1. Use the method of Lagrange multipliers to optimize each function subject to the given constraint.

(a) \( f(x, y) = x^2 + y^2 \) on the curve \( 4x^2 + y^2 = 4 \)

Lagrange multiplier equations: \( 2x = 8\lambda x, 2y = 2\lambda y \)

Note that if \( x \) and \( y \) are both nonzero, these equations give us \( \lambda = \frac{1}{4} \) and \( \lambda = 1 \), which is a contradiction. The only solutions are the points where either \( x = 0 \) or \( y = 0 \), namely \((1, 0), (-1, 0), (0, 2), \) and \((0, -2)\). The function \( f \) has a maximum value of 4 at \((0, 2)\) and \((0, -2)\) and a minimum value of 1 at \((1, 0)\) and \((-1, 0)\).

(b) \( f(x, y) = x + y \) on the curve \( x^4 + y^4 = 1 \)

Lagrange multiplier equations: \( 1 = 4\lambda x^3, 1 = 4\lambda y^3 \)

Solving for \( \lambda \) in the first equation and substituting this into the second equation, we find that \( x = y \). This happens at two places on the constraint curve \( x^4 + y^4 = 1 \), at the points \((2^{-1/4}, 2^{-1/4})\) and \((-2^{-1/4}, -2^{-1/4})\). Evaluating \( f \) at these two points, we see that the function achieves a maximum value of \( 2^{3/4} \) at \((2^{-1/4}, 2^{-1/4})\) and a minimum value of \( -2^{3/4} \) at \((-2^{-1/4}, -2^{-1/4})\).

(c) \( f(x, y, z) = 2x + 2y + z \) on the sphere \( x^2 + y^2 + z^2 = 9 \)

Lagrange multiplier equations: \( 2 = 2\lambda x, 2 = 2\lambda y, 1 = 2\lambda z \)

Solving for \( \lambda \) in the first equation and substituting into each of the other two equations, we obtain \( x = y \) and \( x = 2z \). The points \((x, y, z)\) satisfying these two conditions and the constraint are \((2, 2, 1)\) and \((-2, -2, -1)\). As always, we check the value of \( f \) at these two points to see that it has a maximum value of 9 at \((2, 2, 1)\) and a minimum value of \(-9\) at \((-2, -2, -1)\).

2. Find the maximum and minimum values of \( f(x, y) = x^2 + y^2 \) in the triangular region bounded by the lines \( y = 3 + x, y = 3 - x, \) and \( y = 1 \).
To optimize this function, we need to consider the critical points of $f$, endpoints, and critical points when constrained to each of the three lines making up one of the triangle’s sides. The only critical point of $f$ is at $(0,0)$, but this is outside of the region of interest, so we ignore it. The corners of the triangle are at $(0,3)$, $(2,1)$, and $(-2,1)$. Using Lagrange multipliers on each of the three lines, we turn up the additional boundary critical points $(-\frac{3}{2}, \frac{3}{2})$, $\left(\frac{5}{2}, \frac{3}{2}\right)$, and $(0,1)$, one on each line. The overall maximum value of $f$ is 1 at $(0,1)$, and the minimum value is 9 at $(0,3)$.

If you want even more practice, try reworking the optimization problems 2(a), 2(b), and 2(c) from yesterday using Lagrange multipliers to optimize on the boundary.