

Compact Forms of Homogeneous Spaces and Higher-Rank Semisimple Group Actions

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Abstract

This paper proves that there are no compact forms for a large class of homogeneous spaces admitting actions by higher-rank semisimple Lie groups. It builds on Zimmer's approach for studying such spaces using cocycle superrigidity. For the most natural homogeneous spaces ($J\backslash H$ with H simple, J reductive) admitting higher-rank semisimple group actions, it proves that no compact form exists. For a larger class of spaces, the proof shows that any compact form must be 'standard,' i.e. built via a simple algebraic construction. The proof involves cocycle superrigidity, measure rigidity for unipotent flows, techniques from partially hyperbolic dynamics, and the geometry and pseudo-Riemannian structure of the homogeneous space.

1 Introduction

Let H be a Lie group and J its closed subgroup. The compact forms question for the homogeneous space $J\backslash H$ is the following: does there exist a discrete subgroup $\Gamma < H$ such that $J\backslash H/\Gamma$ is a compact manifold? In this case $J\backslash H/\Gamma$ is a compact Clifford-Klein form (or just 'compact form') of $J\backslash H$.

The question of which pairs (H, J) admit compact forms is extensive and has been addressed by a variety of techniques, ranging from topology of Lie groups to homogeneous dynamics, representation theory, characteristic classes and symplectic geometry. Margulis listed it as one of his "Problems and conjectures in rigidity theory" in [Mar00]. The present paper falls under a dynamical approach pursued by Zimmer, Labourie and Mozes; numerous other contributions are surveyed in [KY05] and [Lab96]. A few basic results mark the boundaries of this subject. If J is compact, the problem is essentially equivalent to finding cocompact lattices in H , and was solved for

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semisimple H by Borel [Bor63]. At the other extreme is the Calabi-Markus phenomenon, which states that if the real ranks of J and H are equal, only a finite Γ can act properly discontinuously on $J\backslash H$ ([CM62], cf. [Kob89]).

This paper proves the following:

Main Theorem. *Let H be a connected, simple Lie group with finite center and J a connected, non-compact, reductive sub-Lie group (in the sense that $\text{ad}_H(J)$ is reductive). Suppose the centralizer in H of J (hereafter $Z_H(J)$) contains a simple Lie group G with real-rank at least two. Then there is no compact form of $J\backslash H$.*

This result extends, at least when H is a simple Lie group and J is reductive, a course of investigation begun by Zimmer utilizing the action of $Z_H(J)$ on $J\backslash H/\Gamma$. Zimmer pioneered this strategy in [Zim94], noticing that if this centralizer is large – i.e. contains a higher-rank semisimple Lie group – cocycle superrigidity can be applied to prove non-existence results. He further pursued this approach with Labourie and Mozes in [LMZ95] and with Labourie in [LZ95]. Their most general result is the following:

Theorem 1.0.1 (Labourie, Mozes and Zimmer, [LMZ95]). *Let H and J be real, unimodular algebraic groups such that there is a real, semisimple group G contained in $Z_H(J)$ each of whose simple factors has real rank at least two. Suppose that G is not contained in a proper, normal subgroup of H and that*

- (i) *the image of every non-trivial homomorphism $\tilde{G} \rightarrow J$ has compact centralizer in J ;*
- (ii) *there is a non-trivial, \mathbb{R} -split, 1-parameter subgroup in $Z_H(JG)$ that is not contained in a normal subgroup of H .*

Then if there is a compact form $J\backslash H/\Gamma$, J is compact.

This theorem is not ideal in that its algebraic conditions, particularly (i), restrict its application significantly. Consequently, there are many homogeneous spaces which admit higher-rank actions and hence to which Zimmer’s cocycle superrigidity can be applied but which [LMZ95] does not address. For example, in the test case $SL_{n-k}(\mathbb{R})\backslash SL_n(\mathbb{R})$, where $SL_{n-k}(\mathbb{R})$ is embedded in the standard way, condition (i) requires that $k \geq n/2$. Cocycle superrigidity, however, applies whenever $k \geq 3$. Labourie and Zimmer must use a separate argument to prove nonexistence for $k \geq 3$ in [LZ95]¹. This argument uses the fact that the Weyl group for $SL_n(\mathbb{R})$ can exchange any two diagonal entries while leaving the rest fixed; it will not work for a general H .

¹Nonexistence for $k = 1$, n even was proven by Benoist in [Ben96], using a Lie theoretic approach; the question is open for $k = 1$, n odd and for $k = 2$ and $n \neq 4, 6$ (see [Sha00] for $n = 4$ and [Kob92] for $n = 4, 6$).

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Thus, there are many other examples for which this theorem's applications are similarly restricted and for which the compact forms question has not been addressed by other means. Theorem 1.0.1 does have the advantage of allowing H and J which are merely unimodular. Note, however, that in all examples provided in [LMZ95] the condition that G not be contained in a proper normal subgroup of H is ensured by taking H simple, and the most natural examples have J reductive.

To the contrary, the Main Theorem, apart from the natural assumptions that H is simple and J is reductive, has few extra conditions. The conditions on G , H and J are those needed to ensure Zimmer's cocycle superrigidity applies; that G is simple and higher-rank ensures that the G -action is irreducible, a necessary condition for the application of cocycle superrigidity (see the Conclusion for a discussion of general higher-rank semisimple G). The condition that J is reductive helps ensure that the geometry of $J \backslash H$ is manageable.

It is easy to produce examples where the Main Theorem applies. The following lists a few of the most natural. The first, as noted above, is due to Labourie and Zimmer and is included here as it is a particularly nice example and will play an illustrative role in the exposition of the proof. Some of the others are listed in [Kob96], but with stronger restrictions on k and l .

- $SL_{n-k}(\mathbb{R}) \backslash SL_n(\mathbb{R})$ for $k \geq 3$.
- $SL_{n-k}(\mathbb{C}) \backslash SL_n(\mathbb{C})$ for $k \geq 3$.²
- $SO(n-k, m-l) \backslash SO(n, m)$ for $k \geq 2, l \geq 3$.
- $PSO(2(n-k), \mathbb{C}) \backslash PSO(2n, \mathbb{C})$ for $k \geq 2$.
- $PSO(2(n-k), \mathbb{C}) \backslash SO(2n+1, \mathbb{C})$ for $k \geq 2$.
- $SO(2(n-k)+1, \mathbb{C}) \backslash SO(2n+1, \mathbb{C})$ for $k \geq 2$.
- $SU(p-k, q-l) \backslash SU(p, q)$ for $k, l \geq 2$.
- $Sp(2(m-k), \mathbb{R}) \backslash Sp(2m, \mathbb{R})$ for $k \geq 2$
- $J' \backslash H$ for any H listed above and J' a non-compact reductive subgroup contained in the corresponding J listed above.

Most of the results leading to the Main Theorem hold, with a few additional assumptions, when H is semisimple. In this case, the proof yields an entirely algebraic description of any compact form. This description is

²Although they do not explicitly claim this, Labourie and Zimmer's work in [LZ95] seems to apply just as well to this case as to the real case.

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a step in the proof of the Main Theorem but is also interesting in its own right. Kobayashi and Yoshino ([KY05]) and Margulis ([Mar00] and [Mar97]) note that all known examples of compact forms for J reductive are based on the following construction. Let L be a subgroup of H such that $JL = H$ and $J \cap L$ is a compact subgroup K . Suppose Γ is a uniform lattice in L intersecting K trivially. Then

$$J \backslash H / \Gamma \cong K \backslash L / \Gamma$$

is a compact form of $J \backslash H$; these are examples of ‘standard forms.’ (More generally, for a standard form we require only that $J \backslash H / L$ is compact.) A natural question is whether all compact forms arise in this way. This is not the case; work of Goldman in [Gol85] and later Kobayashi ([Kob98]) shows that some compact forms which have no such algebraic construction (‘nonstandard forms’) can be obtained by deforming standard forms. Salein has produced further examples ([Sal97], [Sal00]) and this field has recently been further developed by Kassel in [Kas08], [Kas]. Similarly, Oh and Witte-Morris ([OW02]) construct examples of nonstandard compact forms of varying homogeneous spaces of $SO(2, n)$ by deforming J . In all of these examples, however, the homogeneous spaces $J \backslash H$ admit standard forms, leading Kobayashi to conjecture that any homogeneous space admitting a compact form admits a standard one (see [KY05]). This paper proves the following theorem, confirming that the standard form construction is the only option for the homogeneous spaces discussed in this paper.

Characterization Theorem. *Let H be connected, semisimple with finite center, J connected, reductive, and assume there is a semisimple Lie group $G < Z_H(J)$ such that:*

- (1) *All simple factors of G have real-rank at least two*
- (2) *The vector space sum of the Lie algebra for J and the Lie algebra generated by all nonzero weight spaces for a Cartan subgroup $A < G$ is the Lie algebra of H .*

Then if there is a compact form of $J \backslash H$, there exists a Lie subgroup $L < H$ and a uniform lattice Γ in L such that $JL = H$ and $J \cap L = K$ is compact, and

$$J \backslash H / \Gamma \cong K \backslash L / \Gamma.$$

Assumption (1) is needed to ensure Zimmer’s cocycle superrigidity theorem can be directly applied. Assumption (2) might be considered in some sense complimentary to assumption (ii) in Theorem 1.0.1. In their work, Labourie, Mozes and Zimmer use a subgroup which commutes with both J and the acting group essentially. For the Characterization Theorem, one assumes that any such subgroup can be accessed via nonzero weight spaces

for the G -action and can be dealt with using a simple algebraic argument at the end of the proof. This theorem provides what seems to be a very strong algebraic obstruction to the existence of a compact form (see Remark 9.2.6).

The proof of these theorems combines techniques from several central results in dynamics and rigidity: Zimmer’s cocycle superrigidity, Ratner’s measure rigidity for unipotent flows, and the theory of partially hyperbolic dynamical systems. The three most important new ideas involved in the proof are the following. First, superrigidity is used to calculate Lyapunov exponents and find many more nonzero exponents than are *a priori* available. Second, a pseudo-Riemannian structure together with the geometry of $J\backslash H/\Gamma$ is used to study ergodic components of the G -action. This involves careful arguments relating this geometry to Lyapunov exponents; in particular, the effect of unipotent elements on Lyapunov exponents is crucial. The third is a dynamical argument inspired by [EK03]’s use of non-commuting foliations. Together, these ideas allow one to translate the compact forms question into a totally algebraic question and solve it as such.

1.1 Outline of the paper

The proof of the Main Theorem begins with Zimmer’s original approach, examining the action of G on $J\backslash H/\Gamma$ by left-multiplication. Cocycle superrigidity applies to G and can be used to lift an ergodic measure from $J\backslash H/\Gamma$ to H/Γ (see section 3). Working on the single quotient space H/Γ proves a considerable advantage. Ratner’s measure classification theorem shows (section 4) that this lifted measure is algebraic.

Section 5 shows how contributions from smooth dynamics, particularly the theory of partially hyperbolic dynamics, give further information about this measure. The method for using cocycle superrigidity to calculate Lyapunov exponents is discussed in section 5.2.

In section 6 the behavior of the Lyapunov exponents under unipotent flows is examined. Combined with careful consideration of the geometry of $J\backslash H/\Gamma$ in section 7, this yields strong results on the support of a particular G -ergodic measure.

In section 8, ergodicity of the G -action with respect to the volume is proved.

The final steps of the proof are algebraic. Combined with the description via Ratner of the lift of this measure, the ergodicity of the G -action yields the Classification Theorem’s entirely algebraic characterization of any compact form (section 9.1). Section 9.2 proves the Main Theorem by showing that the algebraic description cannot be satisfied. This relies on work by Oniřčík characterizing certain decompositions of simple Lie groups and on some simple algebraic restrictions to the existence of proper actions on homogeneous spaces.

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2 Definitions and basic objects

The paper begins under the assumptions of the Characterization Theorem and the supposition that $J\backslash H/\Gamma$ is compact. Section 9.2 requires that H be simple.

2.1 Spaces

Let H be a semisimple Lie group with finite center and J a closed, non-compact, reductive subgroup of H . German letters ($\mathfrak{h}, \mathfrak{j}, \mathfrak{g}, \mathfrak{z}$, etc.) denote Lie algebras of the corresponding groups. Assume the following:

- (1) $Z_H(J)$ contains a connected, semisimple Lie group G whose simple factors are of real-rank at least two.

Let A be a maximal Cartan subgroup of G . Let Z be the center of $Z_H(J)$ and let G' be the subgroup whose Lie algebra is generated by the nonzero weight spaces for \mathfrak{a} . Assume as well:

- (2) $\mathfrak{h} = \mathfrak{j} + \mathfrak{g}'$.

The basic example of $H = SL_n(\mathbb{R})$, $J = SL_{n-3}(\mathbb{R})$, embedded in the upper left-hand corner of $SL_n(\mathbb{R})$ and $G = SL_3(\mathbb{R})$, satisfies these and will play an illustrative role throughout the discussion. Labourie and Zimmer have previously proven (with a simpler argument) that this space has no compact form [LZ95]; it is used here for its value as a simple example. Note that if H is simple, condition (2) is automatically satisfied, as these weight spaces generate a nonzero ideal in \mathfrak{h} . Thus the Main Theorem falls under the requirements of the Characterization Theorem.

Suppose that Γ is a discrete subgroup of H acting properly discontinuously on $J\backslash H$ on the right. Then the quotient $J\backslash H/\Gamma$ is a manifold naturally modeled on $J\backslash H$; suppose that $J\backslash H/\Gamma$ is compact. Let $\pi : H/\Gamma \rightarrow J\backslash H/\Gamma$ be projection. Denote by $[h]$ the image of h in H/Γ or $J\backslash H/\Gamma$.

2.2 Metrics

As the standard left- and right-invariant Riemannian metrics on H may not be bi-invariant, neither descends to a metric on $J\backslash H/\Gamma$. However, there is one metric structure on $J\backslash H/\Gamma$ with direct relation to the algebraic structure of H : the pseudo-Riemannian metric coming from the (*bi*-invariant) Killing form on \mathfrak{h} . Its existence and signature are easily established by the proposition below. First we note the existence of a ‘horizontal’ distribution \mathcal{H} in TH , transversal to the ‘vertical’ distribution described by \mathfrak{j} ; it will be used throughout the paper. As \mathfrak{j} is $ad(\mathfrak{j})$ -invariant and \mathfrak{j} is reductive, one may take \mathcal{H} as an $ad(\mathfrak{j})$ -invariant complement to \mathfrak{j} . Using right-translation, \mathcal{H} extends to a left- J -invariant distribution over all of TH or $T(H/\Gamma)$.

Proposition 2.2.1. *For a Lie group G , define $d(G)$ to be $\dim(G) - \dim(K)$ where K is a maximal compact subgroup of G . There is a pseudo-Riemannian metric on the homogeneous space $J\backslash H$ of signature*

$$(d(H) - d(J), \dim(H) - \dim(J) - d(H) + d(J)).$$

Proof. Begin with the Killing form on \mathfrak{h} . Restrict the Killing form to $\mathcal{H} \times \mathcal{H}$ and extend it to TH by right-invariance to form a pseudo-Riemannian metric. By $ad(J)$ -invariance, this descends to a well-defined pseudo-Riemannian metric on $J\backslash H/\Gamma$. See [KY05] Proposition 3.2.7 for the calculation of its signature. \square

For $SL_n(\mathbb{R})$, the Killing form is easily calculated: for any A, B in \mathfrak{h} it is $4Tr(AB)$.

Denote the pseudo-Riemannian metric on $J\backslash H/\Gamma$ by g_{PR} . The pseudo-Riemannian structure provides a Levi-Civita connection, denoted by ∇ . Note that $\mathfrak{j} \times \mathfrak{g}$ is also reductive and $ad(\mathfrak{j} \times \mathfrak{g})$ leaves \mathfrak{j} stable, so \mathcal{H} can be taken to be $ad(\mathfrak{g})$ -invariant as well. With such a choice, g_{PR} and ∇ are left- G -invariant.

Any vector field X on $J\backslash H/\Gamma$ has a unique lift \bar{X} to a vector field lying in \mathcal{H} and, by construction, $\pi_* : \mathcal{H} \rightarrow T(J\backslash H/\Gamma)$ is an isometry for the pseudo-Riemannian metrics on these spaces. Finally, the J -fibers themselves are pseudo-Riemannian submanifolds. Therefore, π together with the horizontal distribution \mathcal{H} is a pseudo-Riemannian submersion in the sense given by O’Neill in Def 7.44, [O’N83]. For O’Neill, the horizontal space is specifically the normal space with respect to the pseudo-Riemannian metric; here \mathcal{H} is used instead as it is unclear *a priori* whether $\mathfrak{j} \cap \mathfrak{j}^\perp = \{0\}$. Nonetheless, \mathcal{H} satisfies all the necessary properties for the following:

Lemma 2.2.2. *For any two smooth vector fields X, Y on $J\backslash H/\Gamma$,*

$$(a) \langle \bar{X}, \bar{Y} \rangle = \langle X, Y \rangle \circ \pi,$$

$$(b) \mathcal{H}([\bar{X}, \bar{Y}]) = \overline{[X, Y]},$$

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$$(c) \mathcal{H}(\bar{\nabla}_{\bar{X}}\bar{Y}) = \overline{(\nabla_X Y)},$$

where $\mathcal{H}(\cdot)$ denotes the horizontal part and $\bar{\nabla}$ and ∇ are the Levi-Civita connections for g_{PR} on H/Γ and $J\backslash H/\Gamma$, respectively.

Proof. The proof follows that of O'Neill's Lemma 7.45 almost verbatim. (a) restates that $\pi_* : \mathcal{H} \rightarrow T(J\backslash H/\Gamma)$ is an isometry. (b) follows from the easy fact that $\pi_*([\bar{X}, \bar{Y}]) = [X, Y]$. The third claim will hold if both sides have the same inner product under the Killing form with every horizontal lift \bar{Z} . This strategy needs the observation that the Killing form, restricted to \mathcal{H} , is nondegenerate, which follows easily from (a) and the nondegeneracy of g_{PR} provided by Proposition 2.2.1. Thus, the lemma will follow if one can prove the middle equality in

$$\langle \mathcal{H}(\bar{\nabla}_{\bar{X}}\bar{Y}), \bar{Z} \rangle = \langle \bar{\nabla}_{\bar{X}}\bar{Y}, \bar{Z} \rangle \stackrel{?}{=} \langle \nabla_X Y, Z \rangle \circ \pi = \langle \overline{(\nabla_X Y)}, \bar{Z} \rangle$$

for every smooth vector field Z on $J\backslash H/\Gamma$. The first equality holds as the horizontal space is normal to the J -fibers under the Killing form; the last equality is (a). To complete the proof, use the Koszul formula to expand the middle two terms and note

$$\bar{X}\langle \bar{Y}, \bar{Z} \rangle = \bar{X}(\langle Y, Z \rangle \circ \pi) = \pi_*(\bar{X})\langle Y, Z \rangle = (X\langle Y, Z \rangle) \circ \pi$$

and

$$\langle [\bar{X}, \bar{Y}], \bar{Z} \rangle = \langle \overline{[X, Y]}, \bar{Z} \rangle = \langle [X, Y], Z \rangle \circ \pi$$

using (a) and (b). □

To sum up: there is a well-defined distribution \mathcal{H} complementary to TJ and one can calculate g_{PR} , vector field brackets and the Levi-Civita connection by taking lifts to this distribution.

$J\backslash H/\Gamma$ is a manifold, so by general arguments it also carries a Riemannian metric, g_R , about which one knows very little *a priori*. A main task of this paper will be to extract connections between the dynamical behavior of g_R (namely Lyapunov exponents) and the algebraic structure of H .

2.3 Distributions

Consider the adjoint action of $J \times A$ on \mathcal{H} (recall that \mathcal{H} is invariant under this action). As J is reductive, $J \times A$ is reductive, so this representation is completely reducible. Write \mathcal{H} as a sum of irreducible representations

$$\begin{aligned} \mathcal{H} &= \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \\ &\oplus \mathbb{R}\mathfrak{a}_1 \oplus \cdots \oplus \mathbb{R}\mathfrak{a}_{\text{rank}(G)} \\ &\oplus \mathbb{R}\mathfrak{z}_1 \oplus \cdots \oplus \mathbb{R}\mathfrak{z}_{\dim(Z)} \\ &\oplus \mathbb{R}\mathfrak{u}_1 \oplus \cdots \oplus \mathbb{R}\mathfrak{u}_k \end{aligned}$$

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where $\mathfrak{a}_i, \mathfrak{z}_i$ and \mathfrak{u}_i are bases for the Lie algebras of A, Z and the nilpotent elements in \mathfrak{g} , respectively. Note that the \tilde{V}_i can be chosen so that the \mathfrak{g} -action on them takes a simple form. Specifically, fix any copy of $\mathfrak{sl}_2(\mathbb{R})$ in \mathfrak{g} with diagonal element a and upper and lower unipotents U and L . By beginning with the standard representation theory of this $\mathfrak{sl}_2(\mathbb{R})$ on \mathcal{H} and using the fact that it commutes with $Ad(J)$ one can easily pick the \tilde{V}_i such that $ad(U)\tilde{V}_i$ and $ad(L)\tilde{V}_i$ each equal some other \tilde{V}_j or are zero and, in addition, if $ad(U)\tilde{V}_i = \tilde{V}_j$ then $ad(L)\tilde{V}_j = \tilde{V}_i$. That is, $ad(U)$ and $ad(L)$ act as raising and lowering operators on a sub-collection of the \tilde{V}_i . Choices of the \tilde{V}_i with this property will be used below.

In the basic example, the following irreducible representations work for the $\mathfrak{sl}_2(\mathbb{R})$'s associated to the standard root-space decomposition of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$:

$$\begin{pmatrix} 0 & \cdots & 0 & | & | & | \\ \vdots & \ddots & \vdots & \tilde{V}_1 & \tilde{V}_2 & \tilde{V}_3 \\ 0 & \cdots & 0 & | & | & | \\ - & \tilde{V}_4 & - & 0 & 0 & 0 \\ - & \tilde{V}_5 & - & 0 & 0 & 0 \\ - & \tilde{V}_6 & - & 0 & 0 & 0 \end{pmatrix}.$$

The \tilde{V}_i are right- H -invariant distributions on TH so they are well-defined on H/Γ . As these distributions are also $Ad(J)$ -invariant, they descend to distributions V_i on $J\backslash H/\Gamma$. Note that the V_i are preserved by the A -action on $J\backslash H/\Gamma$.

2.4 Measures and volume forms

Denote by m the Haar measure on H . Since H is unimodular, by well-known arguments m descends to a well-defined measure on $J\backslash H/\Gamma$. This measure, also denoted m , is preserved by the G -action.

Furthermore, the distributions \tilde{V}_i carry natural volume forms. Fix a basis $\{v_1(id), \dots, v_m(id)\}$ of $\tilde{V}_i(id)$ and extend these to right-invariant vector fields $\{v_1, \dots, v_m\}$; these form a basis of \tilde{V}_i at any point in H . Then $\widetilde{vol}_i = v_1 \wedge \cdots \wedge v_m$ is a right-invariant volume form on \tilde{V}_i . If such volume forms, or forms derived from them, are $Ad(J)$ -invariant, they will descend to forms on the V_i . For the example $H = SL_n(\mathbb{R})$, $J = SL_{n-3}(\mathbb{R})$, this is the case as the $Ad(J)$ -action acts by the standard $SL_{n-3}(\mathbb{R})$ -action on distributions isomorphic to \mathbb{R}^{n-3} . The general case requires somewhat more careful work.

For each i consider the irreducible representation $Ad : J \times A \rightarrow GL(\tilde{V}_i)$. $J \times A$ is reductive; under Ad its semisimple part maps into $SL(\tilde{V}_i)$, which preserves \widetilde{vol}_i , and its center maps into the center of $GL(\tilde{V}_i)$. Notice, for later use, that for any $a \in A$, $Ad(a)$ acts by a scalar on \tilde{V}_i . The following theorem is due to Benoist and Labourie:

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Theorem 2.4.1 ([BL92] Corollaire 1). *If $J \backslash H$ admits a compact quotient then J has compact center.*

Let Z_J be the (compact) center of J . Construct the following finite form on \tilde{V}_i :

$$\widehat{vol}_i = \int_{j \in Z_J} j_*(\widetilde{vol}_i) dh_{Z_J}(j)$$

where h_{Z_J} is the Haar measure on Z_J . This construction averages out the volume form, providing a finite volume form invariant under the adjoint action of $Ad^{-1}(C)$, where C is the center of $GL(\tilde{V}_i)$. Therefore $Ad(J)$ preserves \widehat{vol}_i and \widehat{vol}_i descends to a volume form vol_i on V_i .

3 Cocycle superrigidity

The objects detailed above will come into play extensively below, but a rigidity result is the starting point for this proof. This is Zimmer's cocycle superrigidity theorem. This approach to the compact forms problem was first used by Zimmer in [Zim94] and continued in his work with Labourie [LZ95] and Labourie and Mozes [LMZ95]. The theorem is stated below in somewhat less than its full generality. For a full treatment, see [Zim84].

Definition 3.0.2 (Cocycle). For a group G acting on a space X , a cocycle over this action with values in a group J is a map $\alpha : G \times X \rightarrow J$ satisfying:

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 x) \alpha(g_2, x).$$

Definition 3.0.3 (Equivalence of cocycles). Two cocycles α and β are said to be equivalent if there exists a map $P : X \rightarrow J$ such that

$$\beta(g, x) = (P(gx))^{-1} \alpha(g, x) P(x).$$

Cocycles and equivalences are only required to be measurable, and the equations above to hold for almost all values of g, g_1, g_2 and x .

Theorem 3.0.4 (Cocycle superrigidity, see [Zim84]). *Let G be a connected, semisimple Lie group of finite index in the real points of an algebraic \mathbb{R} -group, with \mathbb{R} -rank(G) ≥ 2 and no compact factors. Let X be an irreducible ergodic G -space with finite invariant measure. Let J be a connected, simple, noncompact, of finite index in the real points of an algebraic \mathbb{R} -group, and have trivial center. Suppose $\alpha : G \times X \rightarrow J$ is a cocycle not equivalent to a cocycle taking values in a proper algebraic subgroup $L \subset J$. Then there exists a rational homomorphism $\rho : G \rightarrow J$ defined over \mathbb{R} such that α is equivalent to the cocycle $(g, x) \mapsto \rho(g)$.*

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The application of cocycle superrigidity proceeds as follows. One works with the G -action as the fact that all simple factors of G have rank at least two ensures that the irreducibility condition of superrigidity is satisfied for each simple factor. First, as H has finite center we may assume, up to a finite cover, that it is center-free. Let B be the center of J and consider the following diagram:

$$\begin{array}{ccc}
 J & \longrightarrow & H/\Gamma \\
 \downarrow & & \downarrow \\
 B \backslash J & \longrightarrow & B \backslash H/\Gamma \\
 & & \downarrow \\
 & & J \backslash H/\Gamma
 \end{array}$$

G acts on the J -bundle $H/\Gamma \rightarrow J \backslash H/\Gamma$ and the $B \backslash J$ -bundle $B \backslash H/\Gamma \rightarrow J \backslash H/\Gamma$ by bundle automorphisms, preserving the finite measure m on $J \backslash H/\Gamma$. Note that $B \backslash J$ is a semisimple Lie group. As J is reductive in the center-free H , we may pass to the adjoint representation to see that H , and hence J are finite index in algebraic \mathbb{R} -groups.

Let $\bar{\sigma}$ be a measurable section of $B \backslash H/\Gamma \rightarrow J \backslash H/\Gamma$. This section relates the G actions on $B \backslash H/\Gamma$ and $J \backslash H/\Gamma$:

$$\bar{\sigma}(g \cdot x) = \alpha(g, x)g \cdot \bar{\sigma}(x)$$

where $\alpha(g, x)$ is an element of $B \backslash J$. It is easy to check that α is a cocycle.

Cocycle superrigidity applies to this cocycle over any G -ergodic measure μ after perhaps making the following adjustments. First, one needs to further mod out the center of the algebraic hull of the cocycle from J (or $B \backslash J$). Second, it may be necessary to pass to a finite ergodic cover of the action on $J \backslash H/\Gamma$ to ensure that the algebraic hull is connected. If the algebraic hull is compact, one can move immediately to the averaging construction below; if not, again by taking a finite ergodic cover, we can ensure that the algebraic hull is finite index in J (or $B \backslash J$). The details of these modifications can be found in [Zim94] – clearly none of them affects the existence of a compact form. Translated back into the language of sections the application of cocycle superrigidity provides the following (see, [Zim94] and [LZ95]). There exist:

1. a rational homomorphism $\bar{\rho} : G \rightarrow B \backslash J$
2. a compact subgroup $K < Z_{B \backslash J}(\bar{\rho}(G))$, and
3. a choice of the measurable section $\bar{\sigma} : J \backslash H/\Gamma \rightarrow B \backslash H/\Gamma$ such that

$$\bar{\sigma}(g \cdot x) = g\bar{\rho}(g)\bar{c}(g, x) \cdot \bar{\sigma}(x) \text{ for } \mu\text{-a.e. } x, \quad (1)$$

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where $\bar{c}(g, x) \in K$. Note that $g\bar{\rho}(g)$ lies on the graph of $\bar{\rho}$, denoted henceforth as $gr(\bar{\rho})$.

Take any measurable section of the projection $H/\Gamma \rightarrow B \backslash H/\Gamma$ and compose it with $\bar{\sigma}$ to form a section σ of the J -bundle $H/\Gamma \rightarrow J \backslash H/\Gamma$. Let ρ be the rational homomorphism from G to $J \cong B \times B \backslash J$ defined by $\rho(g) = (id_B, \bar{\rho}(g))$. Let $c(g, x) = (id_B, \bar{c}(g, x))$. Then,

$$\sigma(gx) \in Bg\rho(g)c(g, x) \cdot \sigma(x). \quad (2)$$

The section σ allows one to work with a finite measure on the considerably simpler single coset space H/Γ . The lift $\sigma_*\mu$ is a finite measure on H/Γ . Let

$$\hat{\mu} = \int_{c \in B \times K} c_*(\sigma_*\mu) d\nu_{B \times K}(c)$$

where $\nu_{B \times K}$ denotes the Haar measure on $B \times K < B \times B \backslash J \cong J$. Note that by the result of Benoist-Labourie mentioned above ([BL92] Corollaire 1), B is compact, so $\hat{\mu}$ is a finite measure. (In the same way, moving to a finite cover where H is allowed finite center we still get a finite measure.) It is easy to check, using equation (2) that $\hat{\mu}$ is $gr(\rho)$ -invariant.

Remark 3.0.5. At this point, note a potential strategy for proving that compact forms do not exist, utilized by Labourie and Zimmer in [LZ95]. Consider the measure $\hat{\mu}$. Labourie and Zimmer note that if $Stab(\hat{\mu}) \cap J$ is non-compact there is a contradiction since $\hat{\mu}$ is a finite measure and J acts properly on H/Γ . In addition, for any h in H , one can construct the finite measure $h_*(\hat{\mu})$ which will have stabilizer $hStab(\hat{\mu})h^{-1}$. There will be a contradiction again if for any h , $hStab(\hat{\mu})h^{-1} \cap J$ is non-compact.

4 Ratner's Measure Classification Theorem

The next step in the proof is to apply Ratner's measure classification theorem.

Theorem 4.0.6 (Ratner's Measure Classification Theorem, [Rat91]). *Let μ be a probability measure on H/Γ where H is a Lie group and Γ is a discrete subgroup. Let U be a subgroup of H generated by one-parameter unipotent subgroups and suppose μ is ergodic for the U -action on H/Γ . Then μ is an algebraic measure, that is, there is some $h \in H$ and subgroup $L \supseteq U$ such that $L \cap h\Gamma h^{-1}$ is a lattice in L and μ is the Haar measure on the L -orbit through $[h]$.*

Apply this in the following way. For the acting group U , take $gr(\rho)$. Without loss of generality, we may assume that the higher-rank semisimple group G is generated by unipotent subgroups by restricting, if necessary, to the connected subgroup of G generated by its unipotent one-parameter

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subgroups. This group is clearly still higher-rank semisimple. The homomorphism ρ provided by superrigidity is rational, so it takes unipotent subgroups to unipotent subgroups. Therefore, $gr(\rho)$ is generated by the unipotents. For the measure, take $\hat{\mu}_e$, any ergodic component of $\hat{\mu}$ for the $gr(\rho)$ -action. Let L be the subgroup of H containing $gr(\rho)$ that arises from the application of Ratner's theorem, and denote by \mathfrak{l} the Lie algebra of L . This group is the link that will allow translation of the original problem into an entirely algebraic question.

The ergodic measure $\hat{\mu}_e$ projects to a G -ergodic measure on $J\backslash H/\Gamma$; from its construction it must project to (some multiple of) μ . One then sees that the subgroup L and its Lie algebra \mathfrak{l} reflect the measure μ with which we started. In particular, there should be a connection between the support of μ , and the directions in \mathfrak{l} which are transverse to \mathfrak{j} (i.e. $proj_{\mathcal{H}}\mathfrak{l}$). We now make this relationship precise.

Proposition 4.0.7. *Let $p \in \text{supp}(\mu)$. Let $v \in \mathcal{H}$ at basepoint $\sigma(p)$. Consider the path $\gamma_v(t) = \pi(\exp(tv)\sigma(p))$ formed by taking the integral curve to the vector field represented by v through $\sigma(p)$ and projecting it to $J\backslash H/\Gamma$. If for sufficiently small positive t accumulating at 0, $\gamma_v(t) \in \text{supp}(\mu)$, $v \in proj_{\mathcal{H}}\mathfrak{l}$. In particular, if this holds for a basis of \mathcal{H} , then $proj_{\mathcal{H}}\mathfrak{l} = \mathcal{H}$.*

Proposition 4.0.8. *If $proj_{\mathcal{H}}\mathfrak{l} = \mathcal{H}$ then $\text{supp}(\mu) = J\backslash H/\Gamma$ and $JL = H$.*

We prove the second proposition first:

Proof of Prop. 4.0.8. Let $p_1 : H \rightarrow K\backslash H$ and $p_2 : K\backslash H \rightarrow J\backslash H$ be the natural projections; p_2 is a $K\backslash J$ -fiber bundle. Let Lh be the L -orbit on H provided by Ratner's Theorem. The tangent space to Lh is \mathfrak{l} (considering Lie algebra elements as right-invariant vector fields). It is nearly transverse to the fibers, with the only intersection possibly lying in K , and its projection to \mathcal{H} is by assumption \mathcal{H} in TH . The tangent space to $p_1(Lh)$ therefore projects surjectively onto \mathcal{H} as well (abusing notation slightly – \mathcal{H} descends to $K\backslash H$ as it is $Ad(J)$ -invariant). On $K\backslash H$, $T(p_1(Lh))$ is a full transversal to the fiber direction.

First, $p_2 \circ p_1(Lh)$ is open. This follows from the fact that the tangent space to Lh is a full transversal to the tangent space of the $K\backslash J$ -fibers in $K\backslash H$.

Second, $p_2 \circ p_1(Lh)$ is closed. To prove this, put a right-invariant Riemannian metric on H and let it descend to a right-invariant metric on $K\backslash H$ by averaging over K . Let

$$\theta_0 = \min\{\angle(v, w) : v \in T(p_1(Lh)) - \{0\}, w \in T(K\backslash J) - \{0\}\}.$$

This angle is non-zero as $T(p_1(Lh))$ is transverse to the fiber directions, and it is independent of the basepoint chosen by right-invariance of the metric, \mathfrak{l} and the fiber direction. Let $x \in J\backslash H$ be a point in the closure

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of $p_2 \circ p_1(Lh)$ with sequence $\{y_n\} \subset p_1(Lh)$ such that $\{p_2(y_n)\} \rightarrow x$. Take a closed ball B around x over which the $K \backslash J$ -bundle trivializes and fix a smooth trivialization

$$\begin{aligned} p_2^{-1}(B) &\rightarrow B \times K \backslash J \\ y &\mapsto (\phi_1(y), \phi_2(y)). \end{aligned}$$

Let $ht(y) = d(y, (\phi_1(y), [id]))$ for d the distance in the right-invariant metric. This is a smooth function on $p_2^{-1}(B)$. If there exists a constant A such that for all $y \in p_2^{-1}(B) \cap p_1(Lh)$, $ht(y) < A$ then this portion of the closed set $p_1(Lh)$ is compact. Hence $p_2(p_2^{-1}(B) \cap p_1(Lh))$ is compact, hence closed, hence $x \in p_2 \circ p_1(Lh)$, proving the result. To find the constant A , fix any $y_0 \in p_2^{-1}(B) \cap p_1(Lh)$. For any $y \in p_2^{-1}(B) \cap p_1(Lh)$, there exists a path of length less or equal to the diameter of B joining $p_2(y_0)$ and $p_2(y)$. Given the angle bound θ_0 following a path in $p_1(Lh)$ from y_0 to y covering the path in B connecting their images, we easily calculate:

$$ht(y) \leq ht(y_0) + \frac{\text{diam} B}{\tan \theta_0}$$

giving the desired bound.

That $p_2 \circ p_2(Lh)$ is both open and closed and $J \backslash H$ is connected imply $p_2 \circ p_2(Lh) = J \backslash H$, i.e. $JL = H$. The support of μ contains the projection to $J \backslash H / \Gamma$ of the L -orbit, and hence is all of $J \backslash H / \Gamma$. □

Proof of Prop 4.0.7. For this proof, let $p_1 : H / \Gamma \rightarrow K \backslash H / \Gamma$ and $p_2 : K \backslash H / \Gamma \rightarrow J \backslash H / \Gamma$. Again, \mathcal{H} is well-defined on all these spaces. First, we claim that $proj_{\mathcal{H}} T(p_1(L[h])) = (p_1)_* proj_{\mathcal{H}} l$. Consider any $k \in K$ which takes a point on the L -orbit to another point on the L -orbit. For a generic point, the L orbit through the first point will be $L[h] = \overline{gr(\rho)h\Gamma}$ and for the second will be $\overline{gr(\rho)kh\Gamma} = \overline{kgr(\rho)h\Gamma} = kL[h]$ since K centralizes $gr(\rho)$. As this holds for generic points, we will have that at all points in $L[h]$, left-multiplication by such $k \in K$ preserves the L -orbit. It is then clear that to calculate $proj_{\mathcal{H}} T(p_1(Lh))$ we need only calculate $proj_{\mathcal{H}} l$ and then project to $K \backslash H / \Gamma$ to obtain the desired result.

By the same argument used in the proof of Proposition 4.0.8, $p_2 \circ p_1(L[h])$ is closed and hence is equal to the support of μ . Therefore the path $\gamma_v(t)$ assumed by the proposition implies (for a sequence of times $\{t_n\} \rightarrow 0$) that there exist points

$$\widetilde{\gamma_v(t_n)} \in p_2^{-1}(\gamma_v(t_n)) \cap p_1(L[h])$$

converging to $\tilde{p} \in p_2^{-1}(\gamma_v(0)) \cap p_1(L[h])$. This sequence of points lies on a curve in $p_1(L[h])$ which projects to $\gamma_v(t)$ in $J \backslash H / \Gamma$, so the projection to \mathcal{H} of the initial tangent vector to this curve \tilde{v} is $(p_1)_* v$. Thus, $proj_{\mathcal{H}} T(p_1(L[h]))$

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contains $(p_1)_*v$ so $v \in \text{proj}_{\mathcal{H}}\mathfrak{l}$. As $\text{proj}_{\mathcal{H}}\mathfrak{l}$ is a linear subspace of \mathcal{H} , if it contains a basis, it must equal \mathcal{H} . □

The connection described by these propositions between \mathfrak{l} and the support of μ will be essential below and drives one to a more careful study of the dynamics on $J \backslash H / \Gamma$. Before moving on to this study, note that the use of Ratner's theorem allows one to improve the result of superrigidity.

Proposition 4.0.9. *The section σ provided by superrigidity can be taken to satisfy the following:*

- For every point $p \in \text{supp}(\mu)$, $\sigma(p)$ belongs to the L -orbit supporting $\hat{\mu}_e$
- The equation

$$\sigma(g \cdot p) = gr(\rho)(g) \cdot \sigma(p)$$

holds up to an error given by left-multiplication by some element of the compact K for every $p \in \text{supp}(\mu)$ and all $g \in G$.

This improves on what was known above by specifying where the image of σ lies and by guaranteeing that the key equation holds not just for μ -almost every p and almost all g .

Proof. The projection $H/\Gamma \rightarrow J \backslash H / \Gamma$ takes $\hat{\mu}_e$ to (some multiple of) μ . The L -orbit $L \cdot [h]$ supports $\hat{\mu}_e$, so one can certainly assume that the (still only measurable) section σ takes its values in this L -orbit for all points belonging to the projection of this L -orbit to $J \backslash H / \Gamma$. The arguments in the proof of Proposition 4.0.8 show that this is the support of μ . As $\sigma(g \cdot p)$ lies on $L \cdot [h]$ and $gr(\rho)(g) \in L$, $gr(\rho)(g) \cdot \sigma(p)$ lies in $L \cdot [h]$ for any g and any $p \in \text{supp}(\mu)$. Lying over the same point in $J \backslash H / \Gamma$ is $\sigma(g \cdot p)$, so these two differ by left-multiplication by some element of J ; by the construction of $\hat{\mu}$, that element lies in K . □

5 Dynamics of the A -action

The dynamics of actions by left-multiplication on H/Γ are easily approached using the adjoint representation. The dynamics of the actions on the double coset space $J \backslash H / \Gamma$, however, are considerably more opaque. This section uses several techniques from smooth dynamics and measure rigidity to begin the proof that the G -action is ergodic for the measure m which descends from Haar measure on H . It begins by carefully choosing a G -ergodic component of m which will have a strong relation to the Lyapunov exponents for the A action. This is followed by a technique for calculating those exponents using cocycle superrigidity.

5.1 Conditional measures and entropy

In order to ensure that Lyapunov exponents provide good information about an ergodic measure, that measure must be chosen carefully. That task is carried out here, using some important results from smooth dynamics relating conditional measures and entropy.

Definition 5.1.1 (Lyapunov exponent). For a differentiable flow f_s on a manifold M and tangent vector $v \in TM$ the forward and backward Lyapunov exponents of v for the flow f_s are defined, respectively, by

$$\chi_+(v) = \overline{\lim}_{s \rightarrow \infty} \frac{1}{s} \log \|Df_s(v)\|,$$

$$\chi_-(v) = \overline{\lim}_{s \rightarrow \infty} \frac{1}{s} \log \|Df_{-s}(v)\|.$$

Recall some terminology from smooth dynamical systems. For a smooth flow with non-zero Lyapunov exponents, there are stable and unstable foliations defined almost everywhere (see [BP06]) with the property that leaves of the stable foliation are contracted by the flow in forward time and those of the unstable foliation are contracted in backward time. With respect to a measure μ , conditional measures along these foliations can be defined (see, e.g., [Lin06]) which, roughly speaking, account for how much the measure μ extends along leaves of the foliation. In addition, each leaf carries a natural volume induced by restricting the underlying Riemannian metric to the leaves. The conditional measure is called *absolutely continuous* if it is absolutely continuous with respect to this Riemannian volume. Recall that associated to any measure-preserving flow is a measure-theoretic entropy (see [KH95] for a good exposition). The entropy for a flow f_s with respect to a measure μ is denoted $h_\mu(f)$ here.

Consider the Cartan subgroup $A < G$. Take a basis $\{a_1, \dots, a_{r_k(G)}\}$ for \mathfrak{a} and let $A_1(s), \dots, A_{r_k(G)}(s)$ be the corresponding one-parameter subgroups.

Lemma 5.1.2. *There exists a G -ergodic measure μ , an ergodic component of m , for which conditional measures for the stable and unstable foliations of each A_i -flow are absolutely continuous for the leaves through μ -almost every point.*

Proof. The proof proceeds by recalling some important results on entropy for partially hyperbolic systems. The Pesin entropy formula (see [BP06]), states that for a smooth measure, such as m , the measure theoretic entropy for the A_i -action is

$$h_m(A_i) = \int \sum_{j \leq u(x)} \lambda_j(x) \dim E_j(x) dm(x).$$

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The $\lambda_j(x)$ are the positive Lyapunov exponents at x , $u(x)$ is the number of positive exponents at x and $E_j(x)$ is the distribution corresponding to the exponent $\lambda_j(x)$.

The Ledrappier-Young formula ([LY85b], Theorem C), which holds for any Borel measure and any C^2 flow, states that there exist numbers $\gamma_j(x)$ satisfying $0 \leq \gamma_j(x) \leq \dim E_j(x)$ such that

$$h_m(A_i) = \int \sum_{j \leq u(x)} \lambda_j(x) \gamma_j(x) dm(x).$$

Furthermore, the numbers $\gamma_j(x)$ are constant on ergodic components and are related to the measure theoretic dimension of the conditional measures on the stable and unstable manifolds.

Consider the ergodic decomposition of m , first for a single flow $A_i(s)$. In order for the entropies of the $A_i(s)$ -ergodic components to produce the full entropy for m , for almost every $A_i(s)$ -ergodic component of m , all $\gamma_j(x)$ must equal $\dim E_j(x)$. This same is true for the backward time flow $A_i(-s)$ so for almost every A_i -ergodic component μ_i , all $\gamma_j(x) = \dim E_j(x)$ for the flows in both directions. That is, the Pesin entropy formula holds for μ_i for both the forward and backward A_i -flows. A further result of Ledrappier and Young states that the Pesin entropy formula holds for a measure if and only if its conditional measures are absolutely continuous with respect to the Riemannian volume on the stable foliation ([LY85a], Theorem A). Using the A_i -flow in both directions, this implies that for almost every A_i -ergodic component μ_i the conditional measures for A_i 's stable and unstable foliations are absolutely continuous.

Now consider the ergodic decomposition of m for the full G -action. Each G -ergodic component further decomposes into A_i -ergodic components. For each i , almost all of these sub-ergodic components have absolutely continuous conditional measures. Therefore, there must exist some G -ergodic measure μ (in fact, almost every choice of μ will work) such that almost all of the A_i -ergodic measures (for all choices of i) that compose it satisfy the conditions of the lemma on conditional measures. Such a measure is what the lemma calls for. \square

Lyapunov exponents, stable and unstable distributions and foliations, and conditional measures are only defined on full measure sets to begin with. Thus, nothing has really been lost in the conclusion that the desired result on conditional measures holds μ -almost everywhere. The work in section 3 can be applied to this measure. Quite important to the strategy in this paper is the fact that there is total freedom to choose a G -ergodic measure μ with which to work. A measure fixed ahead of time could conceivably be supported on a closed G -orbit and thus not satisfy the conclusions of Lemma 5.1.2.

5.2 Superrigidity and Lyapunov exponents

Lemma 5.1.2 provides a foothold for understanding the G -ergodic measure μ via smooth dynamics and Lyapunov exponents. This section uses cocycle superrigidity to calculate Lyapunov exponents for the G -action, showing that many are nonzero. In addition, superrigidity can be used to show that the dynamics on $J\backslash H/\Gamma$ are unusually nice. In particular, all points in the support of μ are Lyapunov regular.

Much of the dynamics of the G -action on H/Γ , including its Lyapunov exponents, is easily understood via the use of a right-invariant metric and the adjoint action on \mathfrak{h} . However, because the J -fibers over $J\backslash H/\Gamma$ are non-compact there is no *a priori* way to relate the metric behavior of the dynamics on the two spaces. Using the section σ provided by superrigidity allows this.

The section σ obeys the equation

$$\sigma(g \cdot p) = gr(\rho)(g) \cdot \sigma(p) \quad (3)$$

up to an error in compact subgroup K for every $p \in \text{supp}(\mu)$ and all $g \in G$, thanks to Proposition 4.0.9. To relate Lyapunov exponents in $J\backslash H/\Gamma$ with Lyapunov exponents in H/Γ , one needs a way to lift vectors on $J\backslash H/\Gamma$ to H/Γ . Let X be a vector in $T_p(J\backslash H/\Gamma)$. Define $\bar{\sigma}(X)$ to be the unique lift of X based at $\sigma(p)$ and lying in \mathcal{H} , where the Killing form is used to define the perpendicular space to the J -fibers.

Relating the dynamics on the tangent spaces of $J\backslash H/\Gamma$ and H/Γ is the following analogue to equation (3):

$$\bar{\sigma}(Dg(X)) = Dgr(\rho)(g)\bar{\sigma}(X) \quad (4)$$

up to left-multiplication by an element in K . To prove equation (4), note that the projections of both sides to $J\backslash H/\Gamma$ are the same. In addition, up to the error in K , the basepoints of both sides correspond due to equation (3). Finally, since the left-multiplication by K and $gr(\rho)(G)$ preserve \mathcal{H} , both sides of equation (4) lie in \mathcal{H} . Therefore, they are equal up to the error in K .

Equation (3) guarantees that the dynamics of the $gr(\rho)(G)$ -action on H/Γ over the chosen ergodic component take place entirely in the compact set $KL \cdot [h]$; compactness of $L \cdot [h]$ follows from the arguments of Proposition 4.0.8. This allows comparison of the Riemannian metric g_R on $J\backslash H/\Gamma$ and the right-invariant metric g_{rt} on H/Γ . Particularly, there exists a uniform constant C such that for all p in $\text{supp}(\mu)$, all $X \in T_p(J\backslash H/\Gamma)$ and all $k \in K$

$$\frac{1}{C} \|k_* \bar{\sigma}(X)\|_{g_{rt}} \leq \|X\|_{g_R} \leq C \|k_* \bar{\sigma}(X)\|_{g_{rt}}. \quad (5)$$

From this equation and equation (4) applied to an $A_i(s)$ -flow,

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$$\frac{1}{C} \|gr(\rho)(A_i(s))_* \bar{\sigma}(X)\|_{g_{rt}} \leq \|A_i(s)_* X\|_{g_R} \leq C \|gr(\rho)(A_i(s))_* \bar{\sigma}(X)\|_{g_{rt}}. \quad (6)$$

The final equation shows that the Lyapunov exponent for X under the $A_i(s)$ -flow with respect to g_R is equal to the Lyapunov exponent for $\bar{\sigma}(X)$ under $gr(\rho)(A_i(s))$ with respect to g_{rt} . This argument yields the following conclusion:

Proposition 5.2.1. *Let $\mathcal{A}_i = Dgr(\rho)(a_i)$ (recall, a_i is a Lie algebra generator for $A_i(s)$). For any X with footpoint in the support of μ , consider $\bar{\sigma}(X)$ as a representative of a right-invariant vector field in \mathfrak{h} . If $\bar{\sigma}(X)$ has weight λ for $ad(\mathcal{A}_i)$, the Lyapunov exponent for X is λ .*

Proof. This follows from the preceding discussion and the fact that the weights for the adjoint action are Lyapunov exponents on H/Γ for the right-invariant metric. \square

The ability to calculate Lyapunov exponents using the particularly well-behaved dynamics on H/Γ yields a further result.

Definition 5.2.2 (Lyapunov regular set; see [BP06], section 1). If a flow f_t on a manifold M has forward and backward Lyapunov exponents $\chi_{\pm}^i(p)$ at a point p , let

$$V_{\pm}^i(p) = \{v \in T_p M : \chi_{\pm}^i(v) \leq \chi_{\pm}^i(p)\}.$$

The $V_{\pm}^i(p)$ form filtrations of $T_p M$. One says these filtrations *comply* at p if the numbers of distinct forward and backward Lyapunov exponents are the same and if the subspaces $E_i(p) = V_+^i(p) \cap V_-^i(p)$ form a splitting

$$T_p M = \bigoplus_i E_i(p).$$

The point p is called *Lyapunov regular* if

1. the filtrations comply at p ;
2. for all i and any nonzero $v \in E_i(p)$

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|D_p f_t(v)\| = \chi_+^i(p) = -\chi_-^i(p) \stackrel{def}{=} \chi_i(p)$$

with uniform convergence on $\{v \in E_i(p) : \|v\| = 1\}$;

- 3.

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |\det D_p f_t| = \sum_i \chi_i(p) \dim E_i(p).$$

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The Multiplicative Ergodic Theorem (see [BP06], Theorem 1.2) states that for a diffeomorphism of a smooth manifold the set of Lyapunov regular points is full measure for any invariant Borel measure. However, here one has the following:

Proposition 5.2.3. *All points $p \in \text{supp}(\mu)$ belong to the Lyapunov regular set for the A -action.*

Proof. Let $V(v_1, \dots, v_k)$ denote the k -volume of the parallelepiped spanned by the v_i . The following criterion for Lyapunov regularity of the point p is given in [BP06], Theorem 1.1. A point p is Lyapunov regular if and only if

1. for any $v_1, \dots, v_k \in T_p M$ the limit

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log V(D_p f_t v_1, \dots, D_p f_t v_k)$$

exists and if $v_1, \dots, v_k \in E_i(p)$ and $V(v_1, \dots, v_k) \neq 0$ then

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log V(D_p f_t v_1, \dots, D_p f_t v_k) = k\chi_i(p),$$

2. if $v \in E_i(p)$ and $w \in E_j(p)$ are nonzero and $i \neq j$ then

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |\sin \angle(D_p f_t v, D_p f_t w)| = 0.$$

The requirements of this criterion can be checked using the observation above that over the compact set $KL \cdot [h]$ in which all the dynamics for the $gr(\rho)$ -action take place, the Riemannian metric on $J \backslash H/\Gamma$ and the right-invariant metric for horizontal lifts of vectors are the same up to a uniform multiplicative constant (see equation (5)). Analogous comparisons can be made for the k -volumes of a set of vectors on $J \backslash H/\Gamma$ and of their horizontal lifts, and for the sines of the angles between such lifts. All the constants disappear when taking the limits above. Thus, Lyapunov regularity for all points in $\text{supp}(\mu)$ for the G -action on $J \backslash H/\Gamma$ with respect to the Riemannian metric g_R follows from the fact that all points in H/Γ are Lyapunov regular for the $gr(\rho)$ -action and the right-invariant metric. \square

The scheme of proof for Proposition 5.2.1 also allows one to say something about stable manifolds.

Suppose $Y \in T_p(J \backslash H/\Gamma)$ with p in the support of μ , and suppose $\chi(Y) = \lambda < 0$ for A_s . Fix the lift $\tilde{p} = \sigma(p)$ of p in H/Γ ; $\bar{Y}(\tilde{p})$ denotes the horizontal lift of Y at \tilde{p} . Let \bar{Y} denote as well the right-invariant vector field on H/Γ corresponding to this vector. Let p' be the point on $J \backslash H/\Gamma$ covered by $\tilde{p}' = \exp(\bar{Y})\tilde{p}$. Let $\bar{Y}' = \bar{Y}(\tilde{p}')$.

Lemma 5.2.4. $\chi(Y') = \lambda$ for the A_s -flow, and the path in $J \backslash H/\Gamma$ covered by $\exp(\tau \bar{Y})\tilde{p}$ lies in the stable manifold at p for A_s .

Proof. First of all, over any point in the support of μ , Proposition 5.2.1 holds: the Lyapunov exponent for \bar{Y} for the right-invariant metric g_{rt} and the $gr(\rho)(A_s)$ -action is the same as that for Y with respect to the Riemannian metric g_R and the A_s -action, namely λ . As $\lambda < 0$, under the $gr(\rho)(A_s)$ -flow the magnitude of \bar{Y} for the right-invariant metric decreases; in turn, the length of the path $gr(\rho)(A_s)\exp(\tau \bar{Y})$ for $\tau \in [0, 1]$ decreases. Therefore, the footpoint of $Dgr(\rho)(A_s)\bar{Y}'$ remains a bounded g_{rt} -distance from the compact set $KL \cdot [h]$. This bounded distance neighborhood is compact as well, so again the magnitudes of $Dgr(\rho)(A_s)\bar{Y}'$ for the right-invariant metric and $(A_s)_*Y'$ can be compared as in equation (5), and its Lyapunov exponent can be calculated in the same way.

The upshot of this is twofold. First, \bar{Y}' is in the same right-invariant field as \bar{Y} , so both have the same weight for the adjoint representation of $Dgr(\rho)(a_i)$. Therefore, Y' has the same Lyapunov exponent as Y . This holds for any tangent vector to the path covered by $\exp(\tau \bar{Y})\tilde{p}$, so the second point, that this path covers a path in the stable manifold, follows as well. \square

Remark 5.2.5. In the above, one lifts a vector to H/Γ and uses the exponential map there to produce a path which is then projected back down to $J \backslash H/\Gamma$, tangent to the original vector. In the lemma, it was important to take the lift at the point $\sigma(p)$ to allow comparison of Lyapunov exponents for the two Riemannian metrics \tilde{g}_R and g_{rt} . However, to calculate the path covered by $\exp(\tau \bar{Y})\tilde{p}$ one is free to choose another lift of p . Suppose $j\tilde{p}$ is a second lift of p . The horizontal lift of v to this basepoint is $j_*\bar{v}$. If \bar{V} denotes the right-invariant field with $\bar{V}(\tilde{p}) = \bar{v}$, the right-invariant field corresponding to $j_*\bar{v}$ is $Ad(j)\bar{V}$. The path $\exp(\tau Ad(j)\bar{V})j\tilde{p} = j \exp(\tau \bar{V})\tilde{p}$ is the left-translation by j of the path constructed at \tilde{p} . Therefore both project to the same path on $J \backslash H/\Gamma$. Therefore, one can define $\exp(\tau Y)p$ as the path covered by $\exp(\tau \bar{Y})\tilde{p}$ independent of a specific choice of \tilde{p} (and corresponding choice of \bar{Y}). This freedom in picking a lift will be exploited below.

Together with the Proposition 4.0.7 and Lemma 5.1.2, Proposition 5.2.1 provides the following: if all directions in \mathcal{H} have nonzero weight for $Dgr(\rho)(a_i)$ for some i , $proj_{\mathcal{H}}\mathfrak{l} = \mathcal{H}$. The task from here is to deal with the possibility of directions in \mathcal{H} with zero weight. This begins with a consideration of the action of unipotent elements.

6 The dynamics under unipotent elements

The goal of this section is to understand how Lyapunov exponents for the $A_i(s)$ -flows are affected by unipotent elements in G . Once again the situation

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is quite simple for a single quotient H/Γ with the right-invariant metric but rather opaque at the start for $J\backslash H/\Gamma$.

As G is semisimple, one can pick the Lie algebra elements a_i in a Cartan subalgebra and such that each lies in a copy of $\mathfrak{sl}_2(\mathbb{R})$. Fix one of the corresponding flows $A_i(s)$ and call it A_s for now. Let \bar{U} and \bar{L} be the upper and lower unipotent elements in the $\mathfrak{sl}_2(\mathbb{R})$ corresponding to a and let u_t, l_t be the flows they generate. Note that \bar{U} and \bar{L} are horizontal and descend to well-defined vector fields U and L on $J\backslash H/\Gamma$.

Recall that one can make a careful choice of the \tilde{V}_i as discussed in Section 2.3 and adjust their labeling (to ease notation) so that there is a string V_0, V_1, \dots, V_r where $ad(\bar{L})V_0 = 0$ and $ad(\bar{U})V_r = 0$, $ad(\bar{U})V_i = V_{i+1}$ and $ad(\bar{L})V_i = V_{i-1}$. One works with strings of length at least two below; all strings containing a V_i with non-zero weight for A are at least length two by standard $\mathfrak{sl}_2(\mathbb{R})$ representation theory. To make what follows somewhat more concrete, here is the picture for $SL_{n-3}(\mathbb{R})\backslash SL_n(\mathbb{R})$, where $r = 1$:

$$\begin{pmatrix} 0 & \cdots & 0 & | & | & 0 \\ \vdots & \ddots & \vdots & \tilde{V}_0 & \tilde{V}_1 & \vdots \\ 0 & \cdots & 0 & | & | & 0 \\ 0 & \cdots & 0 & 0 & \bar{U} & 0 \\ 0 & \cdots & 0 & \bar{L} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this section, the effect of unipotents on Lyapunov exponents will be addressed in two ways. The effect of the operator ∇_U , defined using the Levi-Civita connection for the pseudo-Riemannian metric, will be addressed first. Then one addresses how $(u_t)_*$ changes the exponents. These two operations will be played against one another in section 7 to get further information about μ .

6.1 The Levi-Civita connection and Lyapunov exponents

This section describes a way to use the Levi-Civita connection to gain more information about Lyapunov exponents.

Let v be a vector in $T_p(J\backslash H/\Gamma)$. To make sense of the object $\nabla_U v$ one must extend v locally to a smooth vector field V . One can do this in a canonical way as follows. Fix a lift \tilde{p} of p and lift v to a horizontal vector \bar{v} at \tilde{p} . Consider the right-invariant vector field corresponding to \bar{v} and specifically look at its restriction to a small segment around \tilde{p} of the path $u_t \tilde{p}$. Project this down to $J\backslash H/\Gamma$, obtaining a vector field along a small segment around p of the path $u_t p$. First, note that this process is independent of the choice of lift. If $\tilde{p}' = j\tilde{p}$, $\bar{v}' = j_* \bar{v}$ (using left- J -invariance of \mathcal{H}). The right-invariant field containing \bar{v}' is j_* of the right-invariant field containing \bar{v} . As j and u_t commute, $u_t \tilde{p}' = j u_t \tilde{p}$. Thus, the vector fields

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restricted to these path segments are related by j_* and project to the same thing on $J\backslash H/\Gamma$.

Extend the result of this projection locally to a smooth vector field on $J\backslash H/\Gamma$, denoted V . Then $V(p) = v$ and the values of $V(u_t p)$ – the only values which matter for the calculation of $\nabla_U V$ – are canonical. Define $\nabla_U v := (\nabla_U V)(p)$.

The connection can be calculated explicitly using the fact that $H/\Gamma \rightarrow J\backslash H/\Gamma$ is a pseudo-Riemannian submersion and the formula for the Levi-Civita connection for a bi-invariant metric on H/Γ (see Lemma 2.2.2 or [O’N83] Chapter 7 on pseudo-Riemannian submersions and [O’N83] Chapter 11 for bi-invariant metrics). This proceeds as follows. Lift U and V to horizontal vector fields on H/Γ . There one calculates their bracket and $2\nabla_U V$ is the projection of the bracket back down to $J\backslash H/\Gamma$. Specifically, look at these lifts at \tilde{p} . By construction, \tilde{V} has value \tilde{v} here, and (locally) along $u_t \tilde{p}$, corresponds to a right-invariant field. Likewise, \tilde{V} has values along $u_t \tilde{p}'$ corresponding to j_* of this right-invariant field. As only the values of \tilde{V} along these paths matter for the calculation of $\nabla_U V(p)$ one can take \tilde{V} to be a right-invariant vector field for the purposes of the bracket calculation (and *only* these purposes, as \tilde{V} is not right-invariant). Then the Lie algebra structure can be used to compute the bracket: $\frac{1}{2}[\tilde{U}, \tilde{V}](\tilde{p})$ projects down to $\nabla_U v$. To ensure this is well-defined, note that in the Lie algebra, $j_* \tilde{V}$ corresponds to $Ad(j)\tilde{V}$ and $[\tilde{U}, Ad(j)\tilde{V}] = [Ad(j)\tilde{U}, Ad(j)\tilde{V}] = Ad(j)[\tilde{U}, \tilde{V}]$. The value of $Ad(j)[\tilde{U}, \tilde{V}]$ at $\tilde{p}' = j\tilde{p}$ is $j_*([\tilde{U}, \tilde{V}](\tilde{p}))$. Thus both yield the same $\nabla_U v$.

One result of this is that the operation of ∇_U on $J\backslash H/\Gamma$ is covered by the operation of $\frac{1}{2}ad(\tilde{U})$ on right-invariant vector fields on H/Γ . As $ad(\tilde{U})$ maps \tilde{V}_i bijectively to \tilde{V}_{i+1} , one can write any vector v in V_{i_0} as $\nabla_U^{i_0} Y$ for a unique vector Y in V_0 .

The next lemma accounts for how ∇_U affects the Lyapunov exponents, and will be extremely useful below.

Lemma 6.1.1. *For $Y \in V_i$ with $i < r$,*

$$\chi_+(\nabla_U Y) = \chi_+(Y) + 2 \text{ and } \chi_-(\nabla_U Y) = \chi_-(Y) - 2.$$

Likewise for $i > 0$,

$$\chi_+(\nabla_L Y) = \chi_+(Y) - 2 \text{ and } \chi_-(\nabla_L Y) = \chi_-(Y) + 2.$$

Proof. These come from the fact that the Lyapunov exponent for U is 2 and for L is -2, facts easily seen from the commutation relations among A_s , u_t and l_t . Below is the argument for forward Lyapunov exponents and ∇_U ; the arguments for backward exponents and for ∇_L are analogous.

$$(A_s)_*(\nabla_U Y) = \nabla_{(A_s)_* U} (A_s)_* Y$$

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by the left-invariance of ∇ . Assume $\chi_+(Y) = \lambda$. Then, for s tending to $+\infty$, and any $\delta > 0$,

$$\left| \frac{1}{D_\delta} e^{(\lambda+2-\delta)s} \nabla_{\widehat{(A_s)_*U}}(\widehat{(A_s)_*Y}) \right| \leq |\nabla_{(A_s)_*U}(A_s)_*Y| \leq |D_\delta e^{(\lambda+2+\delta)s} \nabla_{\widehat{(A_s)_*U}}(\widehat{(A_s)_*Y})|$$

for some constant D_δ depending on δ . Here $\widehat{}$ denotes the unit vector in the relevant direction. Note that $\widehat{(A_s)_*U}$ can be taken to be U , with the right choice of a Riemannian metric. For any unit vector u in V_i , using the procedure for locally extending u as above, $\nabla_U u$ lies in V_{i+1} and is nonzero. Using compactness and the canonical definition of $\nabla_U u$, one can bound

$$\frac{1}{E} \leq |\nabla_U u| \leq E.$$

Combining this with the above produces

$$\frac{1}{D_\delta E} e^{(\lambda+2-\delta)s} \leq |\nabla_{(A_s)_*U}(A_s)_*Y| \leq D_\delta E e^{(\lambda+2+\delta)s}$$

as $s \rightarrow \infty$. As such a bound holds for all positive δ ,

$$\chi_+(\nabla_U Y) = \lambda + 2.$$

□

6.2 The unipotent flows and Lyapunov exponents

The next step is to consider how the unipotent flow $(u_t)_*$ for $t > 0$ affects Lyapunov exponents. The argument begins by showing that $\chi_+((u_t)_*Y) \geq \lambda + 2r$ using the A_s -flow in backward time. It then uses growth rates for volume forms on the V_i distributions to show that strict equality must hold. Recall that E_λ is the Lyapunov space for the exponent λ .

Proposition 6.2.1. *Suppose $Y \in E_\lambda$, $Y \in V_0(p)$. Then for any t , the forward Lyapunov exponent $\chi_+((u_t)_*Y) = \lambda + 2r$.*

Proof. Under A_{-s} as $s \rightarrow \infty$, $u_t p$ goes to $p_s := A_{-s} u_t p = u_{te-2s} A_{-s} p$. Likewise, $(u_t)_* Y$ goes to $(u_{te-2s})_*(A_{-s})_* Y$ (see Figure 1).

Let d^* be the metric on $T^1(J \backslash H / \Gamma)$ which is the product metric of the Riemannian metric for the base space and angular distance on the fibers.

Estimate the growth of $\|(A_{-s} u_t)_* Y\|$ as follows. As $s \rightarrow \infty$,

$$\frac{1}{C_\delta} e^{(-\lambda-\delta)s} \leq \|(A_{-s})_* Y\| \leq C_\delta e^{(-\lambda+\delta)s} \quad (7)$$

for any $\delta > 0$ and some constant C_δ depending on δ ; recall that λ is the Lyapunov exponent for Y . As $s \rightarrow \infty$, $u_{te-2s} \rightarrow id$ so, using compactness, there must exist a uniform $K > 0$ such that

$$\frac{1}{K} \|(A_{-s})_* Y\| \leq \|(u_{te-2s})_*(A_{-s})_* Y\| \leq K \|(A_{-s})_* Y\| \quad (8)$$

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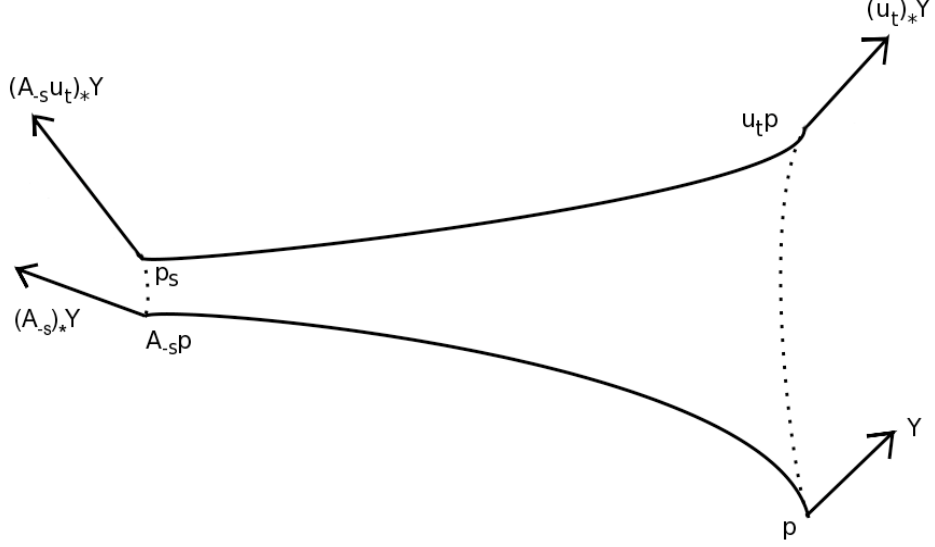


Figure 1: Translating backwards with A_{-s} .

as $s \rightarrow \infty$. Combining (7) and (8) produces

$$\frac{1}{C_\delta K} e^{(-\lambda-\delta)s} \leq \| (u_{te^{-2s}})_* (A_{-s})_* Y \| \leq C_\delta K e^{(-\lambda+\delta)s}$$

as $s \rightarrow \infty$. This allows calculation of the backward Lyapunov exponent for $(u_t)_* Y$ under the A_s -flow. As the equation holds for any $\delta > 0$,

$$\chi_-((u_t)_* Y) = -\lambda.$$

Lift Y to a horizontal vector $\bar{Y}(\tilde{p})$ in $T(H/\Gamma)$ with footpoint \tilde{p} . Extend this to a right-invariant vector field \bar{Y} and consider it as an element of \mathfrak{h} . Then $(u_t)_*(\bar{Y}(\tilde{p}))$ will be $(Ad(u_t)\bar{Y})(u_t\tilde{p})$. As $u_t = \exp(t\bar{U})$, $Ad(u_t) = e^{t ad(\bar{U})}$. Therefore,

$$(u_t)_*(\bar{Y}(\tilde{p})) = \left(\bar{Y} + t ad(\bar{U})\bar{Y} + \frac{t^2}{2} ad(\bar{U})^2 \bar{Y} + \cdots + \frac{t^r}{r!} ad(\bar{U})^r \bar{Y} \right) (u_t\tilde{p}).$$

Recalling that $ad(\bar{U})$ covers $2\nabla_U$, one sees that $(u_t)_* Y$ has the form

$$w + t2\nabla_U w + \frac{t^2}{2} (2\nabla_U)^2 w + \cdots + \frac{t^r}{r!} (2\nabla_U)^r w$$

for some w in $V_0(u_t p)$. The terms in this sum are all related by application of ∇_U . Due to Lemma 6.1.1, the w -term will have the largest backward-time Lyapunov exponent and will provide the backward-time exponent for $(u_t)_* Y$. Therefore,

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$$\chi_-(w) = -\lambda.$$

Applying Lemma 6.1.1 repeatedly,

$$\chi_-(\nabla_U^i w) = -\lambda - 2i.$$

Therefore, $\nabla_U^i w = X' + X''$, where $X' \in E_{\lambda+2i}$ is nonzero, and $X'' \in \sum E_\alpha$ for $\alpha > \lambda + 2i$; this uses the Lyapunov decomposition of the A -invariant distribution V_i . From this one can draw the first conclusion about a forward Lyapunov exponent. It is an easy fact about Lyapunov exponents that

$$\chi_+(\nabla_U^i w) \leq \max\{\chi_+(X'), \chi_+(X'')\}$$

with strict ' $<$ ' a possibility only if $\chi_+(X') = \chi_+(X'')$. Here this is not the case so

$$\chi_+(\nabla_U^i w) \geq \chi_+(X') = \lambda + 2i.$$

Likewise, for $(u_t)_* Y = w + t2\nabla_U w + \cdots + \frac{t^r}{r!}(2\nabla_U)^r w$,

$$\chi_+((u_t)_* Y) \leq \max\{\chi_+(w), \dots, \chi_+(\nabla_U^r w)\}.$$

Again, using Lemma 6.1.1, $\nabla_U^r w$ provides the largest exponent. Therefore,

$$\chi_+((u_t)_* Y) \geq \lambda + 2r. \tag{9}$$

The next step is to show equality; this is accomplished using volume forms whose growth rates are easily calculated. The basic idea is that if equality is ever violated in equation (9) it will cause a certain volume form to grow faster than it should.

Recall that the V_i carry volume forms vol_i (see section 2.4). One can consider as well the restriction of the Riemannian metric on $J \setminus H/\Gamma$ to these distributions and use an orthonormal basis to form another volume form, which will be denoted ν_i . These are smooth, nonzero forms and as $J \setminus H/\Gamma$ is compact, there is a constant C such that

$$\frac{1}{C} vol_i(p) \leq \nu_i(p) \leq C vol_i(p) \text{ for all } p \in J \setminus H/\Gamma. \tag{10}$$

The following calculation is simple to check. Recall that $ad(a)$ has weight λ_0 on \tilde{V}_0 and $\lambda_0 + 2i$ on \tilde{V}_i . Let m be the dimension of the V_i . For $p \in J \setminus H/\Gamma$,

$$(A_s)_* vol_i(p) = e^{m(\lambda_0+2i)s} vol_i(A_s p).$$

Hence, using (10):

$$\frac{1}{C^2} e^{m(\lambda_0+2i)s} \nu_i(A_s p) \leq (A_s)_* \nu_i(p) \leq C^2 e^{m(\lambda_0+2i)s} \nu_i(A_s p). \tag{11}$$

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Note that

$$(A_s)_* \nu_i(p) = \det((A_s)_*(p)|_{V_i(p)}) \nu_i(A_s p) \quad (12)$$

and recall that, for all Lyapunov regular p – all $p \in \text{supp}(\mu)$ in this case –

$$\lim_{s \rightarrow \pm\infty} \frac{1}{s} \log |\det(A_s)_*(p)|_{V_i(p)} = \sum_{l=1}^m \chi_l^{(i)}(p) \quad (13)$$

where $\chi_l^{(i)}$ are the Lyapunov exponents on V_i for the A_s -flow and are listed with multiplicity according to the dimension of their Lyapunov subspace (cf. Definition 5.2.2).

Equations (11) - (13) imply

$$\sum_{l=1}^m \chi_l^{(i)}(p) = m(\lambda_0 + 2i);$$

in particular,

$$\sum_{l=1}^m (\chi_l^{(0)}(p) + 2r) = \sum_{l=1}^m \chi_l^{(r)}(p). \quad (14)$$

Note that this last equation is just as easily implied by the earlier work on how Lyapunov exponents behave under ∇_U (see section 6.1).

Recall that $(u_t)_* Y$ has the form

$$w + t2\nabla_U w + \frac{t^2}{2}(2\nabla_U)^2 w + \cdots + \frac{t^r}{r!}(2\nabla_U)^r w \quad (15)$$

for some $w \in V_0(u_t p)$ and that the forward Lyapunov exponent will be that of $\nabla_U^r w$. Fix a basis f_1, \dots, f_m of $V_r(u_t p)$ with each f_l having Lyapunov exponent $\chi_l^{(r)}$. Let w_1, \dots, w_m have the form of equation (15) and be such that the V_r component of w_l is in the f_l direction. Then,

$$\sum_{l=1}^m \chi_+(w_l) = \sum_{l=1}^m \chi_l^{(r)} = \sum_{l=1}^m (\chi_l^{(0)} + 2r).$$

From equation (9), one has that $\chi_+((u_{-t})_* w_l) \leq \chi_+(w_l) - 2r$. Thus, using as well equation (14),

$$\sum_{l=1}^m (\chi_+((u_{-t})_* w_l) + 2r) \leq \sum_{l=1}^m \chi_+(w_l) = m(\lambda_0 + 2r).$$

However, the $(u_{-t})_* w_l$ are a basis for $V_0(p)$. The exponential growth rate for the volume form ν_0 is less than or equal to the sum of forward Lyapunov exponents for any basis of $V_0(p)$. This growth rate for ν_0 is $m\lambda_0$. This gives the first inequality in the following:

$$m(\lambda_0 + 2r) \leq \sum_{l=1}^m (\chi_+((u_{-t})_* w_l) + 2r) \leq \sum_{l=1}^m \chi_+(w_l) = m(\lambda_0 + 2r).$$

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Therefore, equality holds throughout and due to equation (9) this can only be the case if

$$\chi_+(w_l) = \chi_+((u_{-t})_*w_l) + 2r \text{ for all } l.$$

The above states that strict equality holds in equation (9) for a basis on $V_0(p)$. Since it holds for a basis it must hold for *all* of $V_0(p)$. For suppose $Y \in \text{span}\{(u_{-t})_*w_1, \dots, (u_{-t})_*w_d\}$ with the w_i ordered by increasing forward exponent and suppose Y does not belong to $\text{span}\{(u_{-t})_*w_1, \dots, (u_{-t})_*w_{d-1}\}$. Likewise $(u_t)_*Y$ is in $\text{span}\{w_1, \dots, w_d\}$ and not in $\text{span}\{w_1, \dots, w_{d-1}\}$. Then $\chi_+((u_t)_*Y) = \chi_+(w_d) = \chi_+((u_{-t})_*w_d) + 2r = \chi_+(Y) + 2r$, completing the proof of Proposition 6.2.1. □

7 The V_i -distributions and $\text{supp}(\mu)$

This section provides stronger information still about the support of μ . It shows that the measure μ extends in all V_i -directions which have nonzero weight for A ; equivalently, $\text{proj}_{\mathcal{H}}\mathfrak{l}$ contains the relevant \tilde{V}_i . The proof proceeds by carefully using the geometry of the distributions, and playing the two results on unipotents and Lyapunov exponents against one another. Throughout it is essential that the measure μ is the image of an algebraic measure on H/Γ .

Consider as before the string V_0, V_1, \dots, V_r which is acted on by ∇_U and ∇_L as raising and lowering operators for the $\mathfrak{sl}_2(\mathbb{R})$ spanned by \bar{U}, \bar{L} and a . By the choice of the measure μ , if \tilde{V}_i is not in $\text{proj}_{\mathcal{H}}\mathfrak{l}$, it must be that the Lyapunov decomposition of V_i contains a Lyapunov space for the exponent zero. One proceeds, then, by showing that any direction in \tilde{V}_i with Lyapunov exponent 0 is in $\text{proj}_{\mathcal{H}}\mathfrak{l}$ using geometric arguments.

First, assume that the vector v with Lyapunov exponent zero does not lie in V_0 . If it does, the argument below can be run exchanging the roles of U and L and exchanging V_0 with V_r , etc. Suppose $v \in V_{i_0}(p)$ with $p \in \text{supp}(\mu)$. Let Y be the vector in $V_0(p)$ such that $(2\nabla_U)^{i_0}Y = v$. Then $\chi(Y) = -2i_0$. Fix a lift \tilde{p} of p in H/Γ . One uses the horizontal lifts of vectors with basepoint \tilde{p} to conduct explicit calculations in H/Γ , where the geometry of the Lie group H makes these calculations possible. Again, $\bar{Y}(\tilde{p})$ denotes the horizontal (i.e. in \mathcal{H}) lift of $Y(p)$; let \bar{Y} denote as well the right-invariant vector field on H/Γ corresponding to this vector. Let p' be the point on $J \backslash H/\Gamma$ covered by $\tilde{p}' = \exp(\bar{Y})\tilde{p}$ (i.e. $p' = \exp(Y)p$, using the terminology of Remark 5.2.5). Let $\bar{Y}' = \bar{Y}(\tilde{p}')$.

From Lemma 5.2.4 and the choice of μ , one knows that p , the path $\exp(\tau Y)p$ and its endpoint p' all lie in the support of μ , and that $\chi(Y') = \chi(Y) = -2i_0$. Push this whole picture forward by the unipotent flow u_t (see Figure 2). As u_t preserves the measure, $u_t p$, $u_t p'$ and the path $u_t \exp(\tau Y)p$ are in the support of μ . The tangent vectors to this path, $(u_t)_*Y$ and $(u_t)_*Y'$,

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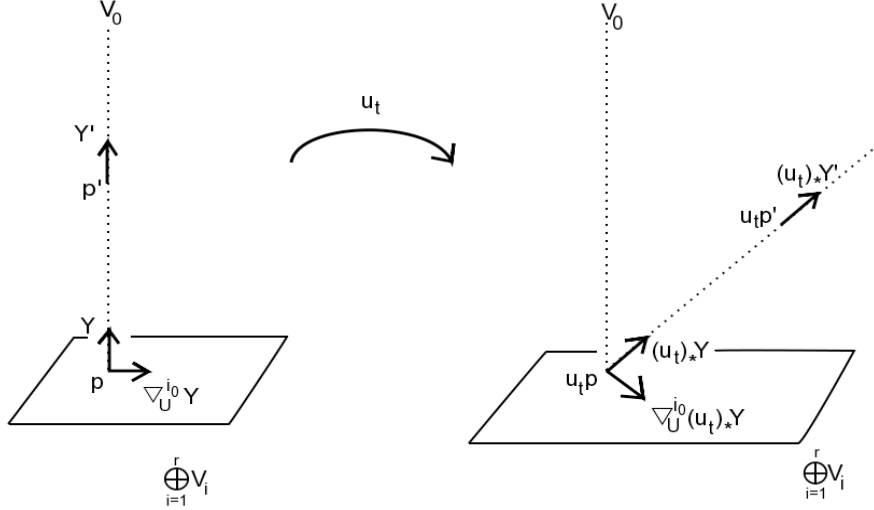


Figure 2: Geometry of the V_i under u_t .

both have forward Lyapunov exponent $-2i_0 + 2r$ by Proposition 6.2.1 and backward Lyapunov exponent $-2i_0$ (seen in the proof of that proposition). The image of $(2\nabla_U)^{i_0} Y$ under $(u_t)_*$ is $(2\nabla_U)^{i_0} (u_t)_* Y$ and if one can show this direction is in the support of μ at $u_t p$ one will be able to reach the desired conclusion by applying u_{-t} .

Using horizontal lifts to H/Γ , one calculates that

$$(u_t)_*(\bar{Y}(\tilde{p})) = \bar{Y}(u_t \tilde{p}) + \frac{t}{1!} ad(\bar{U}) \bar{Y}(u_t \tilde{p}) + \cdots + \frac{t^r}{r!} ad(\bar{U})^r \bar{Y}(u_t \tilde{p})$$

and that

$$(u_t)_* \bar{Y}' = \bar{Y}'(u_t \tilde{p}') + \frac{t}{1!} ad(\bar{U}) \bar{Y}'(u_t \tilde{p}') + \cdots + \frac{t^r}{r!} ad(\bar{U})^r \bar{Y}'(u_t \tilde{p}').$$

Given this form for $(u_t)_* \bar{Y}'$ and using Lemma 6.1.1, one can conclude that the forward exponent for $(u_t)_* Y'$ must be controlled by the projection via π_* to $J \backslash H/\Gamma$ of the final summand, so

$$\chi_+(\pi_*(ad(\bar{U})^r \bar{Y}'(u_t \tilde{p}'))) = -2i_0 + 2r.$$

Applying $ad(L)$ or ∇_L to this successively shows

$$\chi_+(\pi_*(ad(\bar{U})^i \bar{Y}'(u_t \tilde{p}'))) = -2i_0 + 2i. \quad (16)$$

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Likewise, the backward Lyapunov exponent is controlled by the projection of the first summand in the expression for $(u_t)_*\bar{Y}$, so

$$\chi_-(\pi_*(\bar{Y}(u_t\tilde{p}))) = 2i_0.$$

Apply $ad(\bar{U})$ or ∇_U to this successively to conclude

$$\chi_-(\pi_*(ad(\bar{U})^i\bar{Y}(u_t\tilde{p}))) = 2i_0 - 2i. \quad (17)$$

Note that for $i < i_0$, $\chi_+(\pi_*(ad(\bar{U})^i\bar{Y}(u_t\tilde{p})))$ is negative. Lemma 5.2.4 and Remark 5.2.5 show that a lift of a stable vector in these directions may be exponentiated to produce a path lying above a stable manifold on $J \setminus H/\Gamma$. (Here, one exploits the freedom provided by Remark 5.2.5 to use $u_t\tilde{p}'$ as the basepoint for computing this path.) Apply that as follows. Consider the vector

$$\bar{W} := -\bar{Y}(u_t\tilde{p}') - \frac{t}{1!}ad(\bar{U})\bar{Y}(u_t\tilde{p}') - \dots - \frac{t^{i_0-1}}{(i_0-1)!}ad(\bar{U})^{i_0-1}\bar{Y}(u_t\tilde{p}').$$

This vector covers a stable vector, given equation (16). Thus its exponential covers a path lying in the support of μ . Therefore the point

$$\exp(\bar{W})\exp((u_t)_*\bar{Y})u_t\tilde{p}$$

lies over the support of μ . One can run the above arguments using $s\bar{Y}$ in place of \bar{Y} . For all s ,

$$\gamma_s := \exp(s\bar{W})\exp(s(u_t)_*\bar{Y})u_t\tilde{p}$$

lies over the support of μ . The tangent vector to this smooth path at time $s = 0$ will provide a direction in $proj_{\mathcal{H}}\mathfrak{l}$. This tangent vector can be calculated using the Cambell-Baker-Hausdorff formula:

$$\exp(sX)\exp(sY) = \exp(sX + sY + \frac{s^2}{2}[X, Y] + \text{higher order terms in } s).$$

The tangent vector is $X + Y$; here one finds that the

$$\bar{W} + (u_t)_*\bar{Y} = \frac{t^{i_0}}{i_0!}ad(\bar{U})^{i_0}\bar{Y} + \frac{t^{i_0+1}}{(i_0+1)!}ad(\bar{U})^{i_0+1}\bar{Y} + \dots + \frac{t^r}{r!}ad(\bar{U})^r\bar{Y} \quad (18)$$

direction at the point $u_t\tilde{p}$ covers a direction in the support of μ .

When $i > i_0$, equation (17) implies that the $ad(\bar{U})^i\bar{Y}$ directions are unstable; therefore, they cover directions in the support of μ . As the directions in the support of μ are described by the linear subspace $proj_{\mathcal{H}}\mathfrak{l}$, one can form a linear combination of these with the direction produced in equation (18) and find that

$$\left(ad(\bar{U})^{i_0} \bar{Y} + \frac{t}{1!} ad(\bar{U})^{i_0+1} \bar{Y} + \cdots + \frac{t^{r-i_0}}{(r-i_0)!} ad(\bar{U})^{i_0+(r-i_0)} \bar{Y} \right) (u_t \tilde{p})$$

covers a direction in the support of μ . This is precisely the image of $ad(\bar{U})^{i_0} \bar{Y}(\tilde{p})$ under $(u_t)_*$. As the flow u_t preserves μ , it must be that the support of μ extends in the direction $v = \nabla_{\bar{U}}^{i_0} Y$ at the point p . This is the direction one hoped to recover at the start.

All of this work, using crucially the fact that the measure μ is described by $proj_{\mathcal{H}} \mathfrak{l}$, implies the following proposition, another key step in understanding \mathfrak{l} :

Proposition 7.0.2. *The projection of \mathfrak{l} onto \mathcal{H} contains all \tilde{V}_i which have nonzero weight for A .*

8 Ergodicity of the G -action

The goal of this section is to demonstrate that the volume measure m on $J \backslash H / \Gamma$ is ergodic for the G -action. This is accomplished by showing the support of the ergodic component μ is in fact all of $J \backslash H / \Gamma$. This section is the first for which assumption (2) for the Characterization Theorem is necessary. The work above has shown that $proj_{\mathcal{H}} \mathfrak{l}$ contains all \tilde{V}_i which have nonzero weight for A . Using this we have the following:

Lemma 8.0.3. *Under the assumptions of the Characterization Theorem, any μ satisfying the requirements of Lemma 5.1.2 has support equal to $J \backslash H / \Gamma$.*

Proof. Assumption (2) for the Characterization Theorem is that the nonzero weight spaces for \mathfrak{a} generate a Lie algebra \mathfrak{v} such that $\mathfrak{j} + \mathfrak{v} = \mathfrak{h}$. As noted above, under the assumption that H is simple this holds as these non-zero weight spaces generate a non-trivial ideal of \mathfrak{h} and \mathfrak{h} is simple. The idea of the proof is to use Proposition 4.0.8 that $proj_{\mathcal{H}} \mathfrak{l}$ determines the support of μ .

If v and w belong to \tilde{V}_i 's which have nonzero weight, then $v + J_v$ and $w + J_w$ belong to \mathfrak{l} for some J_v and J_w in \mathfrak{j} . One calculates,

$$[v + J_v, w + J_w] = [v, w] + [J_v, w] + [v, J_w] + [J_v, J_w] \in \mathfrak{l}.$$

The second and third summands above lie in the \tilde{V}_i which have already been shown part of $proj_{\mathcal{H}} \mathfrak{l}$. The final summand lies in \mathfrak{j} and as $proj_{\mathcal{H}} \mathfrak{l}$ is a linear space, $[v, w]$ must project to an element of $proj_{\mathcal{H}} \mathfrak{l}$ as well. Therefore, \mathfrak{l} contains $[v, w] + J_{[v,w]}$. Continue: for any u in the \tilde{V}_i with nonzero weight for A ,

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$$[u + J_u, [v, w] + J_{[v,w]}] = [u, [v, w]] + [u, J_{[v,w]}] + [J_u, [v, w]] + [J_u, J_{[v,w]}]$$

belongs to \mathfrak{l} . The second summand lies in $\text{proj}_{\mathcal{H}}\mathfrak{l}$, the final in \mathfrak{j} and for the third:

$$[J_u, [v, w]] = -[w, [J_u, v]] - [v, [w, J_u]].$$

The right-hand side consists of single brackets of elements in those \tilde{V}_i which have nonzero A -weight. The previous calculation showed that these project to $\text{proj}_{\mathcal{H}}\mathfrak{l}$ as well, so $[u, [v, w]]$ projects to $\text{proj}_{\mathcal{H}}\mathfrak{l}$. Continuing in this manner, one can show inductively that the Lie algebra generated by the \tilde{V}_i with non-zero A -weight projects to $\text{proj}_{\mathcal{H}}\mathfrak{l}$; therefore, $\mathfrak{j} + \text{proj}_{\mathcal{H}}\mathfrak{l} = \mathfrak{h}$. This is exactly what one needs to ensure that $\text{supp}(\mu) = J \backslash H / \Gamma$. As this measure is an ergodic component of the volume measure, we see that the G -action is ergodic for the volume. \square

9 Completion of the proof

The proof of the Main Theorem is completed by discussing the Characterization Theorem's algebraic characterization of any compact form and then showing that for H simple it cannot be satisfied.

9.1 Characterization of compact forms

Characterization Theorem. *Let H be connected, semisimple with finite center, J reductive, $G < Z_H(J)$ semisimple and assume*

- (1) *All simple factors of G have real-rank at least two,*
- (2) *If \mathfrak{g}' is the Lie algebra generated by all non-zero weight spaces of \mathfrak{a} then $\mathfrak{j} + \mathfrak{g}' = \mathfrak{h}$.*

Then if there is a compact form of $J \backslash H$, there exists a Lie subgroup $L < H$ and a uniform lattice Γ in L such that $JL = H$ and $J \cap L = K$ is compact, and

$$J \backslash H / \Gamma \cong K \backslash L / \Gamma.$$

Proof. This follows from the work above proving $\text{proj}_{\mathcal{H}}\mathfrak{l} = \mathcal{H}$ and Proposition 4.0.8. \square

9.2 Non-existence of compact forms

Together with the Characterization Theorem, the following completes the proof of the Main Theorem.

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Theorem 9.2.1. *Under the conditions of the Main Theorem, there is no Lie subgroup $L < H$ such that $JL = H$ and $L \cap J$ is compact.*

The proof will be accomplished in a series of steps.

Lemma 9.2.2. *Any L such that $JL = H$ is semisimple.*

Proof. Because the radical of L , $Rad(L)$, is solvable it fixes a point in a maximal boundary H/P of H (P is a minimal parabolic in H). This minimal parabolic is contained in a maximal parabolic Q which contains J . This can be seen via the following argument. Let P_2 be a proper parabolic containing J ; we claim that $PP_2 \neq H$ after which one can take Q to be a maximal parabolic containing PP_2 . Suppose $PP_2 = H$; using this, every H -conjugate of P is a P_2 conjugate. Replacing P with a P_2 conjugate, we may assume $P \cap P_2$ is parabolic. Now choose $h \in H$ such that $P^h \cap P_2$ is not parabolic. Taking $h \in P_2$, calculate $P^h \cap P_2 = (P \cap P_2)^h$ which is parabolic, a contradiction.

Let $\bar{F} \subset H/P$ be the fixed set of $Rad(L)$. Since $P \subset Q$, H/P maps onto H/Q ; let F be the image of \bar{F} under this map.

First, note that L acts transitively on H/Q . This is because H acts transitively, J has a fixed point (as $J \subset Q$) and $JL = H$. Second, note that L fixes \bar{F} , and hence F , setwise, as \bar{F} is the fixed point set for a normal subgroup of L . These two facts imply together that $F = H/Q$, that is, $Rad(L)$ fixes all of H/Q . For $Rad(L)$ to fix a point hQ in H/Q means that $Rad(L) \subseteq h^{-1}Qh$. Thus

$$Rad(L) \subseteq \bigcap_{h \in H} h^{-1}Qh;$$

therefore, $Rad(L)$ is contained in a proper normal subgroup of H . As H is simple, such a subgroup must be trivial; this implies that L is semisimple. \square

Thus, one has a decomposition of the simple Lie group H as a product of reductive and semisimple groups J and L . One also knows the following facts about the ranks of J and L , and the dimensions of the symmetric spaces associated to them:

Lemma 9.2.3. *The real rank of J is at least two less than the real rank of H .*

Proof. As $Z_H(J)$ has rank at least 2, this is clear. This is a special case of the formula $\mathbb{R}\text{-rank}(L) + \mathbb{R}\text{-rank}(J) \leq \mathbb{R}\text{-rank}(H)$ which must be satisfied if L is to act properly on $J \backslash H$ (see [Kob89]). \square

Definition 9.2.4. For a real, reductive group G with maximal compact subgroup K set

$$d(G) = \dim(G/K),$$

the dimension of the symmetric space attached to G .

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Theorem 9.2.5 ([Kob89] Theorem 1.7). *Let G be a real reductive Lie group and H_1, H_2 reductive subgroups such that H_2 acts properly on $H_1 \backslash G$ and $H_1 \backslash G / H_2$ is compact. Then*

$$d(H_1) + d(H_2) = d(G).$$

This implies that $d(J) + d(L) = d(H)$.

The scheme of proof now is to show by exhaustion that no subgroup L satisfying the criteria of the Characterization Theorem and these two dimension requirements exists. The most convenient way to do this turns out to be a bit backward. One begins with Oniščik's study of Lie group triples (G, G', G'') for simple G and reductive G' and G'' such that $G = G'G''$ (see [Oni69]). Simple dimension counting arguments together with Lemma 9.2.3 and Theorem 9.2.5 rule out most of these possibilities. Finally, one shows that the surviving possibilities never admitted a higher-rank semisimple action in the first place and hence could not fall under the conditions of this paper.

For ease of comparison with Oniščik's work and notation, in what follows G and G' are no longer higher-rank semisimple groups as above, but are now general Lie groups.

In his work on this problem, Oniščik first shows that to understand the decompositions $G = G'G''$ where G is reductive and G' and G'' are reductive in G , it is enough to understand the corresponding Lie algebras and to find decompositions $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$ (his Theorem 3.1). He then shows (Theorem 3.2) that for this decomposition to hold, a corresponding decomposition of the semisimple part of \mathfrak{g} into the semisimple parts of \mathfrak{g}' and \mathfrak{g}'' must hold. Oniščik classifies these decompositions for \mathfrak{g} simple. He approaches them under two cases according to whether or not \mathfrak{g} is a complex simple Lie algebra.

When \mathfrak{g} is a complex simple Lie algebra, Oniščik's Theorem 4.2 states that \mathfrak{g}' and \mathfrak{g}'' are complex simple also, or \mathfrak{g} is the complex form of D_4 , \mathfrak{g}' is the complex form of B_3 and \mathfrak{g}'' is $\mathfrak{so}(1, 7)$ or $\mathfrak{so}(3, 4)$. In the first case, the decompositions available are the complex forms underlying the classification in his Table 2 (see Cor. 3.2 in [Oni69]). This table lists the Lie algebra of $\mathfrak{g}' \cap \mathfrak{g}''$ as well; it is easy to verify that the complex forms of all those listed are noncompact, so none of these will provide a compact form of a homogeneous space. One need now only examine the case $\mathfrak{h} = D_4 = \mathfrak{so}(8, \mathbb{C})$. Oniščik finds that in the case for $\mathfrak{so}(3, 4)$, this lie algebra intersects B_3 in a noncompact Lie algebra (the noncompact form of G_2); therefore it is ruled out. For the other possibility, the real rank of $\mathfrak{so}(1, 7)$ is one so it cannot be \mathfrak{l} . Examine then the embedding of $\mathfrak{so}(1, 7)$ in $\mathfrak{so}(8, \mathbb{C})$. Using a conjugacy in $SO(8, \mathbb{C})$ any real form of signature $(1, 7)$ can be put in the standard form. It is easy to check that under the standard embedding of $\mathfrak{so}(1, 7)$ into $\mathfrak{so}(8, \mathbb{C})$ the centralizer of $\mathfrak{so}(1, 7)$ is trivial. Therefore this case cannot give

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rise to a compact form under the hypotheses here. Note, however, that a compact form of this algebraic type does exist (see [KY05], Table 3.3).

Oniščik lists decompositions for \mathfrak{g} not complex simple in Table 2 of [Oni69]. For the cases under consideration in this paper, Theorem 9.2.5 rules out many from giving rise to a construction of a compact form, as does the restriction on ranks $\mathbb{R}\text{-rank}(L) + \mathbb{R}\text{-rank}(J) \leq \mathbb{R}\text{-rank}(H)$. The following table lists those that remain:

	\mathfrak{g}	\mathfrak{g}'	\mathfrak{g}''
1	$\mathfrak{su}(2n, 2)$	$\mathfrak{sp}(n, 1)$	$\mathfrak{su}(2n, 1)$
2	$\mathfrak{su}(1, 1)$	$\mathfrak{sp}(1, \mathbb{R})$	$\mathfrak{su}(1)$
3	$\mathfrak{so}(4, 3)$	G_2	$\mathfrak{so}(4, 1)$
4	$\mathfrak{so}(2n, 2)$	$\mathfrak{so}(2n, 1)$	$\mathfrak{su}(n, 1)$
5	$\mathfrak{so}(2, 2)$	$\mathfrak{so}(2, 1)$	$\mathfrak{sl}(2, \mathbb{R})$
6	$\mathfrak{so}(4n, 4)$	$\mathfrak{so}(4n, 3)$	$\mathfrak{sp}(n, 1)$
7	$\mathfrak{so}(8, 8)$	$\mathfrak{so}(8, 7)$	$\mathfrak{so}(8, 1)$
8	$\mathfrak{so}(4, 4)$	$\mathfrak{so}(4, 3)$	$\mathfrak{so}(4, 1)$
9	$\mathfrak{so}(4, 4)$	$\mathfrak{so}(4, 3)$	$\mathfrak{so}(4, 1) \times \mathfrak{so}(3)$

Table 1: Relevant decompositions of simple, non-complex Lie algebras

Lemma 9.2.3 implies that the real rank of \mathfrak{j} is at least 2 smaller than that of H – considering only the semisimple part of J , this still holds. This restriction rules out all possibilities but number 3 with $\mathfrak{j} = \mathfrak{so}(4, 1)$, number 6 with $\mathfrak{j} = \mathfrak{sp}(n, 1)$, number 7 with $\mathfrak{j} = \mathfrak{so}(8, 1)$, number 8 with $\mathfrak{j} = \mathfrak{so}(4, 1)$, and number 9 with $\mathfrak{j} = \mathfrak{so}(4, 1) \times \mathfrak{so}(3)$.

#3: For $\mathfrak{so}(4, 1) \hookrightarrow \mathfrak{so}(4, 3)$, complexify this embedding and note that the centralizer of $\mathfrak{so}(5, \mathbb{C})$ in $\mathfrak{so}(7, \mathbb{C})$ will contain the complexification of the centralizer for the real forms. The centralizer in the complexified version is at largest $\mathfrak{so}(2, \mathbb{C})$ which has real-rank one, so there is certainly no higher-rank semisimple group in the centralizer of the real forms.

#6: Likewise for $\mathfrak{sp}(n, 1) \hookrightarrow \mathfrak{so}(4n, 4)$, examining the complexifications one sees that the centralizer of $\mathfrak{sp}_{2(n+1)}(\mathbb{C})$ in $\mathfrak{so}_{4(n+1)}(\mathbb{C})$ is trivial, so there is no higher-rank semisimple group in the centralizer.

#7: The representation of $\mathfrak{so}(8, 1)$ into $\mathfrak{so}(8, 8)$ which Oniščik lists, when complexified, is the spin representation of $\mathfrak{so}(9, \mathbb{C})$ into $\mathfrak{so}(16, \mathbb{C})$. This representation is irreducible, again ruling it out for the cases under consideration here.

#8 & #9: Finally, for $\mathfrak{so}(4, 1) \hookrightarrow \mathfrak{so}(4, 4)$ complexify the embedding and find that the centralizer is at largest $\mathfrak{so}(3, \mathbb{C})$. The real lie algebras

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that complexify to $\mathfrak{so}(3, \mathbb{C})$ are $\mathfrak{so}(2, 1)$ and $\mathfrak{so}(3)$, neither of which contains a higher-rank semisimple group.

These considerations complete the proof of Theorem 9.2.1 and the Main Theorem.

Table 1 should be compared to Table 3.3 in [KY05] in which Kobayashi and Yoshino list some homogeneous spaces that *do* admit compact forms via the simple Lie group construction. Many of the Lie algebra triples arising in Table 1 are found in their list as well. However, these spaces do not admit actions of the type under consideration in this paper.

Remark 9.2.6. An assumption that H is only semisimple does not work for the algebra in this section, but there may be another algebraic approach to this case. At any rate, the algebraic conditions imposed by the Characterization Theorem and Remark 3.0.5 seem strong, at least when G is not contained in any proper normal subgroup of H .

10 Conclusion

To close, return to the motivating example of $SL_{n-k}(\mathbb{R}) \backslash SL_n(\mathbb{R})$. The compact forms question remains open for $k = 2$ ($n \neq 4, 6$) and $k = 1$, n even. The techniques of this paper may provide a way to approach the $k = 2$ case, utilizing a $GL_2(\mathbb{R})$ -action which is similar to the higher-rank semisimple actions used crucially above. For $k = 1$ there is less idea of how to proceed. Benoist's approach for odd n yields no results and the work here contributes little. Using ideas related to those above, it is possible to show that the flow on $SL_{n-1}(\mathbb{R}) \backslash SL_n(\mathbb{R}) / \Gamma$ given by the centralizer of J has non-zero Lyapunov exponents. One question the author would pose is whether this flow must be Anosov. Beyond this, though, many new ideas would be necessary and the approach via dynamics may not contribute to this problem.

A second question is the following. To ensure that the irreducibility condition for cocycle superrigidity holds, this paper takes as acting group only the higher-rank semisimple factors in $Z_H(J)$. What about general higher-rank semisimple groups? Can one prove that if the semisimple part of $Z_H(J)$ is, for example, $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ its action on $J \backslash H / \Gamma$ is irreducible? If so, the path to using the arguments of this paper is clear.

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