## Continued Fractals

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## Euclidean algorithm

Given two positive integers $p, q$, how can we find the greatest common factor (gcf)? That is, the biggest integer which divides both $p$ and $q$.

Euclid (325-265 BC) proposed the following computer program:

E1. Is $p=q$ ? If yes, return $p$ and quit

E2. Is $p>q$ ? If no, interchange $p$ and $q$

E3. Set $p$ equal to $p-q$ and go to E1

Example: $p=7, q=5$. This gives:

$$
\begin{gathered}
(7,5) \rightarrow(2,5) \rightarrow(5,2) \rightarrow(3,2) \rightarrow \\
\rightarrow(1,2) \rightarrow(2,1) \rightarrow(1,1) \rightarrow 1=\operatorname{gcf}(7,5)
\end{gathered}
$$

Steps E2 and E3 in the algorithm can be recoded numerically as follows. Encode the pair $(p, q)$ as the fraction $z:=\frac{p}{q}$.

In this scheme, the algorithm becomes:

$$
\text { If } z<1 \text {, do } z \rightarrow \frac{1}{z} \text {. If } z>1 \text {, do } z \rightarrow z-1
$$

Example: the "itinerary" of the initial choice $\frac{7}{5}$ is

$$
\frac{7}{5} \rightarrow \frac{2}{5} \rightarrow \frac{5}{2} \rightarrow \frac{3}{2} \rightarrow \frac{1}{2} \rightarrow \frac{2}{1} \rightarrow \frac{1}{1}
$$

## Continued fractions

There is a convenient notation for keeping track of the steps in the algorithm. This is the notation of a continued fraction.

Example:

$$
\begin{aligned}
\frac{7}{5}= & 1+\frac{2}{5}=1+\frac{1}{\frac{5}{2}}=1+\frac{1}{2+\frac{1}{2}}= \\
& =1+\frac{1}{2+\frac{1}{2}}=1+\frac{1}{2+\frac{1}{2}}
\end{aligned}
$$

By truncating this partial fraction, we get successive approximations to $\frac{7}{5}$ :

$$
1,1+\frac{1}{2}=\frac{3}{2}, 1+\frac{1}{2+\frac{1}{2}}=\frac{7}{5}
$$

Euclid's algorithm shows how to encode any positive rational number as a continued fraction.

More generally, continued fractions can be finite or infinite. An infinite continued fraction is necessarily irrational.

Example:

$$
e=2+\frac{1}{1+} \frac{1}{2+} \frac{1}{1+} \frac{1}{1+} \frac{1}{4+} \frac{1}{1+} \frac{1}{1+} \frac{1}{6+} \ldots
$$

Example:

$$
\pi=3+\frac{1}{7+} \frac{1}{15+} \frac{1}{1+} \frac{1}{292+} \frac{1}{1+} \frac{1}{1+} \ldots
$$

which gives rational approximations

$$
3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \cdots \rightarrow \pi
$$

Example:

$$
\begin{gathered}
\phi=1+\frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \ldots \\
\therefore \phi=1+\frac{1}{\phi} \\
\therefore \phi^{2}-\phi-1=0 \\
\therefore \phi=\frac{1+\sqrt{5}}{2} \quad(\text { because } \phi>1)
\end{gathered}
$$

This particular quadratic irrational $\phi$ is known as the golden ratio. The successive rational approximations to $\phi$ are ratios of Fibonacci numbers:

$$
1, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \cdots \rightarrow \phi
$$

A continued fraction expansion is a convenient way of expressing a real number, similar in some ways to a decimal expansion.

In fact, it has some advantages over decimal expansion, as well as some disadvantages.

|  | Dec. | CF |
| :---: | :---: | :---: |
| finite | $p / 10^{n}$ | rational |
| periodic | rational | quad.irrational |
| unique | no | yes |
| ,,$+- \times, /$ | easy | hard |
| rat'l approx. | poor | good |

Algorithms for performing addition, subtraction, multiplication and division on continued fractions were developed by R. W. Gosper.

## Gauss' transformation

The two transformations $z \rightarrow 1 / z$ and $z \rightarrow z-1$ can be collapsed to a single transformation:
G. Given $z \in(0,1)$, replace $z$ by the fraction--al part of $\frac{1}{z}$

We denote the fractional part of a real number $z$ by $\{z\}$. Example: $\{3.1415926\}=0.1415926$

The transformation is denoted symbolically by

$$
G: z \rightarrow\left\{\frac{1}{z}\right\}
$$

Note that $G\left(\frac{1}{n}\right)=0$ for any integer $n$. For convenience, we define $G(0)=0$ by convention, and then think of $G$ as a function from the half-open interval $[0,1)$ to itself.
C. F. Gauss (1777-1855) studied the properties of the transformation

$$
G:[0,1) \rightarrow[0,1)
$$

He showed that for almost every choice of number $r \in[0,1)$, the sequence

$$
r, G(r), G^{2}(r), G^{3}(r), \ldots
$$

gets close to every point in $[0,1)$, with density

$$
\frac{1}{(1+x) \ln (2)}\left(=\frac{d}{d x} \log _{2}(1+x)\right)
$$

It follows that for almost every real number $r$, the integer $n$ appears in the continued fraction expansion of $r$ with frequency

$$
\operatorname{freq}(n)=\log _{2}\left(1+\frac{1}{n}\right)-\log _{2}\left(1+\frac{1}{n+1}\right)
$$

Example: the number 1 appears about 41.5\% of the time.

Rational approximation and geometry

On the real number line, put a circle of diameter $\frac{1}{q^{2}}$ at each reduced rational number of the form $p / q$ :


The circles are tangent, and are tightly packed, but they do not cross.

For any real number $r$ draw a vertical line which hits the real number line at $r$. The circles it intersects on the way down correspond to the rational numbers of the form $p / q$ such that

$$
|r-p / q| \leq \frac{1}{2 q^{2}}
$$

Note that if $r$ is irrational, there are infinitely many such $p / q$.


## Hyperbolic geometry

Hyperbolic geometry takes place above the real number line in the plane. As objects move towards the real line, they shrink, in proportion to the (usual Euclidean) distance to the line.


Since footsteps shrink the closer one moves to the line, the "shortest distance" between two points (i.e. fewest footsteps necessary) is actually the arc of a circle perpendicular to the real axis.

The pattern of circles from before is part of a symmetric tessellation in hyperbolic geometry:


Each (black) semicircle lands on the real line at two rational numbers, and passes through the point where the associated (grey) circles are tangent.

A vertical line crosses a sequence of semicircles on the way down. These circles pivot left or right in sequence. The number of consecutive pivots in each direction are the terms in the continued fraction.


$$
\frac{71}{100}=0+\frac{1}{1+} \frac{1}{2+} \frac{1}{2+} \frac{1}{4+} \frac{1}{3}
$$

pivot sequence: RLLRRLLLLRRR

## Hyperbolic origami

If we "fold up" the real line plus infinity into a circle, the hyperbolic plane folds up into a disk:


If we include the real numbers in the complex plane, then the hyperbolic plane includes as a "slice" through hyperbolic 3-space.


The space $\mathcal{Q \mathcal { F }}$

Many beautiful fractals can be obtained by bending and folding this copy of the hyperbolic plane in hyperbolic 3-space, according to precise instructions.

The set of all ways of doing this is parameterized by a certain space called Quasi-Fuchsian space, or $\mathcal{Q F}$ for short.

In 2004, J. Brock, R. Canary and Y. Minsky proved Thurston's Ending Lamination Conjecture, which gives a very good picture of the space $\mathcal{Q F}$.
$\mathcal{Q F}$ is high-dimensional, but we can study it by looking at finite dimensional "slices" through it.

## Maskit's slice

When a sequence of folded planes approaches the boundary of $\mathcal{Q} \mathcal{F}$, they start to bump into themselves. On each slice, a point on the boundary is parameterized by a real number $r$. The Ending Lamination Conjecture says that the corresponding fractal can be reconstructed from the continued fraction expansion of $r$.


