MARTIN-LÖF RANDOMNESS AND GALTON-WATSON PROCESSES

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Abstract. The members of Martin-Löf random closed sets under a distribution studied by Barmpalias et al. are exactly the infinite paths through Martin-Löf random Galton-Watson trees with survival parameter $\frac{2}{3}$. To be such a member, a sufficient condition is to have effective Hausdorff dimension strictly greater than $\gamma = \log_2 \frac{3}{2}$, and a necessary condition is to have effective Hausdorff dimension greater than or equal to γ.

Keywords: random closed sets, computability theory.

Contents

1. Introduction

Classical probability theory studies intersection probabilities for random sets. A random set will intersect a given deterministic set if the given set is large, in some sense. Here we study a computable analogue: the question of which real numbers are "large" in the sense that they belong to some Martin-Löf random closed set.

Barmpalias et al. [2] introduced algorithmic randomness for closed sets. Subsequently Kjos-Hanssen [7] used algorithmically random Galton-Watson trees to obtain results on infinite subsets of random sets of integers. Here we show that the distributions studied by Barmpalias et al. and by Galton and Watson are actually equivalent, not just classically but in an effective sense.

For $0 \leq \gamma < 1$, let us say that a real x is a $\textsc{Member}_\gamma$ if x belongs to some Martin-Löf (ML-) random closed set according to the Galton-Watson distribution (defined below) with survival parameter $p = 2^{-\gamma}$. We show that for $p = \frac{2}{3}$, this is equivalent to x being a member of a Martin-Löf random closed set according to the distribution considered by Barmpalias et al.

In light of this equivalence, we may state that (i) Barmpalias et al. showed that in effect not every MEMBER_γ is ML-random, and (ii) Joe Miller and Antonio Montálban showed that every ML-random real is a MEMBER_γ; the proof of their result is given in the paper of Barmpalias et al. [2] The way to sharpen these results goes via *effective Hausdorff dimension.* Each ML-random real has effective Hausdorff dimension equal to one. In Section 3 we show that (i′) a MEMBER_γ may have effective Hausdorff dimension strictly less than one, and (ii′) every real of sufficiently large effective Hausdorff dimension (where some numbers strictly less than one are "sufficiently large") is a MEMBER_γ.

2. Equivalence of two models

We write $\Omega = 2^{<\omega}$, and 2^ω, for the sets of finite and infinite strings over $2 = \{0, 1\}$, respectively. If $\sigma \in \Omega$ is an initial substring (a prefix) of $\tau \in \Omega$ we write $\sigma \preceq \tau$; similarly $\sigma \prec x$ means that the finite string σ is a prefix of the infinite string $x \in 2^\omega$. The length of σ is $|\sigma|$. Concatenation of strings σ, τ is written $\sigma\tau$ or $\sigma^\frown\tau$, the empty string is written $\langle\,\rangle$ and strings of length one are written $\langle i\rangle$ or simply i, where $i = 0, 1$. We use the standard notation $[\sigma] = \{x : \sigma \prec x\}$, and for a set $U \subseteq \Omega$, $[U]^\preceq := \bigcup_{\sigma\in U}[\sigma]$. Let $\mathcal{P}$ denote the power set operation, $\mathcal{P}(X) = 2^X$. For a real number $0 \le \gamma < 1$, $\lambda_{1,\gamma}$ denotes the distribution with sample space $\mathcal{P}(\Omega)$ such that for each $\sigma \in \Omega$,

$$\lambda_{1,\gamma}(\{S : \sigma \in S\}) = 2^{-\gamma},$$

and the events $\{S : \sigma \in S\}$ are mutually independent for distinct σ. Let λ^*_γ be the distribution with sample space $\mathcal{P}(\Omega)$ such that for each J, writing $p = 2^{-\gamma}$,

$$\lambda^*_\gamma(\{S : S \cap \{\sigma 0, \sigma 1\} = J\} = \begin{cases} 1-p & \text{if } J = \{\sigma 0\} \text{ or } J = \{\sigma 1\}; \\ 2p-1 & \text{if } J = \{\sigma 0, \sigma 1\}, \end{cases}$$

and the events $\{S : S \cap \{\sigma 0, \sigma 1\} = J\}$ are mutually independent for distinct σ. The notation $\lambda_{1,\gamma}$ is consistent with earlier usage [7] and is also easy to distinguish visually from λ^*_γ.

The closed set Γ_S determined by $S \subseteq \Omega$ is defined by

$$\Gamma_S = \{x \in 2^\omega : (\forall \sigma \prec x)\, \sigma \in S\}.$$

The *Galton-Watson (GW) distribution for survival parameter* $2^{-\gamma}$, also known as the $(1, \gamma)$-induced distribution [7], and as the distribution of a *percolation limit set* [12], is a distribution $\mathbb{P}_{1,\gamma}$ on the set of all closed subsets of 2^ω defined by

$$\mathbb{P}_{1,\gamma}(\{\Gamma : \Gamma \in E\}) = \lambda_{1,\gamma}\{S : \Gamma_S \in E\}.$$

Thus, the probability of a property E of a closed subset of 2^ω is the probability according to $\lambda_{1,\gamma}$ that a random subset of Ω determines a tree whose set of infinite paths has property E. Similarly, let

$$\mathbb{P}^*_\gamma(\{\Gamma : \Gamma \in E\}) = \lambda^*_\gamma\{S : \Gamma_S \in E\}.$$

A Σ^0_1 *subset of* $\mathcal{P}(\Omega)$ is the image of a Σ^0_1 subset of $\mathcal{P}(\omega) = 2^\omega$ via an effective isomorphism between Ω and ω.

Definition 2.1 (Martin-Löf randomness)**.** *A set of strings $S \in \mathcal{P}(\Omega)$ is called* $\lambda_{1,\gamma}$-ML-random *if for each uniformly Σ^0_1 sequence $\{U_n\}_{n\in\omega}$ of subsets of $\mathcal{P}(\Omega)$ with $\lambda_{1,\gamma}(U_n) \le 2^{-n}$, we have $S \notin \bigcap_n U_n$.*

A closed set Γ *is called* $\mathbb{P}_{1,\gamma}$-ML-random *if* $\Gamma = \Gamma_S$ *for some* $\lambda_{1,\gamma}$*-ML-random set of strings* S*.*

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Lemma 2.2 (Axon [1])**.** *Let* $2^{-\gamma} = \frac{2}{3}$*. A closed set* $\Gamma \subseteq 2^\omega$ *is* $\mathbb{P}^*_\gamma$*-ML-random if and only if* Γ *is a Martin-Löf random closed set under the distribution studied by Barmpalias et al.*

A probability space $(M, \mathcal{M}, \mu)$ consists of a set M, a σ-algebra $\mathcal{M}$ on M, and a measure μ defined on each set in $\mathcal{M}$. For each probability space there is a unique canonical M-valued random variable X such that for any $A \in \mathcal{M}$, the probability that $X \in A$ is $\mu(A)$. In this way, for $\mu = \lambda_{1,\gamma}$ or $\mu = \lambda^*_\gamma$ we get the random variable $S \in \mathcal{P}(\Omega)$. Conversely, if $Y = f(X)$ is a random variable defined deterministically from X then there is a unique canonical measure ν such that $\nu(A)$ is the probability that $Y \in A$, i.e. $\nu(A) = \mu(\{x : f(x) \in A\})$.

From such a random variable S we then define further random variables

$$G_n = \{\sigma : |\sigma| = n \ \ \& \ \ (\forall \tau \preceq \sigma)\ \tau \in S\}, \ \ G = \bigcup_{n=0}^{\infty} G_n,$$

$$G_\infty = \{\sigma \in G : \{\tau \in G : \sigma \prec \tau\} \text{ is infinite}\}.$$

We have $\Gamma_G = \Gamma_S$ and $G_\infty \subseteq G \subseteq S$, and values of G_∞ are in one-to-one correspondence with values of Γ_S.

A value of the random variable G is called a *GW-tree* or a *BBCDW-tree* when S is the canonical random variable associated with $\lambda_{1,\gamma}$ or λ^*_γ, respectively.

Let e be the extinction probability of a GW-tree with parameter $p = 2^{-\gamma}$,

$$e = \mathbb{P}_{1,\gamma}(\varnothing) = \lambda_{1,\gamma}(\{S : \Gamma_S = \varnothing\}).$$

For any number a let $\overline{a} = 1 - a$.

Lemma 2.3.

$$e = \overline{p}/p.$$

Proof. Note that $\Gamma_S = \varnothing$ iff either (1) $\langle\,\rangle \notin S$, or (2) $\langle\,\rangle \in S$ but

$$\Gamma_{S\cap\{\sigma:\langle i\rangle\preceq\sigma \text{ or } \sigma\preceq\langle i\rangle\}} = \varnothing$$

for both $i \in \{0, 1\}$. This gives the equation $e = \overline{p} + pe^2$. □

We use standard notation for conditional probability,

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)};$$

in measure notation we may also write $\lambda(E \mid F) = \lambda(E \cap F)/\lambda(F)$.

The following lemma is the first indication that there is a connection between GW- and BBCDW-trees. We write $\mathbf{1}_A$ for the characteristic function of an event or a set A, i.e., $\mathbf{1}_A = 1$ if A occurs, otherwise $\mathbf{1}_A = 0$.

Lemma 2.4. *For all* $J \subseteq \{\langle 0\rangle, \langle 1\rangle\}$*,*

$$\lambda_{1,\gamma}\{G_\infty \cap \{\langle 0\rangle, \langle 1\rangle\} = J \mid G_\infty \ne \varnothing\} = \lambda^*_\gamma[G_1 = J].$$

Proof. By definition, $\lambda^*_\gamma[G_1 = J]$ equals

$$(2p-1)\cdot \mathbf{1}_{J=\{\langle 0\rangle,\langle 1\rangle\}} + \sum_{i=0}^{1}(1-p)\cdot \mathbf{1}_{J=\{\langle i\rangle\}},$$

so we only need to calculate $\lambda_{1,\gamma}\left\{G_\infty \cap \{\langle 0\rangle,\langle 1\rangle\} = J \mid G_\infty \neq \varnothing\right\}$. By symmetry, and because the probability that $G_1 = \varnothing$ is 0, it suffices to calculate this probability for $J = \{\langle 0\rangle,\langle 1\rangle\}$. Now if $G_1 = \{\langle 0\rangle,\langle 1\rangle\}$ then $\langle\,\rangle$ survives and both immediate extensions are non-extinct. Thus the conditional probability that $G_1 = \{\langle 0\rangle,\langle 1\rangle\}$ is $\frac{p(1-e)^2}{1-e} = p(1-e)$. By Lemma 2.3, this is equal to $2p-1$. □

Let a measure λ_c on $\mathcal{P}(\Omega)$ be defined by

$$\lambda_c(\{S : S \in M\}) = \lambda_{1,\gamma}(M \mid G_\infty \neq \varnothing).$$

Let

$$\lambda_i(\{S : S \in M\}) = \nu(M \mid G_\infty \neq \varnothing)$$

where ν is the canonical measure obtained from G_∞.

Let μ_i and μ_c be the canonical measures obtained from G when S is the canonical random variable obtained from λ_i and λ_c, respectively (so $\mu_i = \lambda_i$).

Let λ_f be the distribution with sample space $\mathcal{P}(\Omega)$ such that for each $\sigma \in \Omega$,

$$\lambda_f(\{S : \sigma \in S\}) = 1 - p,$$

and the events $\{S : \sigma \in S\}$ are mutually independent for distinct σ. Note that this is exactly the definition of $\lambda_{1,\gamma}$, but with $1-p$ in place of p. If the random variable G is defined as before, but on this new space with measure λ_f, then G again is a GW-tree, but now with survival parameter $1 - p \leq \frac{1}{2}$. It turns out that the extinction probability e' of such a tree is 1, so such a tree is almost surely finite.

Lemma 2.5.

$$e' = \lambda_f(\{S \mid \Gamma_S = \varnothing\}) = 1.$$

Proof. As in Lemma 2.3, we observe that $\Gamma_S = \varnothing$ iff either (1) $\langle\,\rangle \notin S$, or (2) $\langle\,\rangle \in S$ but

$$\Gamma_{S\cap\{\sigma:\langle i\rangle\preceq\sigma \text{ or } \sigma\preceq\langle i\rangle\}} = \varnothing$$

for both $i \in \{0,1\}$. This gives the equation $e' = (1-p) + (1-p)(e')^2$, which has solutions 1 and $\frac{p}{1-p}$. Since $\frac{p}{1-p} > 1$, it cannot represent a probability, so $e' = 1$. □

Corollary 2.6. *If $S \subseteq \Omega$ is chosen randomly with respect to λ_f, then for all reals f, there are infinitely many initial segments of f which are not in S.*

Proof. Let $M \subseteq 2^\Omega$ have measure 0, and let T be a fixed, finite set of strings. Define $M' = \{S \mid (\exists S' \in M) S \setminus T = S' \setminus T\}$. Then $\lambda_f(M') \leq \left(\frac{1}{p}\right)^{|T|}\lambda_f(M) = 0$, so M' also has measure 0. Now let T vary, and let

$$M'' = \{S \mid (\exists S' \in M)(\exists T \subset \Omega, |T| < \infty) S \setminus T = S' \setminus T\}.$$

Then M'' is a countable union of measure 0 sets, so has measure 0. If

$$M = \{S \mid \Gamma_S \neq \varnothing\},$$

then M'' is the set we are interested in, which therefore has measure 0. □

Remark 2.7. *Take* $M = \{S \mid \Gamma_S \neq \varnothing\}$*, and fix* T*. Then the set* M' *above is a* Π^0_1 *class of measure 0, and is therefore contained in the intersection of the universal ML-test. Since this is true for all finite* T*, the set* M'' *is also contained in the intersection of the universal ML-test.*

We define a $\mu_i \times \lambda_f \to \mu_c$ measure-preserving map $\psi : 2^\Omega \times 2^\Omega \to 2^\Omega$. The idea is to overlay two sets G_i, S_f, so that G_i specifies the extendible nodes of a tree, and S_f specifies the non-extendible nodes. Let ψ be defined by

$$\psi(G_i, S_f) = \{\sigma : (\forall \tau \preceq \sigma)\tau \in G_i \cup S_f\}.$$

(In other words, since $G_i \cup S_f$ will not necessarily be a tree, we take the set of strings in that set whose predecessors are also all in the set to get the largest possible tree contained in that set of strings.)

Lemma 2.8. *The following identities hold for every string* σ*, and every set* $D \subseteq \{\sigma 0, \sigma 1\}$*:*

$$(\mu_i \times \lambda_f)(G \cap \{\sigma 0, \sigma 1\} = D \mid \sigma \in G) = \mu_c(G_\infty \cap \{\sigma 0, \sigma 1\} = D \mid \sigma \in G_\infty),$$

$$(\mu_i \times \lambda_f)(\psi(G, S) \cap \{\sigma 0, \sigma 1\} = D \mid \sigma \in G) = \mu_c(G \cap \{\sigma 0, \sigma 1\} = D \mid \sigma \in G_\infty),$$

$$\begin{aligned}(\mu_i \times \lambda_f)(\psi(G, S) \cap \{\sigma 0, \sigma 1\} &= D \mid \sigma \in \psi(G, S) \setminus G) \\ &= \mu_c(G \cap \{\sigma 0, \sigma 1\} = D \mid \sigma \in G \setminus G_\infty).\end{aligned}$$

Proof. Note that the event $\sigma \in G_\infty$ implies the event $G_\infty \neq \varnothing$, and the event $\sigma \in G \setminus G_\infty$ implies that any further events cannot affect the probability of the event $G_\infty \neq \varnothing$. Thus we may replace μ_c by $\lambda_{1,\gamma}$ in the above, as μ_c is $\lambda_{1,\gamma}$ conditioned on the event $G_\infty \neq \varnothing$. The rest is a straightforward computation, and is omitted. ☐

Theorem 2.9. *The map* ψ *is* $\mu_i \times \lambda_f \to \mu_c$ *measure preserving.*

Proof. To show that ψ is measure preserving, it suffices to show that it is measure preserving on the basic open sets. We will write $\Omega_n = 2^{<n}$ for the set of strings of length less than n, and given $S \subseteq \Omega$, we will write $S \upharpoonright n$ for $S \cap \Omega_n$, the set of strings in S of length less than n. Recall that the basic open sets in 2^Ω are the sets of the form $N_T = \{S \mid S \upharpoonright n = T\}$ for fixed $n \in \omega$, $T \subseteq \Omega_n$. Thus we must show that $\mu_c(N_T) = (\mu_i \times \lambda_f)(\psi^{-1}(N_T))$ for each T.

Since $\mu_c(\{S \mid S \text{ is not a tree}\}) = 0$, and $\psi(G, S)$ is always a tree, we can ignore elements of 2^Ω which are not trees, and focus our attention on the sets N_T where T is a tree. For $T' \subseteq T$, let $N_{T,T'} = \{S \mid S \text{ is a tree, } S \upharpoonright n = T, \text{ and } S_\infty \upharpoonright n = T'\}$. Observe that N_T is equal to the disjoint union $N_T = \bigcup_{T' \subseteq T} N_{T,T'}$. Thus it suffices to show that $\mu_c(N_{T,T'}) = (\mu_i \times \lambda_f)(\psi^{-1}(N_{T,T'}))$ for each pair T, T'.

Suppose $\psi(G, S) \in N_{T,T'}$. Then either $G \upharpoonright n = T'$, the extendible tree G contains a non-extendible node, or the tree $\psi(G, S)$ contains an extendible node outside of G. The latter event implies that there is a string f such that all but finitely many initial segments of f are elements of S. By Corollary 2.6, such an event has λ_f-measure 0, so (up to measure 0 events), $\psi(G, S) \in N_{T,T'}$ implies that $G \upharpoonright n = T'$.

We will prove, by induction, for each n, and all $T' \subseteq T \subseteq \Omega_n$, that $\mu_c(N_{T,T'}) = (\mu_i \times \lambda_f)(\psi^{-1}(N_{T,T'}))$. First, we see that when $n = 1$, we either have $T = \varnothing$ or $T = \{\langle\rangle\}$, and similarly with T'. We have that μ_c is conditioned on the event $G_\infty \neq \varnothing$, and μ_i is the distribution of a nonempty extendible tree, so both sides are equal to 1 when $T = T' = \{\langle\rangle\}$, and 0 when $T' = \varnothing$, so we have equality when $n = 1$.

Now assume $n > 0$, and equality holds for $T' \subseteq T \subseteq \Omega_n$. Let $U' \subseteq U \subseteq \Omega_{n+1}$ with $U \upharpoonright n = T$, and $U' \upharpoonright n = T'$. We may assume that U, U' are trees. We wish to show that $\mu_c(N_{U,U'}) = (\mu_i \times \lambda_f)(\psi^{-1}(N_{U,U'}))$, given that the same equality holds with T, T' in place of U, U'. Note that, given $N_{T,T'}$, the event $N_{U,U'}$ may be thought of as the intersection, over all $\sigma \in T \cap 2^{n-1}$, of the events $G \cap \{\sigma 0, \sigma 1\} = U \cap \{\sigma 0, \sigma 1\}$ and $G_\infty \cap \{\sigma 0, \sigma 1\} = U' \cap \{\sigma 0, \sigma 1\}$, and these events are independent for distinct σ. Similarly, given $\psi^{-1}(N_{T,T'})$, the event $\psi^{-1}(N_{U,U'})$ may be thought (up to events of measure 0) of as the intersection, over all $\sigma \in T \cap 2^{n-1}$, of the events $\psi(G,S) \cap \{\sigma 0, \sigma 1\} = U \cap \{\sigma 0, \sigma 1\}$ and $G \cap \{\sigma 0, \sigma 1\} = U' \cap \{\sigma 0, \sigma 1\}$, and these events are independent for different σ. By Lemma 2.8, the corresponding probabilities are all equal, so $\mu_c(N_{U,U'} \mid N_{T,T'}) = (\mu_i \times \lambda_f)(\psi^{-1}(N_{U,U'}) \mid \psi^{-1}(N_{S,S'}))$. By induction, ψ is measure-preserving. $\square$

Intuitively, a λ_i-ML-random tree may by van Lambalgen's theorem be extended to a λ_c-ML-random tree by adding finite pieces randomly; we verify this intuition in the following theorem.

Theorem 2.10. *For each ML-random BBCDW-tree H there is a ML-random GW-tree G with $G_\infty = H_\infty$.*

Proof. Let H be an ML-random BBCDW-tree (i.e., let it be ML-random with respect to the measure μ_i). Let $S \subseteq \Omega$ be ML-random relative to H with respect to the measure λ_f. Then, by a suitably generalized version of van Lambalgen's theorem[1], (H, S) is ML-random relative to the measure $\mu_i \times \lambda_f$. Now since ψ is effectively continuous, if (U_n) is a uniformly Σ^0_1 sequence, than so is $(\psi^{-1}(U_n))$. Furthermore, since ψ is $\mu_i \times \lambda_f \to \mu_c$ measure preserving, we have $\mu_c(U_n) = (\mu_i \times \lambda_f)(\psi^{-1}(U_n)$, so ψ^{-1} pulls back μ_c-ML-tests to $\mu_i \times \lambda_f$-ML-tests. Thus, since (H, S) passes every ML-test, so also must $G = \psi(H, S)$, so G is a ML-random GW-tree.

Now, by Corollary 2.6 and Remark 2.7, the set

$$M = \{S' \mid S' \text{ contains all but finitely many initial segments of some real}\}$$

has measure 0, and is contained in the universal λ_f-ML-test. Since S is ML-random with respect to λ_f, for any path $f \in \Gamma_G$, there are infinitely many initial segments of f which are not in the set S. Thus f must contain infinitely many initial segments in H, which means that $f \in \Gamma_H$. Therefore $\Gamma_G \subseteq \Gamma_H$, and $G_\infty \subseteq H_\infty$. But $H \subseteq G$, so we have equality: $G_\infty = H_\infty$. $\square$

We next prove that the live part of every infinite ML-random GW-tree is an ML-random BBCDW-tree.

Theorem 2.11. *For each S, if S is $\lambda_{1,\gamma}$-ML-random then G_∞ is λ^*_γ-random.*

Proof. Suppose $\{U_n\}_{n\in\omega}$ is a λ^*_γ-ML-test with $G_\infty \in \bigcap_n U_n$. Let $\Upsilon_n = \{S : G_\infty \in U_n\}$. By Lemma 2.4, $\lambda_{1,\gamma}(\Upsilon_n) = \lambda^*_\gamma(U_n)$. Unfortunately, Υ_n is not a Σ^0_1 class, but

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[1] To be precise, van Lambalgen's theorem holds in the unit interval $[0,1]$ with Lebesgue measure λ, or equivalently the space 2^ω. If (X, μ) is a measure space then using the measure-preserving map $\varphi : (X, \mu) \to ([0,1], \lambda)$ induced from the Carathéodory measure algebra isomorphism theorem [8], we may apply van Lambalgen's theorem as desired.

we can approximate it. While we cannot know if a tree will end up being infinite, we can make a guess that will usually be correct.

Let e be the probability of extinction for a GW-tree. By Lemma 2.3 we have $e = \frac{\overline{p}}{p}$, so since $p > 1/2$, $e < 1$. Thus there is a computable function $(n, \ell) \mapsto m_{n,\ell}$ such that for all n and ℓ, $m = m_{n,\ell}$ is so large that $e^m \leq 2^{-n}2^{-2\ell}$. Let Φ be a Turing reduction so that $\Phi^G(n, \ell)$, if defined, is the least L such that all the 2^ℓ strings of length ℓ either are not on G, or have no descendants on G at level L, or have at least $m_{n,\ell}$ many such descendants. Let

$$W_n = \{S : \text{ for some } \ell,\ \Phi^G(n, \ell) \text{ is undefined}\}.$$

Let $A_G(\ell) = G_\infty \cap \{0, 1\}^{\leq \ell}$ be G_∞ up to level ℓ. Let the approximation $A_G(\ell, L)$ to $A_G(\ell)$ consist of the nodes of G at level ℓ that have descendants at level L. Let

$$V_n = \{S : A_G(\ell, L) \in U_n \text{ for some } \ell, \text{ where } L = \Phi^G(n, \ell)\}, \text{ and}$$

$$X_n = \{S : \text{ for some } \ell,\ L = \Phi^G(n, \ell) \text{ is defined and } A_G(\ell, L) \neq A_G(\ell)\}.$$

Note that $\Upsilon_n = \{S : \text{for some } \ell,\ A_G(\ell) \in U_n\ \}$, hence $\Upsilon_n \subseteq W_n \cup X_n \cup V_n$. Thus it suffices to show that $\cap_n V_n$, W_n, $\cap_n X_n$ are all $\lambda_{1,\gamma}$-ML-null sets.

Lemma 2.12. $\lambda_{1,\gamma}(W_n) = 0$.

Proof. If $\Phi(\ell)$ is undefined then there is no L, which means that for the fixed set of strings on G at level ℓ, they do not all either die out or reach m many extensions. But eventually this must happen, so L must exist.

Indeed, fix any string σ on G at level ℓ. Let k be the largest number of descendants that σ has at infinitely many levels $L > \ell$. If $k > 0$ then with probability 1, above each level there is another level where actually $k + 1$ many descendants are achieved. So we conclude that either $k = 0$ or k does not exist. □

From basic computability theory, W_n is a Σ^0_2 class. Hence each W_n is a Martin-Löf null set.

Lemma 2.13. $\lambda_{1,\gamma}(X_n) \leq 2^{-n}$.

Proof. Let E_σ denote the event that all extensions of σ on level L are *dead*, i.e. not in G_∞. Let F_σ denote the event that σ has at least m many descendants on $G(L)$.

If $A_G(\ell, L) \neq A_G(\ell)$ then some $\sigma \in \{0, 1\}^\ell \cap G$ has at least m many descendants at level L, all of which are dead. If a node σ has at least m descendants, then the chance that all of these are dead, given that they are on G at level L, is at most e^m (the eventual extinction of one is independent of that of another), hence writing $\mathbb{P} = \lambda_{1,\gamma}$, we have

$$\mathbb{P}\{A_G(\ell, L) \neq A_G(\ell)\} \leq \sum_{\sigma \in \{0,1\}^\ell} \mathbb{P}\{E_\sigma \ \&\ F_\sigma\} = \sum_{\sigma \in \{0,1\}^\ell} \mathbb{P}\{E_\sigma \mid F_\sigma\} \cdot \mathbb{P}\{F_\sigma\}$$

$$\leq \sum_{\sigma \in \{0,1\}^\ell} \mathbb{P}\{E_\sigma \mid F_\sigma\} \leq \sum_{\sigma \in \{0,1\}^\ell} e^m \leq \sum_{\sigma \in \{0,1\}^\ell} 2^{-n}2^{-2\ell} = 2^{-n}2^{-\ell}.$$

and hence

$$\mathbb{P}X_n \leq \sum_\ell \mathbb{P}\{A_G(\ell, L) \neq A_G(\ell)\} \leq \sum_\ell 2^{-n}2^{-\ell} = 2^{-n}.$$

□

X_n is Σ^0_1 since when L is defined, $A_G(\ell)$ is contained in $A_G(\ell, L)$, and $A_G(\ell)$ is Π^0_1 in G, which means that if the containment is proper then we can eventually enumerate (observe) this fact. Thus $\cap_n X_n$ is a $\lambda_{1,\gamma}$-ML-null set.

V_n is clearly Σ^0_1. Moreover $V_n \subseteq \Upsilon_n \cup X_n$, so $\lambda_{1,\gamma}(V_n) \le 2 \cdot 2^{-n}$, hence $\cap_n V_n$ is a $\lambda_{1,\gamma}$-ML-null set. □

3. BEING A MEMBER OF SOME ML-RANDOM CLOSED SET

For a real number $0 \le \gamma \le 1$, the γ-weight $\mathrm{wt}_\gamma(C)$ of a set of strings $C \subseteq \Omega$ is defined by

$$\mathrm{wt}_\gamma(C) = \sum_{w \in C} 2^{-|w|\gamma}.$$

We define several notions of randomness of individual reals.

A *Martin-Löf (ML-)γ-test* is a uniformly Σ^0_1 sequence $(U_n)_{n<\omega}$, $U_n \subseteq \Omega$, such that for all n, $\mathrm{wt}_\gamma(U_n) \le 2^{-n}$.

A *strong ML-γ-test* is a uniformly Σ^0_1 sequence $(U_n)_{n<\omega}$ such that for each n and each prefix-free set of strings $V_n \subseteq U_n$, $\mathrm{wt}_\gamma(V_n) \le 2^{-n}$.

A real is (strongly) γ-random if it does not belong to $\cap_n [U_n]^{\preceq}$ for any (strong) ML-γ-test $(U_n)_{n<\omega}$.

If $\gamma = 1$ we simply say that the real, or the set of integers $\{n : x(n) = 1\}$, is *Martin-Löf random (ML-random)*. For $\gamma = 1$, strength makes no difference.

For a measure μ and a real x, we say that x is *Hippocrates μ-random* if for each sequence $(U_n)_{n<\omega}$ that is uniformly Σ^0_1, and where $\mu[U_n]^{\preceq} \le 2^{-n}$ for all n, we have $x \notin \cap_n [U_n]^{\preceq}$.

Let the ultrametric υ on 2^ω be defined by

$$\upsilon(x, y) = 2^{-\min\{n : x(n) \neq y(n)\}}.$$

The γ-energy [12] of a measure μ is

$$I_\gamma(\mu) := \iint \frac{d\mu(b) d\mu(a)}{\upsilon(a, b)^\gamma}$$

which in expected value ($\mathbb{E}$) notation can be written $I_\gamma(\mu) = \mathbb{E}_{(a,b)} \nu(a, b)^{-\gamma}$. A real x is *Hippocrates γ-energy random* if x is Hippocrates μ-random with respect to some probability measure μ such that $I_\gamma(\mu) < \infty$.

For background on γ-energy and related concepts the reader may consult the monographs of Falconer [4] and Mattila [11] or the on-line lecture notes of Mörters and Peres [12]. The terminology *Hippocrates random* is supposed to remind us of Hippocrates, who did not consult the oracle at Delphi, but instead looked for natural causes. An almost sure property is more effective if it is possessed by all Hippocrates μ-random reals rather than merely all μ-random reals. In this sense Hippocratic μ-randomness tests are more desirable than arbitrary μ-randomness tests.

Effective Hausdorff dimension was introduced by Lutz [9] and is a notion of partial randomness. For example, if the sequence $x_0 x_1 x_2 \cdots$ is ML-random, then the sequence

$$x_0 0 x_1 0 x_2 0 \cdots$$

has effective Hausdorff dimension equal to $\frac{1}{2}$. Let $\dim^1_H x$ denote the effective (or constructive) Hausdorff dimension of x; then we have $\dim^1_H(x) = \sup\{\gamma :$

x is γ-random} (Reimann and Stephan [14]) which we can consequently take as our definition of effective Hausdorff definition.

Theorem 3.1 ([7]). *Each Hippocrates γ-energy random real is a* $\textsc{Member}_\gamma$.

Here we show a partial converse:

Theorem 3.2. *Each* $\textsc{Member}_\gamma$ *is strongly γ-random.*

Proof. Let $\mathbb{P} = \lambda_{1,\gamma}$ and $p = 2^{-\gamma} \in (\frac{1}{2}, 1]$. Let $i < 2$ and $\sigma \in \Omega$. The probability that the concatenation $\sigma i \in G$ given that $\sigma \in G$ is by definition

$$\mathbb{P}\{\sigma i \in G \mid \sigma \in G\} = p.$$

Hence the absolute probability that σ survives is

$$\mathbb{P}\{\sigma \in G\} = p^{|\sigma|} = \left(2^{-\gamma}\right)^{|\sigma|} = \left(2^{-|\sigma|}\right)^{\gamma}.$$

Let U be any strong γ-test, i.e. a uniformly Σ^0_1 sequence $U_n = \{\sigma_{n,i} : i < \omega\}$, such that for all prefix-free subsets $U'_n = \{\sigma'_{n,i} : i < \omega\}$ of U_n, $\mathrm{wt}_\gamma(U'_n) \leq 2^{-n}$. Let U'_n be the set of all strings σ in U_n such that no prefix of σ is in U_n. Clearly, U'_n is prefix-free. Let

$$[V_n]^{\preceq} := \{S : \exists i\, \sigma_{n,i} \in G\} \subseteq \{S : \exists i\, \sigma'_{n,i} \in G\}.$$

Clearly $[V_n]^{\preceq}$ is uniformly Σ^0_1. To prove the inclusion: Suppose G contains some $\sigma_{n,i}$. Since G is a tree, it contains the shortest prefix of $\sigma_{n,i}$ that is in U_n, and this string is in U'_n. Now

$$\mathbb{P}[V_n]^{\preceq} \leq \sum_{i\in\omega} \mathbb{P}\{\sigma'_{n,i} \in G\} = \sum_{i\in\omega} 2^{-|\sigma'_{n,i}|\gamma} \leq 2^{-n}.$$

Thus V is a test for $\lambda_{1,\gamma}$-ML-randomness. Suppose x is a $\textsc{Member}_\gamma$. Let S be any $\lambda_{1,\gamma}$-ML-random set with $x \in \Gamma_S$. Then $S \notin \cap_n [V_n]^{\preceq}$, and so for some n, $\Gamma \cap [U_n]^{\preceq} = \varnothing$. Hence $x \notin [U_n]^{\preceq}$. As U was an arbitrary strong γ-test, this shows that x is strongly γ-random. □

Examples of measures of finite γ-energy may be obtained from the fact that if $\dim^1_H(x) > \gamma$ then x is Hippocrates γ-energy random [7]. If x is strongly γ-random then x is γ-random and so $\dim^1_H(x) \geq \gamma$.

Corollary 3.3. *Let $x \in 2^\omega$. We have the implications*

$$dim^1_H(x) > \gamma \;\Rightarrow\; x \textit{ is a } \textsc{Member}_\gamma \;\Rightarrow\; dim^1_H(x) \geq \gamma.$$

Proof. Reimann [13] defines the notion of γ-capacitability of a real and proves an effective capacitability theorem: x is strongly γ-random if and only if x is γ-capacitable, in his sense. Reimann shows that each real x with $\dim^1_H(x) > \gamma$ is β-capacitable for some $\beta > \gamma$. This implies that x is γ-energy random [7, Lemma 2.5] and in particular x is Hippocrates γ-energy random. This gives the first implication. For the second implication, we use the fact that each strongly γ-random real x satisfies $\dim^1_H(x) \geq \gamma$ (see e.g. Reimann and Stephan [14]). □

The second implication of Corollary 3.3 does not reverse, as not every real with $\dim^1_H(x) \geq \gamma$ is strongly γ-random [14].

The first implication of Corollary 3.3 fails to reverse as well:

Proposition 3.4. *Let $0 < \gamma < 1$. There is a γ-energy random real of effective Hausdorff dimension exactly γ.*

Proof. Consider the probability measure μ on 2^ω such that $\mu([\sigma^\frown 0]) = \mu([\sigma^\frown 1])$ for all σ of even length, and such that $\mu([\sigma^\frown 0]) = \mu([\sigma])$ for each σ of odd length $f(k) = 2k+1$. A computation shows that $I_\gamma(\mu)$ is finite if and only if $\gamma < 1/2$. In detail,[2] writing σ^* for the neighbor string of σ, i.e.

$$\sigma^* = \sigma \restriction_{|\sigma|-1}{}^\frown(1-\sigma(|\sigma|-1)),$$

we have

$$\begin{aligned} I_\gamma(\mu) &= \mathbb{E}_{(a,b)}\nu(a,b)^{-\gamma} = \mathbb{E}_a \sum_{n=0}^\infty 2^{n\gamma}\mu([a\restriction_{n+1}{}^*]) \\ &=^\dagger \sum_{n=0}^\infty 2^{n\gamma}\mathbb{E}_a(\mu([a\restriction_{n+1}{}^*)]) = \sum_{n=0}^\infty 2^{(2k)\gamma}2^{-(k+1)} \\ &= \frac{1}{2}\sum_{k=0}^\infty 2^{k(2\gamma-1)} < \infty \end{aligned}$$

if $2\gamma - 1 < 0$, i.e., $\gamma < \frac{1}{2}$. To justify the step ($\dagger$), note that if $\gamma < 1/2$ then for all a in the support of μ,

$$\sum_{n=0}^\infty 2^{n\gamma}\mu([a\restriction_{n+1}{}^*]) \le \sum_{k=0}^\infty 2^{2k\gamma}2^{-(k+1)}$$

which is a finite constant, so the dominated convergence theorem applies. On the other hand, if $\gamma = 1/2$ then since "$\ge$" always holds in ($\dagger$), we have $I_\gamma(\mu) = \infty$.

We find that μ-almost all reals are μ-random and have effective Hausdorff dimension exactly $1/2$. By modifying $f(k)$ slightly we can get $I_\gamma(\mu) < \infty$ for $\gamma = 1/2$ while keeping the effective Hausdorff dimension of μ-almost all reals equal to $1/2$. Namely, what is needed is that

$$\sum_{k=0}^\infty 2^{f(k)\gamma}2^{-(k+1)} < \infty.$$

This holds if $\gamma = 1/2$ and $f(k) = 2k - 2(1-\varepsilon)\log k$ for any $\varepsilon > 0$ since $\sum_k k^{-(1+\varepsilon)} < \infty$. Since this $f(k)$ is asymptotically larger than $(2-\delta)k$ for any $\delta > 0$, the μ-random reals still have effective Hausdorff dimension $1/2$. The example generalizes from $\gamma = 1/2$ to an arbitrary $0 < \gamma < 1$. □

Writing implication known to be strict as ⇒ and other implication as →, we have

> $\dim^1_H(x) > \gamma \Rightarrow x$ is γ-energy random $\rightarrow x$ is Hippocrates γ-energy random $\rightarrow x$ is a $\textsc{Member}_\gamma \rightarrow x$ is strongly γ-random $\Rightarrow x$ is γ-random $\Rightarrow \dim^1_H(x) \ge \gamma$.

Conjecture 3.5. *There is a strongly γ-random real which is not Hippocrates γ-energy random.*

In a conference version of this article [3] we made the following conjecture.

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[2] This computation corrects a numerical error in the conference version of the present article [3].

Conjecture 3.6. *A real x is a* $\textsc{Member}_\gamma$ *if and only if x is Hippocrates γ-energy random.*

The following considerations make Conjecture 3.6 seem less plausible.[3]

Definition 3.7 (Address)**.** *If Γ is a closed set and $x \in \Gamma$ then the address of x in Γ is the image of x under the lexicographical order preserving isomorphism between Γ and 2^ω. If y is the address of x in Γ then we write $x = \Gamma(y)$.*

For example, the leftmost path in Γ has address $0^\infty = 000\ldots$ An alternative term for *address* sometimes seen in the literature is *signature.*

Theorem 3.8. *If x is Hippocrates γ-energy random then x is a* $\textsc{Member}_\gamma$ *of a closed set Γ which is ML-random relative to the address of x in Γ.*

Proof. Suppose x is never $\Gamma(y)$ where Γ is ML-random relative to y. Then

$$\{(\Gamma, y) \mid x = \Gamma(y)\} \subseteq \{(\Gamma, y) \mid \Gamma \in V_n^y\}$$

for all n, where V_n is a universal oracle test. Let

$$U_n = \{\hat{x} \mid \{(\Gamma, y) \mid \hat{x} = \Gamma(y)\} \subseteq \{(\Gamma, y) \mid \Gamma \in V_n^y\}\}.$$

The class of reals U_n is Σ^0_1, as follows from compactness upon considering a no-dead-ends tree representation of Γ. As shown in an earlier paper [7], if x is γ-energy random as witnessed by a measure μ, then

$$\frac{\mu(U_n)^2}{c} \leq \mathbb{P}\{\Gamma : \Gamma \cap U_n \neq \varnothing\} \leq (\mathbb{P} \times \nu_n)\{(\Gamma, y) \mid \Gamma(y) \in U_n\}$$

(where ν_n almost surely picks out an element of $\Gamma \cap U_n$ if one exists)

$$= (\mathbb{P} \times \nu_n) \bigcup_{\hat{x} \in U_n} \{(\Gamma, y) \mid \Gamma(y) = \hat{x}\} \leq (\mathbb{P} \times \nu_n)\{(\Gamma, y) \mid \Gamma \in V_n^y\} \leq 2^{-n}.$$

Thus $\cap_n U_n$ is a Martin-Löf null set for the measure μ. Thus x is not μ-random after all. ☐

4. Changing the quantifier

In this section our attention turns away from the types of reals that belong to *some* ML-random closed set, and toward the types of reals can be found in *all* ML-random closed sets.

Given any set $Z \subseteq \omega$ we can form the tree

$$T_Z = \{\sigma : (\forall n < |\sigma|)(Z(n) = 0 \to \sigma(n) = 0)\},$$

and the corresponding closed set $[T_Z] = \{x : (\forall \sigma \prec x)\, \sigma \in T_Z\}$.

Lemma 4.1 ([8, Lemma 4.11])**.** *Suppose given a real number $\gamma \in (0, 1)$, and $\varepsilon > 0$ such that $\gamma + \varepsilon$ is a rational number p/q. If $A = [T_Z]$ with $Z = \{n : n \mod q < p\}$ then there is a probability measure μ on A such that $I_\gamma(\mu) < \infty$, and such that for all for $\sigma \in \Omega$, $\mu([\sigma]) > 0 \leftrightarrow [\sigma] \cap A \neq \varnothing$.*

Lemma 4.2. *Let A be as in Lemma 4.1. For each $x \in A$, $\dim^1_H(x) \leq p/q$.*

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[3] The ideas here are related to selection theorems in the theory of random closed sets, which were introduced to us by David Ross at University of Hawaiʻi in May 2009.

Proof. Let $\mathcal{H}^{p/q}_\varepsilon$ denote the usual ε-approximation to p/q-dimensional Hausdorff measure $\mathcal{H}^{p/q}$. Note that we can cover A with 2^{mp} many cones $[\sigma]$ with $|\sigma| = mq$, and hence if $\varepsilon = 2^{-mq}$ then $\mathcal{H}^{p/q}_\varepsilon(A) \leq 2^{mp}(2^{-mq})^{p/q} = 1$. As $m \to \infty$, $\varepsilon \to 0$ and so $\mathcal{H}^{p/q}(A) \leq 1$ and thus $\dim(A) \leq p/q$. $\square$

Theorem 4.3. *For each $\varepsilon > 0$, each $\mathbb{P}^*_\gamma$-ML-random closed set for $2^{-\gamma} = 2/3$ contains a real x of effective Hausdorff dimension at most $\log_2(\frac{3}{2}) + \varepsilon$.*

Proof. Fix $\varepsilon > 0$. We may assume $\gamma + \varepsilon \in \mathbb{Q}$. Let A be as in Lemma 4.1. It follows from [8, Theorem 4.10] that A has Hausdorff dimension strictly greater than γ.

Let $U := \{\Gamma : (\exists n)(\forall \sigma \in G_n)\,[\sigma] \cap A = \varnothing\}$, which is a Σ^0_1 class. Indeed, there are only finitely many $\sigma \in 2^n$ to check for a given n, and for our choice of set $A = [T_Z]$, $\{\sigma \in \Omega : [\sigma] \cap A = \varnothing\}$ is computable.

As shown by Hawkes [6], U has measure strictly less than 1. In fact, one way to see this is to observe that if U had measure 1 then it would contain all ML-random closed sets Γ (even all Kurtz random closed sets), contradicting Corollary 3.3 and the fact that each set of Hausdorff dimension strictly greater than γ contains a real of effective Hausdorff dimension strictly greater than γ.

Let Γ be a random closed set, let ℓ be its leftmost path, and let n_i be the ith zero of ℓ (so $\ell(n_i) = 0$) and $\ell_i = (\ell \upharpoonright n_i)^\frown 1$. Using the notation $\sigma X = \{\sigma^\frown x : x \in X\}$, we have

$$\Gamma = \bigcup_{i \in \omega} \ell_i \Gamma_i$$

where Γ_i is again a random closed set.

Let U_n be defined by $\Gamma \in U_n$ iff $\Gamma_n \in U$. Then the U_n are independent and have measure $u < 1$ each. Hence the measure of their intersection is $\lim_{n\to\infty} u^n = 0$. Because $u^n \to 0$ effectively, $\cap_n U_n$ is in fact a Martin-Löf null set. Thus if Γ is ML-random, then there is an i for which $A \cap \Gamma_i \neq \varnothing$, or equivalently $(\ell_i A) \cap \Gamma \neq \varnothing$. Thus Γ contains a shift of a member of A.

By Lemma 4.2, each member of A has effective dimension $\leq \gamma + \varepsilon$. Thus each random closed set Γ contains a shift y of a real x with $\dim^1_H(x) \leq \gamma + \varepsilon$ and hence in fact contains y with $\dim^1_H(y) \leq \gamma + \varepsilon$. $\square$

5. APPLICATIONS

5.1. **Approximation properties.**

Proposition 5.1. *Let $0 < \gamma < 1$. If x is a real such that the function $n \mapsto x(n)$ is f-computably enumerable for some computable function f for which $\sum_{j<n} f(i)2^{-n\gamma}$ goes effectively to zero, then x is not γ-random.*

Proof. Suppose $n \mapsto x(n)$ is f-c.e. for some such f, and let $F(n) = \sum_{j<n} f(n)$. Let α be any computable function such that $\alpha(n,i) \neq \alpha(n,i+1)$ for at most $f(n)$ many i for each n, and $\lim_{i\to\infty} \alpha(i,n) = x(n)$. Let $c(n,j)$ be the jth such i that is discovered for any $k < n$; so c is a partial recursive function whose domain is contained in $\{(n,j) : j \leq F(n)\}$. For a fixed i, α defines a real α_i by $\alpha_i(n) = \alpha(i,n)$. Let $V_n = \{x : \exists j \leq F(n)\ x \upharpoonright n = \alpha_{c(n,j)} \upharpoonright n)\}$. Since V_n is the union of at most $F(n)$ many cones $[x \upharpoonright n]$,

$$\mathrm{wt}_\gamma(V_n) \leq \sum_{j=1}^{F(n)} 2^{-n\gamma} = F(n)2^{-n\gamma}$$

which goes effectively to zero by assumption. Thus there is a computable sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\mathrm{wt}_\gamma(V_{n_k}) \le 2^{-k}$. Let $U_k = V_{n_k}$. Then U_k is Σ^0_1 uniformly in k, and $x \in \cap_k U_k$. Hence x is not γ-random. ☐

Corollary 5.2 ([2]). *No member of a ML-random closed set under the BBCDW distribution is f-c.e. for any polynomial-bounded f.*

Proof. If f is polynomially bounded then clearly $\sum_{j<n} f(i)2^{-n\gamma}$ goes effectively to zero. Therefore if x is f-c.e., x is not γ-random, hence not a MEMBER$_\gamma$ for any $0 < \gamma < 1$, and thus not a member of a ML-random closed set under the BBCDW distribution. ☐

5.2. **Randomness for Bernoulli measures.** Our results characterize the Bernoulli measures for which random sequences are MEMBERS. Suppose $0 \le p \le 1$. The Bernoulli measure μ_p on 2^ω is uniquely defined by the properties (i) $\mu_p\{A : A(n) = 1\} = p$ and (ii) the events $\{A : A(n) = 1\}$ are mutually independent for distinct $n \in \omega$. An infinite binary sequence $A \in \Omega$ is ML-random for the Bernoulli measure μ_p, or for short μ_p-random, if for each uniformly $\Sigma^0_1(p)$ sequence of open sets U_n, $n \in \omega$, with $\mu_p(U_n) \le 2^{-n}$, we have $A \notin \cap_n U_n$. This notion was related to (the martingale characterization of) effective Hausdorff dimension by Lutz.

Theorem 5.3 (Lutz [9]). *For each μ_p-random sequence A, the effective Hausdorff dimension of A is*

$$H(\mu_p) := -(p \log p + \overline{p} \log \overline{p}).$$

To find the values of p for which a Bernoulli μ_p-random sequence is a MEMBER$_\gamma$, note that $H(\mu_p) > \gamma$ if and only if $(p)^p(\overline{p})^{\overline{p}} < 2^{-\gamma}$. For the value $2^{-\gamma} = \frac{2}{3}$ studied by Barmpalias et al., a numerical calculation on the web site Wolfram Alpha yields that this inequality is equivalent to

$$0.140276506997464... < p < 0.859723493002535...$$

5.3. **Computing Brownian slow points.** A function $f : \omega \to \omega$ is *diagonally non-recursive* (DNR) if for each n, $f(n)$ is not equal to $\varphi_n(n)$, the value (if any) of the nth partial recursive function on input n. A real A is *Kurtz random relative to an oracle B* if it does not belong to any $\Pi^0_1(B)$ subset of 2^ω of fair-coin measure zero. Furthermore, B is Low(ML, Kurtz) if each real A that is ML-random is Kurtz random relative to B.

A starting point for the present paper was the observation (*) that each non-DNR Turing degree is Low(ML, Kurtz). A proof of this result due and credited to Kjos-Hanssen is given by Greenberg and Miller [5]; they prove that the converse holds as well. This can be used to show that each *slow point* (see Mörters and Peres [12]) of any ML-random Brownian motion must be of DNR Turing degree. The *fast* points on the other hand form a dense G_δ set, so there are fast points that are 1-generic and hence do not Turing compute any slow points.

In any case, the idea was initially to use the result (*) to understand members of random closed sets. However, as it turned out one could use the work of Hawkes [6] and Lyons [10] to better effect.

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