P_4$: For any induced subgraph $H$ of $G_n$, $e(H) = \frac{p}{2} |H|^2 + o(n^2)$.

• $P_5$: For any induced subgraph $H$ of $G_n$ with $|H| = \lfloor \frac{n}{2} \rfloor$, $e(H) = \left( \frac{p}{8} + o(1) \right) n^2$.

• $P_6$: For $u, v \in V(G_n)$, let $s(u, v)$ be the number of vertices in $G_n$ that are either adjacent to both $u$ and $v$ or adjacent to neither $u$ nor $v$. Then $\sum_{u, v \in V(G_n)} |s(u, v) - (p^2 + (1 - p)^2)n| = o(n^3)$.

• $P_7$: $\sum_{u, v \in V(G_n)} ||N(u) \cap N(v)|| - p^2 n| = o(n^3)$

If $(G_n)$ satisfies these properties, we say it is quasirandom.

Perhaps the most striking of these conditions is $P_2(4)$, which says that a sequence of graphs is quasirandom if and only if it has approximately the same number of edges and at most the number of 4-cycles as expected in $G(n, p)$. This is enough to imply $P_1(s)$ for all $s$, which essentially says that the number of copies of every graph in $G_n$ is approximately what we would expect from a random graph. The restriction that $P_2(t)$ must hold for an even cycle is necessary. To see that $P_2(3)$ does not imply quasirandomness, consider a graph $G_n$ on $n$ vertices which is a complete tripartite graph with $\frac{n}{3}$ vertices in each part. The number of edges in $G_n$ is $\frac{1}{2}n^2$, and the number of labeled 3-cycles in $G_n$ is $\frac{2}{3}n^3$, so $(G_n)$ satisfies property $P_2(3)$ with $p = \frac{2}{3}$. However, $G_n$ contains an independent set of size $\frac{1}{3}n$, meaning it does not satisfy $P_4$ and the sequence is not quasirandom. A more general construction works for any odd $t$.

This shows that not every property satisfied by a random graph almost surely is enough to guarantee quasirandomness. Conversely, quasirandom graphs need not possess every property that a random graph almost surely does. For instance, the Paley graph (see section 3.2) can have a largest clique of size greater than the largest clique size of $(1 + o(1)) \log_2(n)$ expected of a random graph in $G(n, \frac{1}{2})$.

One issue with the property $P_2(t)$ that forces it to be restricted to even $t$ is that it is not hereditary—induced subgraphs of $G_n$ need not satisfy $P_2(t)$. The list of equivalent quasirandom properties was extended by Simonovits and Sós in [SS97] to include the following much cleaner hereditary property, which holds for any graph $L$, not just even cycles:

**Theorem 3.4.** Let $(G_n)$ be a sequence of graphs, $L$ be a fixed graph on $s$ vertices, and $0 < p < 1$. If, for any induced subgraph $H \subseteq G_n$ on $h$ vertices,

$$N_H(L) = p^{|E(L)|} h^s + o(n^s)$$

then $(G_n)$ is quasirandom.

Interestingly, this theorem does not hold without further restrictions if we replace the number of not necessarily induced copies $N_H(L)$ with the number of induced copies $N^*_H(L)$ [SS03].
3.2 Examples

A simple example of a quasirandom graph sequence is the sequence of even intersection graphs $G_{2n}$, defined by taking the vertices of $G_{2n}$ to be the subsets of $[n]$, and placing an edge between two vertices $X$ and $Y$ if and only if $|X \cap Y| \equiv 0 \pmod{2}$. A straightforward counting argument shows that for any two vertices $X$ and $Y$, $s(X,Y) = 2^{n-1} - c$, where $c = 0, 1,$ or $2$. This implies $(G_{2n})$ satisfies property $P_6$ with $p = \frac{1}{2}$, so it is quasirandom [CG90].

Another example is the Paley graph, constructed as follows. Let $q$ be a prime number with $q \equiv 1 \pmod{4}$. Define a graph $G_q$ on the vertex set $\mathbb{Z}_q$ (integers mod $q$) by placing an edge between $x$ and $y$ if and only if $x - y$ is a quadratic residue of $q$. The sequence of graphs $(G_q)$ is also quasirandom with $p = \frac{1}{2}$ [CGW89]. To prove this, observe from the properties of quadratic residues that $s(x,y) = \frac{1}{2}(q-3)$ and $G_q$ satisfies $P_0$.

Many more examples of quasirandom graphs appear in [KS06], arising from such diverse areas as number theory, group theory, and geometry.

3.3 Extensions

Thus far, we have only addressed the case where the edge density $p$ is constant. In the sparse case, with $p(n) = o(1)$, we do not necessarily expect to find any copies of large induced subgraphs in a random graph $G(n,p(n))$, which makes generalizing Theorem 3.3 difficult. However, Chung and Graham [CG02] showed that it is possible to find several implications between versions of the properties $P_1$–$P_7$, and showed they are equivalent if the sequence of graphs satisfies certain additional conditions.

Another method of generalizing quasirandom graphs is to base them on a different random graph model. In place of the distribution $G(n,p)$, we define a generalized random graph $G(n,H)$ for a complete weighted graph $H$, including loops at each vertex, with node weights $0 \leq \alpha_i \leq 1$ that sum to 1 and edge weights $0 \leq \beta_{ij} \leq 1$. To obtain a random graph in this model, we partition each of the $n$ vertices into sets $V_i$, where each vertex is independently placed in the set $V_i$ with probability $\alpha_i$. Then an edge is independently placed between each vertex $u \in V_i$ and $v \in V_j$ with probability $\beta_{ij}$.

Lovász and Sós [LS08] call a sequence of graphs $(G_n)$ $H$-quasirandom if the proportion of maps from any unweighted graph $L$ to $G_n$ that are graph homomorphisms approaches the weighted homomorphism count

$$\text{hom}(L,H) = \sum_{\psi: V(L) \to V(H)} \left( \prod_{v \in V(L)} \alpha_{\psi(v)} \prod_{uv \in E(L)} \beta_{\psi(u)\psi(v)} \right)$$

as $n \to \infty$. In fact, it suffices that this condition holds for all graphs $L$ with at most $|H| + (10|H|)^{|H|}$ vertices. They also show that a sequence of graphs is $H$-quasirandom if and only if the vertices of each graph can be partitioned into sets $V_i$, each containing a fraction of approximately $\alpha_i$ of
the vertices, such that each set $V_i$ induces a quasirandom graph with edge density $\beta_{ii}$, and that the edges between each pair of distinct sets $V_i, V_j$ form a quasirandom bipartite graph (a bipartite graph satisfying property $P_1$ for all bipartite graphs $L$) with edge density $\beta_{ij}$. Thus, we can think of an $H$-quasirandom graph sequence as being built up from quasirandom graph sequences with different edge densities.

4 Quasirandom Hypergraphs

4.1 Definitions

In this section, we will use the terms “hypergraph” and “$k$-hypergraph” to refer exclusively to $k$-uniform hypergraphs (where the edges are $k$-element subsets of the vertex set), and will focus on the case corresponding to a random hypergraph with edge probability $p = \frac{1}{2}$ to simplify some of the formulas.

The most obvious way to extend the concept of quasirandomness from ordinary graphs to hypergraphs is to take the properties $P_1$–$P_7$ of quasirandom graphs and try to generalize them to hypergraphs in a way that preserves their equivalence. In order to do so, we will need a few definitions.

The $k$-octahedron $O_k$ is a $k$-hypergraph on $2^k$ vertices $\{x_0^0, x_1^0, \ldots, x_{k-1}^0, x_0^1, x_1^1, \ldots, x_{k-1}^1\}$, with the $2^k$ edges $\{x_\alpha^1, x_\beta^2, \ldots, x_{k-1}^\gamma\}$, $\alpha, \beta \in \{0, 1\}$. For $k = 2$, an octahedron is a 4-cycle, which suggests that the importance of $k$-octahedra in quasirandom $k$-hypergraphs is analogous to the special role 4-cycles play in quasirandom graphs, as we will soon see. An even partial octahedron is a graph with the same vertex set as $O_k$ that has an even number of edges of $O_k$. Let $O^E_k$ denote the set of all even partial octahedra.

To generalize the function $s(u, v)$ appearing in $P_6$, we define the sameness $(k - 1)$-graph $G(u, v)$ of a hypergraph $G$. Fix $u, v \in V(G)$. Let $G(u, v)$ be the $(k - 1)$-hypergraph with vertex set $V(G) \setminus \{u, v\}$ and edges $e' \in \binom{V(G) \setminus \{u, v\}}{k-1}$ in $G(u, v)$ if and only if either both $e' \cup \{x\}$ and $e' \cup \{y\}$ are in $G$ or neither of these two edges is in $G$. For $k = 2$, the sameness “1-graph” $G(u, v)$ is simply the set of vertices counted by $s(u, v)$.

Chung and Graham obtained the following partial generalization of Theorem 3.3, thereby offering a definition of quasirandomness for hypergraphs [CG90]:

**Theorem 4.1.** Let $(G_n)$ be a sequence of $k$-hypergraphs. The following properties are equivalent:

- $Q_1(s)$: For all $k$-hypergraphs $L$ on $s \geq 2k$ vertices,
  $$N^*_G(L) = O((1 + o(1))2^{-\binom{k}{2}}n^s)$$

- $Q_2$: Let $N^*_G(O^E_k)$ be the total number of induced even partial octahedra in $G_n$. Then
  $$N^*_G(O^E_k) \leq \frac{1}{2}(1 + o(1))n^{2k}$$

- $Q_3$: For almost all $u, v \in V(G_n)$, the sameness $(k - 1)$-graph $G_n(u, v)$ is quasirandom, in the sense that it satisfies property $Q_1(2(k - 1))$. 

5
• \(Q_4:\) For almost all \(u,v \in V(G_n)\), the sameness \((k-1)\)-graph \(G_n(u,v)\) satisfies

\[
N_{G_n(u,v)}(K_r^{(k-1)}) = (1 + o(1))2^{-\binom{k}{r}n^r}
\]

for all \(1 \leq r \leq 2k - 1\).

The condition given in \(Q_4\) should not be mistaken for an additional quasirandomness condition. While it is true that almost all sameness \((k-1)\)-graphs are quasirandom if and only if almost all of them have the proper numbers of \(K_1^{(k-1)}\), \ldots, \(K_{2k-1}^{(k-1)}\), counting the occurrences of these complete graphs in a single hypergraph is not enough to determine whether it is quasirandom.

Absent from this list is the obvious generalization of the property \(P_4\), which says that for any induced subgraph \(H\) of \(G_n\), \(e(H) = \frac{1}{2}\binom{|H|}{2} + o(n^k)\). The reason for its absence is that it is not equivalent to any of the \(Q_i\) — it is possible to construct hypergraphs which satisfy this property, but are not quasirandom. One such hypergraph is adapted from an example of Gowers [Gow06]. Let \(G\) be a random graph in \(G(n, \frac{1}{2})\), and let \(G'\) be a 3-hypergraph whose edges are all of the triangles in \(G\). The probability of any set of three vertices being an edge in \(G'\) is \(\frac{1}{4}\), so the hypergraph induced by any subset of the vertices of \(G'\) of size \(\Theta(n)\) will almost surely have about half of its possible edges. However, \(G'\) contains no copies of a complete hypergraph on \(s\) vertices with one edge removed for any \(s \geq 4\), because a copy of \(K_s\) with one edge removed in the underlying graph \(G\) corresponds to a complete graph with two edges removed in \(G'\). This is a clear violation of \(Q_1(s)\), and hence \(G'\) is not quasirandom.

The notion of quasirandomness for hypergraphs can be extended from the binary condition of Theorem 4.1 to a hierarchy of classes of quasirandom \(k\)-hypergraphs \(\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_k\) described in [Chu90]. To define these, let \(G\) be a \(k\)-hypergraph, and define a function \(\mu_G : V(G)^k \to \{-1, 0, 1\}\) by

\[
\mu_G(v_1, \ldots, v_k) = \begin{cases} 0 & \text{if } v_i = v_j \text{ for some } i, j \\ -1 & \text{if } \{v_1, \ldots, v_k\} \in E(G) \\ 1 & \text{otherwise} \end{cases}
\]

Now take \(0 \leq i \leq k\), and define \(\Pi_G^{(i)} : V(G)^{k+i} \to \{-1, 0, 1\}\) by

\[
\Pi_G^{(i)}(u_1, u_2, \ldots, u_{2i}, v_{i+1}, \ldots, v_k) = \prod_{\alpha_1} \cdots \prod_{\alpha_i} \mu_G(\alpha_1, \ldots, \alpha_i, v_{i+1}, \ldots, v_k)
\]

where \(\alpha_j \in \{u_{2j-1}, u_{2j}\}\). Note that \(\Pi_G^{(k)}(u_1, \ldots, u_{2k}) = 1\) if the graph induced by the vertices \(u_1, \ldots, u_{2k}\) is an even partial octahedron, and is equal to \(-1\) if it’s an odd partial octahedron.

Finally, we can define the \(i\)-deviation of the hypergraph \(G\) by

\[
dev_i(G) = \frac{1}{n^{k+i}} \sum_{u_1, \ldots, u_{k+i} \in V(G)} \Pi_G^{(i)}(u_1, \ldots, u_{k+i})
\]

which takes values between \(-1\) and \(1\). The class \(\mathcal{A}_i\) is defined to be the set of all sequences of graphs \((G_n)\) with \(dev_i(G_n) = o(1)\). Therefore, a hypergraph is a member of the class \(\mathcal{A}_k\) if and only if it satisfies the property \(Q_2\), which is part of our earlier definition of quasirandomness. It is straightforward to show from the definition of \(i\)-deviation that if a sequence \((G_n)\) has \(dev_i(G_n) = o(1)\), it must also have \(dev_{i-1}(G_n) = o(1)\), so each class \(\mathcal{A}_i\) is a subset of the previous one.
Theorem 4.2. The classes of quasirandom $k$-hypergraphs form a chain of proper subsets: $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_k$.

Examples showing that all of these inclusions are proper will be discussed in the next section.

Chung [Chu90] also provides another property satisfied by graphs in the class $\mathcal{A}_2$. For a $k$-hypergraph $G$ and an $\ell$-hypergraph $H$ on the same vertex set with $\ell < k$, let the set $E(G,H)$ of edges of $G$ induced by $H$ be

$$E(G,H) = \left\{ e \in E(G) \mid \left( \begin{array}{c} \ell \\ 1 \end{array} \right) \subseteq E(H) \right\}$$

and take $e(G,H) = |E(G,H)|$. For $0 \leq i \leq k$, define the $i$-discrepancy of $G$ by

$$disc_i(G) = \max_{H \text{ an } (i-1)\text{-graph on } V(G)} \frac{|e(G,H) - e(\bar{G},H)|}{|G|^k}$$

Once again, it can be shown from this definition that if $disc_i(G_n) = o(1)$, then $disc_{i-1}(G_n) = o(1)$. The following theorem relates the $i$-deviation and the $i$-discrepancy:

**Theorem 4.3.** Let $(G_n)$ be a sequence of $k$-hypergraphs, and let $2 \leq i \leq k$. If $dev_i(G_n) = o(1)$ (so that $(G_n)$ is in the quasirandom class $\mathcal{A}_i$), then $disc_i(G_n) = o(1)$. More specifically, $disc_i(G_n) < dev_i(G_n)^{1/2}$.

The reverse implication does not hold unless $i = k$. The definition of discrepancy doesn’t apply to $i = 0$ or $i = 1$, but there are simple properties that are equivalent to membership in the class $\mathcal{A}_i$ in each of these cases. For $i = 0$, we have $G_n \in \mathcal{A}_0$ if and only if $e(G_n) = \binom{n}{2} + o(n)$. For $i = 1$, we have $G_n \in \mathcal{A}_1$ if and only if

$$\sum_{u_1, \ldots, u_k-1 \in V(G_n)} (|\{v \in V(G_n) \mid \{u_1, \ldots, u_k-1, v\} \in E(G)\}| - |\{v \in V(G_n) \mid \{u_1, \ldots, u_k-1, v\} \notin E(G)\}|)^2 = o(n^k+1)$$

which essentially says that the “degree” of almost all sets of $(k-1)$ vertices is close to $\frac{1}{2}n$. The $i = 2$ case is covered in Theorem 4.3, but merits further explanation. For $i = 2$, the number of edges $e(G,H)$ is simply the number of edges in the subgraph of $G$ induced by a set of vertices $H$. The condition $disc_2(G_n) = o(1)$ implies $e(H) = \frac{1}{2} \binom{n}{2} + o(n)$ for all induced subgraphs $H$, which is exactly the generalization of the condition $P_d$ discussed above. Of course, when $k = 2$, the set of graph sequences with $disc_2(G_n) = o(1)$ is the class $\mathcal{A}_2 = \mathcal{A}_k$, which explains why this property is one of the equivalent ones in Theorem 3.3 but not in Theorem 4.1.

As the lack of equivalence in Theorem 4.3 suggests, the hierarchy of quasirandom hypergraph classes $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_k$ is actually part of a much larger poset of quasirandom classes, the relations between which are described by Lenz and Mubayi [LM15]. For a $k$-hypergraph $G$, a proper partition $\pi = k_1 + \cdots + k_t$ of $k$, and sets $S_i \subseteq \binom{V(G)}{k_i}$ for $1 \leq i \leq t$, define

$$h(S_1, \ldots, S_t) = |\{(s_1, \ldots, s_t) \in S_1 \times \cdots \times S_t \mid s_1 \cup \cdots \cup s_t \in E(G)\}|$$

A sequence of $k$-hypergraphs $(G_n)$ has the property $Expand[\pi]$ if, for all $S_1, \ldots, S_t$,

$$h(S_1, \ldots, S_t) = \frac{1}{2} \prod_{i=1}^{t} |S_i| + o(n^k)$$
Letting all of the subsets $S_i$ be the same, we can see that $\text{Expand}[1 + \cdots + 1]$ is equivalent to $\text{disc}_2(G_n) = \Omega(1)$ by inclusion-exclusion. For the relationship between the different properties $\text{Expand}[\pi]$, Lenz and Mubayi show that $\text{Expand}[\pi]$ implies $\text{Expand}[\pi']$ if the partition $\pi'$ is a refinement of $\pi$. To do this, let $\pi$ be the partition $k_1 + \cdots + k_t$ and $\pi'$ be $m_1 + \cdots + m_r$, where $k_i = \sum_{\phi(j)=i} m_j$ for some surjection $\phi : [r] \to [t]$. Let $S'_1, \ldots, S'_r$ be disjoint sets, with $S'_i \subseteq (V(G_n))$. Then define $S_i = \{\cup_{\phi(j)=i} X_j \mid X_j \in S'_j\}$. Since each set $S_i$ is constructed by combining corresponding vertex sets under the refined partition $\pi'$, and the sets $S'_1, \ldots, S'_r$ are disjoint, we have

$$h(S'_1, \ldots, S'_r) = h(S_1, \ldots, S_t) = \frac{1}{2} \prod_{i=1}^{t} |S_i| + o(n^k) = \frac{1}{2} \prod_{i=1}^{r} |S'_i| + o(n^k)$$

The proof can be completed by showing that it suffices to consider disjoint subsets in the definition of $\text{Expand}[\pi]$.

Every hypergraph sequence in the class $\mathcal{A}_2$ satisfies $\text{Expand}[\pi]$ for each proper partition $\pi$, so all of the properties $\text{Expand}[\pi]$ lie between $\text{dev}_2(G_n) = \Omega(1)$ and $\text{disc}_2(G_n) = \Omega(1)$ in strength. Examining discrepancy instead of deviation, the property $\text{disc}_i(G_n) = \Omega(1)$ implies $\text{Expand}[k_1 + \cdots + k_t]$ if and only if $i - 1 \geq \max_{1 \leq i \leq t} k_i$. All of the implications between the quasirandom properties of hypergraphs we have discussed can be summarized in the following diagram, taking $k = 6$ as an example [LM15]. (Here we let $E[\pi]$ be short for $\text{Expand}[\pi]$ and $\text{disc}(i)$ denote the property $\text{disc}_i(G_n) = \Omega(1)$.)
4.2 Examples

For graphs in the smallest class $\mathcal{A}_k$, we can generalize the two examples of quasirandom graphs in Section 3.2 [Chu90]. The Paley $k$-hypergraph is defined on the vertex set $\mathbb{Z}_q$ by letting $\{x_1, \ldots, x_k\}$ be an edge of the graph if and only if $x_1 + \cdots + x_k$ is a quadratic residue of $q$. The even intersection graph can be generalized by taking the same vertex set of subsets of $[n]$, and letting $\{X_1, \ldots, X_k\}$ be an edge if and only if $|X_1 \cap \cdots \cap X_k| = 0 \pmod{2}$.

We will now construct a sequence of $k$-hypergraphs that is in the class $\mathcal{A}_i$, but not in the class $\mathcal{A}_{i+1}$, where $0 \leq i < k$. Let $H$ be an $i$-hypergraph in the class $\mathcal{A}_i$ on $n$ vertices (for instance, one of the two types of graphs described above). Define the $k$-hypergraph $G$ by taking $V(G) = V(H)$ and

$$E(G) = \{ e \in \binom{V(G)}{k} \mid \left| \binom{e}{i} \cap E(H) \right| \equiv 1 \pmod{2} \}$$

That is, we make each subset of $k$ vertices an edge of $G$ if and only if the number of edges of $H$ composed of vertices in that subset is odd. We will first show that $G \in \mathcal{A}_i$. By construction, for $k$ distinct vertices in $V(G)$, we have

$$\mu_G(\alpha_1, \ldots, \alpha_{i+1}, \ldots, v_k) = \prod_{\{z_1, \ldots, z_i\} \in \binom{[k]}{i}} \mu_H(z_1, \ldots, z_i)$$

The $i$-deviation of $G$ is

$$dev_i(G) = \frac{1}{n^{k+i}} \sum_{v_{i+1}, \ldots, v_k} \sum_{u_{i+1}, \ldots, u_{2i}} \prod_{\alpha_1} \cdots \prod_{\alpha_i} \prod_{\{z_1, \ldots, z_i\} \in \binom{[k]}{i}} \mu_H(z_1, \ldots, z_i)$$

If $\{z_1, \ldots, z_i\} \neq \{\alpha_1, \ldots, \alpha_i\}$, then the term $\mu_H(z_1, \ldots, z_i)$ will occur an even number of times in the product, so it will not affect its value. As $\mu_H(\alpha_1, \ldots, \alpha_i)$ is the only term that does affect the value, this simplifies to

$$dev_i(G) = \frac{1}{n^{k+i}} \sum_{v_{i+1}, \ldots, v_k} \sum_{u_{i+1}, \ldots, u_{2i}} \prod_{\alpha_1} \cdots \prod_{\alpha_i} \mu_H(\alpha_1, \ldots, \alpha_i)$$

$$= \frac{1}{n^{k+i}} \sum_{v_{i+1}, \ldots, v_k} n^{2i} dev_i(H)$$

As $H \in \mathcal{A}_i$, we have $dev_i(H) = o(1)$, and thus $dev_i(G) = o(1)$ as well. This gives us $G \in \mathcal{A}_i$. To show that $G \notin \mathcal{A}_{i+1}$, we will consider its $(i+1)$-discrepancy:

$$disc_{i+1}(G) \geq \frac{|e(G, H) - e(G, H)|}{n^k}$$

If $\binom{k}{i}$ is odd, then $e(G, H) = 0$, because every edge of $G$ must contain an even number of edges of $H$. The fact that $\binom{k}{i}$ is odd also ensures that each copy of the complete graph $K^i_k$ in $H$ corresponds to an edge in $G$. By quasirandomness of $H$,

$$e(G, H) = N^*_H(K^i_k) = (1 + o(1))2^{-\binom{k}{i}} n^k$$

which implies $disc_{i+1}(G) \geq (1 + o(1))2^{-\binom{k}{i}} > o(1)$. If $\binom{k}{i}$ is even, then $e(G, H) = 0$, and we can perform the same calculation using $e(G, H)$. Applying Theorem 4.3, we find that $G \notin \mathcal{A}_{i+1}$.
4.3 Extensions

An important application of quasirandom hypergraphs is in communication complexity. Suppose $k$ players with unlimited computational power wish to compute the value of a boolean function $f : V^k \rightarrow \{-1, 1\}$ for a set of inputs $(x_1, \ldots, x_k)$, and player $i$ knows all of the inputs except $x_i$. The players can communicate with each other by writing bits on a blackboard visible to all players. We can think of the function $f$ as an ordered $k$-hypergraph on the vertex set $V$ (or just a normal hypergraph if $f$ is symmetric), which lets us define the $i$-deviation and $i$-discrepancy for functions in the same way we did for hypergraphs. Then Theorem 4.3 applies to functions $f$ as well as to hypergraphs $G_n$ [CG90].

The communication complexity $C(f)$ of the function $f$ is defined as the minimum number of bits that must be communicated between the players to compute $f$ in the worst case. Babai, Nisan, and Szegedy [BNS89] proved that the communication complexity of any function $f$ is bounded below by

$$C(f) \geq \log \frac{1}{\text{disc}_k(f)}$$

Functions corresponding to quasirandom hypergraphs in the class $\mathcal{A}_k$ have $\text{disc}_k(f) = o(1)$, so they have high communication complexity. A generalized version of communication complexity was shown to have a lower bound of $\log(\frac{1}{\text{disc}_i(f)})$ for $2 \leq i \leq k$ by Chung and Tetali [CT93], which is unbounded for functions representing quasirandom hypergraphs in the class $\mathcal{A}_i$.

5 Other Quasirandom Objects

The study of quasirandomness isn’t limited to graphs and hypergraphs. Other objects that quasirandom properties have been defined for include subsets of $\mathbb{Z}_n$ [CG92], tournaments [CG91], oriented graphs [Gri13], and permutations [Coo04, KP13]. Each of these comes with a long list of equivalent properties that define quasirandomness, in the spirit of Theorem 3.3. Here, we will focus on those that relate to the graph and hypergraph properties already discussed.

Quasirandom subsets of $\mathbb{Z}_n$, in particular, are very closely related to quasirandom graphs. Perhaps the most intuitive property that defines a quasirandom subset $S \subseteq \mathbb{Z}_n$ is strong translation, which says that for any subset $T \subseteq \mathbb{Z}_n$ and almost all $x \in \mathbb{Z}_n$,

$$S \cap (T + x) = \frac{|S||T|}{n} + o(n)$$

A subset $S \subseteq \mathbb{Z}_n$ is quasirandom if and only if the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{x, y\} \in \binom{\mathbb{Z}_n}{2} \mid x + y \in S\}$ is quasirandom. This means that the quasirandom properties $P_1$–$P_7$ for this graph can be converted into quasirandom properties for the set $S$. In Section 3.2, we showed that the Paley graph is quasirandom, so the set of quadratic residues is a quasirandom subset of $\mathbb{Z}_q$ for primes $q \equiv 1 \pmod{4}$.

For tournaments, the list of quasirandom properties closely parallels Theorem 3.3. Roughly speaking, we can form analogues of the properties $P_1(s)$, $P_4$, $P_5$, $P_6$, and $P_7$ for tournaments by replacing “graph” with “tournament”, “edges containing a given vertex” with “edges directed away from a given vertex”, and “non-edges containing a given vertex” with “edges directed toward a given vertex”.

10
vertex”. For example, the version of $P_4$ for a tournament $T$ on $n$ vertices says that for any induced subtournament $T'$ of $T$,
\[ \sum_{v \in V(T')} |d^+_T(v) - d^-_T(v)| = o(n^2) \]
where $d^+_T(v)$ is the outdegree of $v$ in $T'$ and $d^-_T(v)$ is the indegree. These properties are all equivalent for tournaments, which we take to be the definition of quasirandomness.

A tournament is also quasirandom if it has the number of even 4-cycles (4-cycles with an even number of edges oriented in the same direction going around the cycle) expected in a random tournament, similar to property $P_2(4)$ for graphs. We can draw a direct link between quasirandom tournaments and quasirandom graphs by considering orderings of a tournament $T$ on $n$ vertices, which are bijections $\pi : V(T) \to [n]$. For any ordering, we can construct an undirected graph $T^+_\pi$ on the vertices of $T$ by letting $uv$ be an edge if and only if $\pi(u) < \pi(v)$ and there is an edge in $T_n$ from $u$ to $v$. A tournament $T$ is quasirandom if and only if the graph $T^+_\pi$ is quasirandom for every ordering $\pi$, which is also equivalent to at least one such ordering existing.

Quasirandom oriented graphs are generalizations of quasirandom tournaments. Here, the numbers of 4-cycles with different edge orientations are once again crucial—the number of even 4-cycles determines the number of copies of any oriented graph. The primary difference between tournaments and oriented graphs is that quasirandom conditions on oriented graphs are defined relative to the number of copies of unoriented graphs. For instance, an oriented graph $G$ is quasirandom if and only if
\[ N_G(H) = 2^{-e(H)} N_G(\tilde{H}) + o(|H|) \]
for all oriented graphs $H$, where $\tilde{H}$ is the unoriented version of $H$.

Finally, we will consider quasirandom permutations. Consider a sequence of permutations $(\sigma_n)$, with $\sigma_n : [n] \to [n]$. Define the discrepancy $d(\sigma_n)$ to be
\[ d(\sigma_n) = \max_{A,B \subseteq [n]} \left| \frac{|A||B|}{n^2} - \frac{|\sigma_n(A) \cap B|}{n} \right| \]
where $A$ and $B$ are intervals in $[n]$. A sequence of permutations is defined to be quasirandom if $d(\sigma_n) = o(1)$, which captures the idea that a random permutations should spread intervals out equally through all of $[n]$. As usual, this is equivalent to another notion of quasirandomness involving the number of occurrences of smaller permutations. For any permutation $\pi : [m] \to [m]$ with $m \leq n$, we take $t(\pi, \sigma_n)$ to be the probability that the permutation $\sigma_n$ restricted to a random set of $m$ points is isomorphic to $\pi$. Let $P(k)$ be the property that $t(\pi, \sigma_n) = \frac{1}{m!} + o(1)$ for all permutations $\pi \in S_m$. The discrepancy definition of quasirandomness is equivalent to a sequence of permutations satisfying $P(k)$ for all $k$. In fact, it should come as no surprise at this point that it is only necessary for $P(4)$ to hold to imply quasirandomness.

References


