IMAGINARY ENTROPY FOR HUBBARD TREES

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Abstract.

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1. Introduction

1.1. Statement of results.

1.2. **Acknowledgements.** I would like to thank Sarah Koch, Curt McMullen, Dylan Thurston and Giulio Tiozzo for some helpful suggestions and insights. Danny Calegari was supported by NSF grant DMS 1005246.

2. Embeddable endomorphisms of planar trees

2.1. **Itinerary generating function.** In this section we consider endomorphisms of planar trees.

Definition 2.1.1 (Piecewise monotone)**.** Let X be a compact tree. An endomorphism $f : X \to X$ is *piecewise monotone* if X admits a subdivision into finitely many intervals on each of which the restriction of f is monotone to its image.

A *center* for f is a choice of point $* \in X$.

One natural choice for a center is a fixed point; other natural choices include critical points. Every endomorphism of a compact tree has a fixed point, but such a point is typically not unique.

Definition 2.1.2 (Itinerary generating function)**.** Let X be a compact tree, and $f : X \to X$ piecewise monotone with center $*$. Let S be the set of components of $X - *$, let $\hat{S} = S \cup *$, and let $\mathbb{R}[\hat{S}]$ be the real vector space with basis the elements of $\hat{S}$.

For $x \in X$ and a non-negative integer j, define $s_j(x) \in \hat{S}$ to be equal to $*$ if $f^j(x) = *$, and to be equal to the component of $X - *$ containing $f^j(x)$ otherwise;

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and define $I(x;t)$ to be the *itinerary generating function* of x:

$$I(x;t) := \sum_{i=0}^{\infty} s_j(x)t^j$$

Note that I is a formal power series with coefficients in $\mathbb{R}[\hat{S}]$.

Now suppose ν is probability measure on X. We can define

$$I(\nu;t) := \int_X I(x;t)d\nu(t)$$

i.e. $I(\nu;t)$ is the formal power series whose coefficients $s_j(\nu)$ are simply the ν-expectation of the $\hat{S}$-valued function $s_j(x)$.

Lemma 2.1.3. *Suppose ν is f-invariant. Then $I(\nu;t) = s_0(\nu)/(1-t)$.*

Proof. For any measure ν (not necessarily f-invariant) and any $j \geq 1$ we have

$$s_j(\nu) = \int_X s_j(x)d\nu(x) = \int_X s_{j-1}(f(x))d\nu(x) = \int_X s_{j-1}(x)d(f_*\nu)(x) = s_{j-1}(f_*\nu)$$

so if ν is f-invariant we have $I(\nu;t) = s_0(\nu)(1+t+t^2+\cdots) = s_0(\nu)/(1-t)$. □

For $|t| < 1$ the power series $I(x;t)$ is absolutely convergent, and takes values in $\mathbb{R}[\hat{S}]$. We are thus motivated to define

$$\sigma(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} s_j(x)$$

providing this limit exists.

Informally, we think of $\sigma(x)$ as the "residue" of $I(X;t)$ at $t=1$.

Lemma 2.1.4. *Suppose ν is f-invariant and ergodic. Then for ν-a.e. x the limit $\sigma(x)$ exists and is equal to $s_0(\nu)$.*

Proof. This follows from the Birkhoff ergodic theorem, except that we must be slightly careful about the center $*$. If $*$ has zero measure, then the same is true for its preimages, and we may restrict attention to the space $X - \cup_j f^{-j}(*)$ on which the tautological S-valued function on $X - *$ is continuous, so that the usual Birkhoff theorem applies. Otherwise, since ν is ergodic, ν is supported on a finite orbit which includes $*$, and the lemma is obvious. □

In practice, we will be only be interested in attracting f-invariant measures of two kinds:

(1) an atomic measure supported on an attracting periodic orbit; or
(2) an absolutely continuous invariant measure with full support.

Both possibilities occur with positive measure in the parameter spaces we consider (see Jakobson [2]).

2.2. **Planar trees.** Now, suppose X is a *planar* tree, and $f : X \to X$ is piecewise monotone with center $*$. The set $X - *$ decomposes into q connected components, which we call the *spokes*; the planar embedding lets us cyclically order the spokes as $X_{i/q}$ for $0 \le i < q$. The choice of labeling is not unique; rather the set of labels should be thought of as an affine space for $\mathbb{Z}/q\mathbb{Z}$. In some contexts, however, there is another natural choice of center $*'$ on X, and by convention we let X_0 denote the component containing $*'$.

Andre de Carvalho [1] has shown how planar graphs can be decorated with *infinitesimal edges and loops* so that their endomorphisms are carried by generalized traintrack maps which can be approximated by planar embeddings. We use de Carvalho's language of *thick graphs* and *thick graph maps*; see [1], Definitions 1 and 2.

Definition 2.2.1. An endomorphism $f : X \to X$ of a planar tree is *embeddable* if there is a planar thick graph N, an embedding $F : N \to N$ which is a thick graph map, and a retraction $\pi : N \to X$ satisfying the following properties:

(1) the fibers of π are the leaves and junctions of N;
(2) the map π semiconjugates $F : N \to N$ to $f : X \to X$;
(3) the map π is compatible with the planar structures on N and X, in the sense that the circular ordering coming from the planar structures on links of higher order junctions of N resp. vertices of X are preserved by π.

A choice of $F : N \to N$ as above is called an *embedding* of $f : X \to X$.

Now suppose $f : X \to X$ is embeddable, and $F : N \to N$ is an embedding. de Carvalho [1], Lemma 1 implies that the invariant set $\Lambda := \cap_j F^j(N)$ is homeomorphic to the inverse limit of $f : X \to X$, and the restriction of F to Λ is conjugate to the inverse limit map. Thus, thickening gives us a way to realize the action on the inverse limit as the restriction of a planar homeomorphism to an invariant set.

Now suppose $*$ is a fixed point for f. Identifying Λ with the inverse limit of $f : X \to X$ lets us choose a canonical lift of $*$ to $\Lambda \subset F$ (namely the element $(\cdots, *, *, *, *)$) which we denote $*$ by abuse of notation. Thus F restricts to a homeomorphism of the annulus obtained by deleting $*$ from the plane, and every invariant measure for F (which is necessarily supported on Λ) has a well-defined rotation number in $\mathbb{R}/\mathbb{Z}$.

In fact, we may even extend this rotation number to the fixed point $*$ itself: the link of $*$ is circularly ordered, and the condition of embeddability implies that the restriction of f to the link of $*$ is monotone with respect to this circular order; thus there is a well-defined rotation number there.

3. Quadratic rational maps of the interval

3.1. **Definitions.**

Definition 3.1.1. A map $f : I \to I$ is *unimodal* if it has exactly one critical point c and is elsewhere locally injective.

Milnor-Thurston [3] define the *kneading determinant* as follows:

Definition 3.1.2 (Kneading determinant)**.** Suppose $x \in I$ is not a preimage of the critical point c. Define $\theta_{-1}(x) = 1$ and inductively,

$$\theta_i(x) = \begin{cases} \theta_{i-1}(x) \text{ if } f^i(x) < c \\ -\theta_{i-1}(x) \text{ if } f^i(x) > c \end{cases}$$

Define a formal power series $\theta(x,t) = \sum_{i\ge 0} \theta_i(x)t^i$, and then the *kneading determinant* $D(t)$ of f is the formal power series $D(t) = \theta(c^-, t)$; i.e. the limit of $\theta(x,t)$ as $x \to c$ from below.

Now, for any center $\alpha \in I$ we can define $\phi_i(x,\alpha)$ by $\phi_0(x,\alpha) = 0$, and inductively for $i > 0$ by

$$\phi_i(x,\alpha) = \begin{cases} 1 \text{ if } (f^i(x) - \alpha)(f^{i-1}(x) - \alpha) < 0 \\ -1 \text{ otherwise} \end{cases}$$

and then define a formal power series $\phi(x,\alpha;t) = \sum_{i\ge 0} \phi_i(x,\alpha)t^i$. and $R(\alpha;t) = \phi(c^-, \alpha; t)$ as above. Then if $R(\alpha;t)$ has a simple pole at 1 we define $\rho(\alpha)$ (or just ρ if α is understood) to be the residue there.

Remark 3.1.3. For $\alpha = c$, there is a close relationship between $\theta(x,t)$ and $\phi(x,c,t)$; namely the coefficient ϕ_i is the product of the coefficients $\theta_{i-2}\theta_{i-1}\theta_i$. In other words, formally we can write $\phi = \theta * t\theta * t^2\theta$, where $*$ denotes "logarithmic convolution"; i.e.

$$(f * g)(z) = \frac{i}{2\pi} \int_\gamma f(w) g\left(\frac{z}{w}\right) \frac{dw}{w}$$

where γ is a sufficiently small loop about 0.

Example 3.1.4. A unimodal map is *postcritically finite* if the critical point c is eventually periodic under iteration, with period q. In this case we have $R(\alpha;t) = s(t) + p(t)/(1 - t^q)$ where p is a polynomial of degree $q - 1$, and the residue at 1 is $p(1)/q$, which is the rotation number (up to a factor of 2) of the endomorphism with respect to the center α.

Example 3.1.5. Consider the real quadratic map $f_c : z \to z^2 + c$. For $c \in [-2, -1.6]$ this takes the interval $[c, \beta]$ into itself, where 0 is the critical point, and $\beta = (1 + \sqrt{1-4c})/2$. The other fixed point is $\alpha = (1 - \sqrt{1-4c})/2$. We take this fixed point α as the center. Then the graph of the rotation number $\rho(\alpha)$ is illustrated in Figure 1.

Proposition 3.1.6. *Let α be the fixed point of f. Then ρ achieves its supremum on α.*

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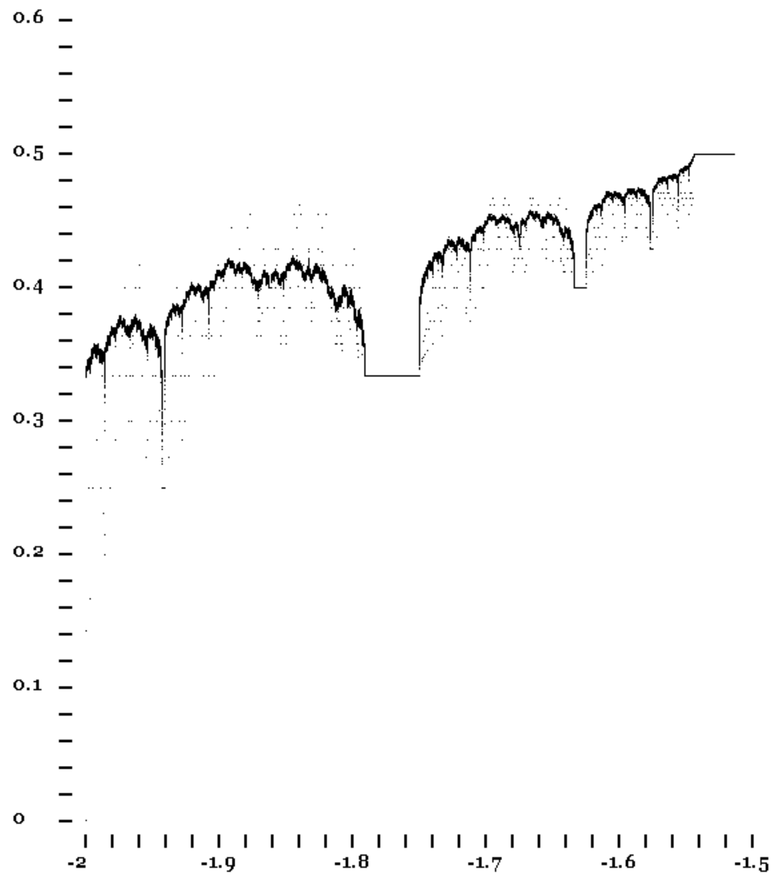

FIGURE 1. Rotation number for real unimodal maps $z \to z^2 + c$ for $c \in [-2, -1.6]$.

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