Kleinian groups and 3-Manifolds

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February 28 2014

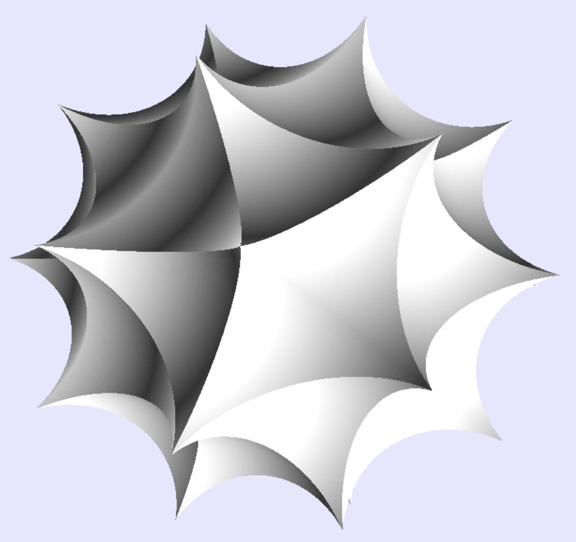
1. Kleinian groups

A Kleinian group is a finitely generated and discrete group of conformal symmetries of the sphere, where

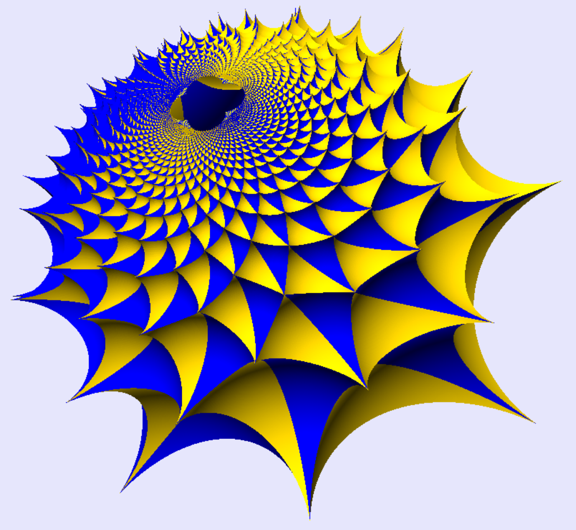
1. "the sphere" means the round unit sphere in Euclidean 3-space; and
2. "conformal" means smooth maps which preserve angles.

The collection of all conformal symmetries of the sphere is a Lie group; "discrete" means discrete as a subset of this group.

A finite group of rotations is a Kleinian group.

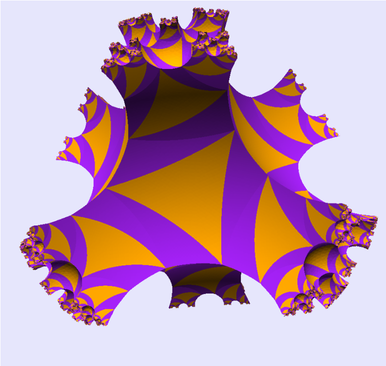


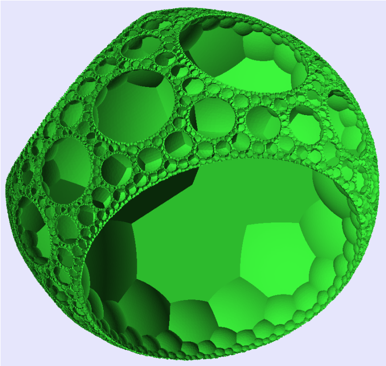
The symmetry group of a Euclidean tessellation is a Kleinian group by stereographic projection.



A Kleinian group preserving a round circle on the sphere is a Fuchsian group.

And there are many other examples.





If we identify the unit sphere with the Riemann sphere

$$S^2 = \mathbb{CP}^1 := \mathbb{C} \cup \infty$$

then (orientation-preserving) conformal symmetries are fractional linear transformations

$$z \to \frac{az+b}{cz+d}$$

and the group of all such transformations is $\mathrm{PSL}(2, \mathbb{C})$.

2. Differential Equations

Kleinian groups arise in nature as monodromy groups of differential equations.

Euler introduced the hypergeometric equation

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0$$

where w is a function of the variable z, and a, b, c are real constants.

For Euler, w and z were real; but for us they can be complex numbers.

The space of solutions to the hypergeometric equation is a complex vector space V of dimension 2.

There are regular singular points at 0, 1 and ∞, and solutions may be analytically continued around these points.

If f and g are a basis for V, the map

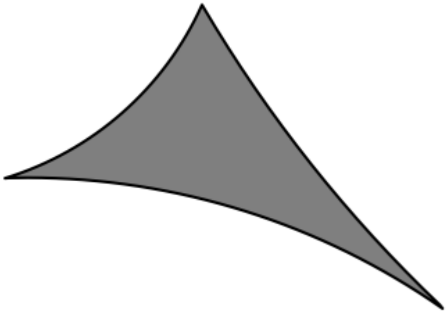
$$D : z \to f(z)/g(z)$$

is well-defined in the upper half plane

$$\mathcal{H} := \{z \in \mathbb{C} \text{ with positive imaginary part}\}$$

Schwarz showed that the image $D(\mathcal{H})$ is a curvilinear triangle T in $\mathbb{C}$ whose sides are segments of round circles or straight lines and whose angles are $|1-c|\pi$, $|c-a-b|\pi$ and $|a-b|\pi$.

The corners of T are the images of 0, 1 and ∞.

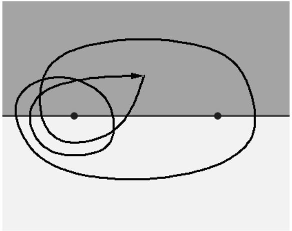


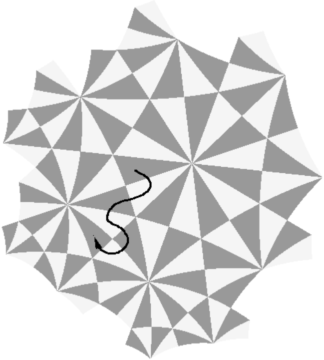
If we analytically continue D across a segment of $\mathbb{R} - \{0, 1, \infty\}$, it maps the lower half plane $\overline{\mathcal{H}}$ onto a triangle obtained from T by inversion in the corresponding circular side (or reflection in a straight side).

Continuing D around loops, we get a representation

$$D_* : \pi_1(\mathbb{CP}^1 - \{0, 1, \infty\}) \to \mathrm{PSL}(2, \mathbb{C})$$

If the angles of the curvilinear triangle T are of the form π/n for integers n, the image of D_* is discrete, and hence is a Kleinian group called a triangle group.





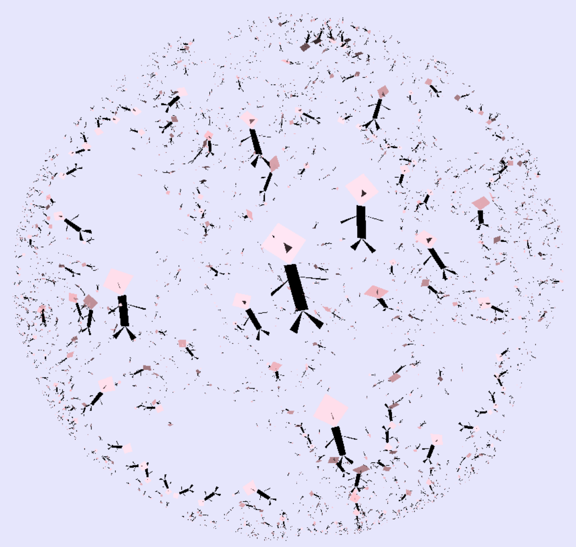
3. Hyperbolic geometry

Poincaré realized that $\mathrm{PSL}(2, \mathbb{C})$ is also the group of (orientation-preserving) isometries of hyperbolic 3-space $\mathbb{H}^3$.

In Poincaré's model, $\mathbb{H}^3$ is the interior of the unit ball. The hyperbolic metric is obtained by rescaling the Euclidean metric by a factor of $2/(1 - r^2)$ where r is the distance to 0.

Straight lines in the Poincaré metric are arcs of Euclidean straight lines or circles perpendicular to the boundary sphere.

In the Poincaré metric, objects near the boundary sphere look smaller.



If Γ is a torsion-free Kleinian group, then Γ acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries.

Thus the quotient $M := \mathbb{H}^3/\Gamma$ is a hyperbolic manifold with universal cover $\mathbb{H}^3$, and fundamental group $\pi_1(M) \cong \Gamma$.

Conversely, every 3-manifold M with a complete hyperbolic metric has universal cover $\widetilde{M}$ isometric to $\mathbb{H}^3$, realizing the deck group $\pi_1(M)$ as a Kleinian group Γ.

Thurston conjectured, and Perelman proved, the

Hyperbolization Theorem: a closed 3-manifold M admits a hyperbolic structure if and only if

1. every smooth embedded sphere in M bounds a ball;
2. $\pi_1(M)$ is infinite; and
3. $\pi_1(M)$ contains no $\mathbb{Z}^2$ subgroup.

4. Dynamics

If Γ is a Kleinian group, the sphere S^2 decomposes into

1. the Limit set Λ where Γ acts ergodically, and
2. the Domain of discontinuity Ω where Γ acts properly discontinuously.

Λ is closed, and Ω is open. For torsion-free Kleinian groups, Λ is the closure of the set of fixed points of Γ.

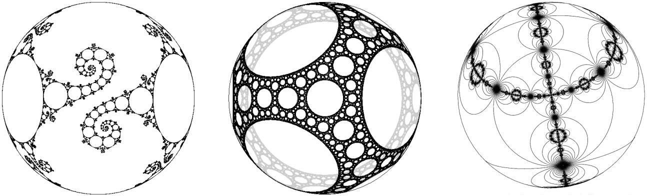
The quotient Ω/Γ is a Riemann surface. Ahlfors showed it is of finite type; i.e. it is homeomorphic to a compact surface minus finitely many points.

A random walk in the Euclidean plane is recurrent. In higher dimensions, it diverges, but very slowly and chaotically.

In hyperbolic space of any dimension, the negative curvature freezes the trajectory of a random walk, so that it converges to a definite point on the sphere at infinity.

If Γ is a Kleinian group, measurable functions on Ω/Γ are the asymptotic values of harmonic functions on the hyperbolic manifold $M := \mathbb{H}^3/\Gamma$.

So whenever Λ is not equal to S^2 there are many harmonic functions on M.



limit set images by Curt McMullen

If M is a closed hyperbolic manifold, there are no nonconstant harmonic functions on M, by the maximum principle. So $\Lambda = S^2$.

But it *is* possible for $\Lambda = S^2$ even if M is noncompact!

Example: Σ is a surface, and $\phi : \Sigma \to \Sigma$ is a diffeomorphism. Define the mapping torus

$$M_\phi := \Sigma \times [0, 1]/(s, 1) \sim (\phi(s), 0)$$

which is topologically a bundle over S^1 with fiber Σ:

$$\Sigma \to M_\phi \to S^1$$

There is a corresponding short exact sequence of fundamental groups

$$0 \to \pi_1(\Sigma) \to \pi_1(M_\phi) \to \mathbb{Z} \to 0$$

In particular, $\pi_1(\Sigma)$ is a normal subgroup of $\pi_1(M_\phi)$.

If ϕ is sufficiently complicated, Thurston showed M_ϕ is hyperbolic.

Let $\widehat{M}_\phi$ be the infinite cover of M_ϕ with fundamental group $\pi_1(\Sigma)$. Topologically, $\widehat{M}_\phi = \Sigma \times \mathbb{R}$.

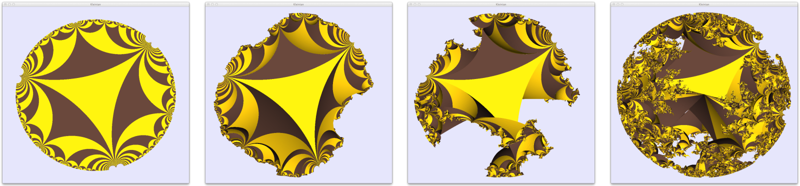
Since $\pi_1(\Sigma)$ is normal in $\pi_1(M_\phi)$, it follows that $\Lambda(\widehat{M}_\phi)$ is invariant under all of $\pi_1(M_\phi)$; in fact

$$\Lambda(\widehat{M}_\phi) = \Lambda(M_\phi)$$

But M_ϕ is closed, so $\Lambda(M_\phi) = S^2$.

Each fiber Σ is a surface, whose universal cover $\widetilde{\Sigma}$ is topologically a plane, which is properly embedded in $\widetilde{M}_\phi \cong \mathbb{H}^3$.

This plane accumulates on $\Lambda(\widehat{M}_\phi) = S^2$; i.e. it limits to all of S^2!



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5. Ahlfors' Conjecture and Tameness

In 1966, Ahlfors formulated his

Ahlfors Measure Conjecture: If Γ is a Kleinian group, then either $\Lambda = S^2$ or Λ has measure zero.

If Γ violates Ahlfors conjecture, we can build a function χ_Λ on S^2 which is 1 on Λ and 0 on Ω, and a nonconstant harmonic function h_Λ on $M := \mathbb{H}^3/\Gamma$ which is zero on Ω/Γ.

A random walk preserves the value of a harmonic function on average. If Γ violates Ahlfors conjecture, a random walk on $M := \mathbb{H}^3/\Gamma$ has a definite chance of failing to converge to Ω/Γ.

Where else can a random walk go? It has to go into one of the ends of M and stay there.

The manifold $M := \mathbb{H}^3/\Gamma$ has $\pi_1(M) \cong \Gamma$ which is finitely generated.

The Scott Core Theorem says that there is a compact core $C \subset M$ such that $C \to M$ is a homotopy equivalence.

The ends of M are the components of $M - C$.

Some of these ends limit to components of Ω/Γ; these are the geometrically finite ends.

Example: The cyclic cover $\widehat{M}_\phi = \Sigma \times \mathbb{R}$ has two geometrically infinite ends. The surfaces $\Sigma \times t$ have bounded area, so a random walk on $\widehat{M}_\phi$ looks like a random walk on $\mathbb{R}$.

But a random walk on $\mathbb{R}$ is recurrent! So the random walk on $\widehat{M}_\phi$ will not stay in either of the ends, but keeps coming back to any compact region.

Thurston and Canary showed that to prove Ahlfors Conjecture, it suffices to show that the ends of $M := \mathbb{H}^3/\Gamma$ are all topologically products $\Sigma \times \mathbb{R}^+$. Such ends are said to be tame.

In 1974, Marden conjectured that all hyperbolic manifolds $M := \mathbb{H}^3/\Gamma$ are tame; equivalently, they are all homeomorphic to the interior of a compact 3-manifold, possibly with boundary.

In 2004, Marden's Tameness Conjecture was proved independently by Agol and by Calegari–Gabai; thus by the combined work of many people, we know that Ahlfors' Conjecture is true.

References

- D. Mumford, C. Series and D. Wright, *Indra's Pearls: The Vision of Felix Klein*.
- W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. AMS, 1982