

DIPOLES AND PIXIE DUST

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ABSTRACT. Every closed subset of the Riemann sphere can be approximated in the Hausdorff topology by the Julia set of a rational map.

1. DIPOLES

In [3], Kathryn Lindsey gives an elegant construction to show that any Jordan curve in the complex plane can be approximated in the Hausdorff topology by Julia sets of polynomials (another such construction was given subsequently by Oleg Ivrii [2]), and further, that any finite collection of disjoint Jordan domains can be approximated by the basins of attraction of a rational map. The proof depends on an interpolation result due to Curtiss [1].

In this note, we give a direct geometric (and computation-free) proof that any closed subset of the Riemann sphere can be approximated in the Hausdorff topology by the Julia set of a rational map.

In the theory of electromagnetism, a *dipole* refers to a pair of oppositely charged particles with charges of equal magnitude. The electric field of a dipole falls off at a rate of $1/r^3$ because of the approximate cancellation of the fields at distances large compared to the separation of the particles (unlike the usual inverse square law for a system with nonzero net charge). The point is that the dipole is effectively electrically neutral on large scales.

By analogy, we define a *dipole* to be a degree 1 rational function $(z - a)/(z - b)$ with a zero and pole at distinct non-zero a, b , for which $|a - b|$ is “small”. The dipole is uniformly close to 1 outside a small disk containing a and b . We now explain how to use dipoles to build designer Julia sets.

Let’s suppose we want to build a rational function whose Julia set approximates X , a closed subset of the Riemann sphere. For simplicity, suppose X is disjoint from the unit circle.

First, start with the map $f : z \rightarrow z^N$ where N is some fixed big integer. The Julia set of f is the unit circle, and f has 0 and ∞ as superattracting fixed points.

Second, pick some finite collection Y of discrete points (“pixels”) which approximates X closely in the Hausdorff sense. We build a new rational function g_ϵ which is the product of f with a dipole centered at each point in Y , where for each dipole the zero and pole are within distance ϵ of each other. As $\epsilon \rightarrow 0$, the Julia sets of g_ϵ converge uniformly in the Hausdorff topology to \hat{Y} , which is equal to the union of Y together with its N^k th roots (i.e. its preimages under f), together with the unit circle. To see this, observe that the dynamics of g_ϵ converge uniformly to f on compact subsets of the complement of this set, while the presence of the dipole near

each point y guarantees a point in the Julia set of g_ϵ near y , and near its preimages under f . If N is very big, \hat{Y} is very close in the Hausdorff topology to $Y \cup S^1$.

But now we are basically done: in place of $f : z \rightarrow z^N$ we could use a map $f_{C,p} : (z - p) \rightarrow C(z - p)^N$ whose Julia set is an arbitrarily small circle centered at an arbitrary point p (for instance, a point in X). Repeating the construction above, we get rational maps with Julia sets as close as we like in the Hausdorff sense to $Y \cup p$ for Y an arbitrary finite set and p arbitrary.

Figure 1 shows an example of a sequence of Julia sets constructed by this method.

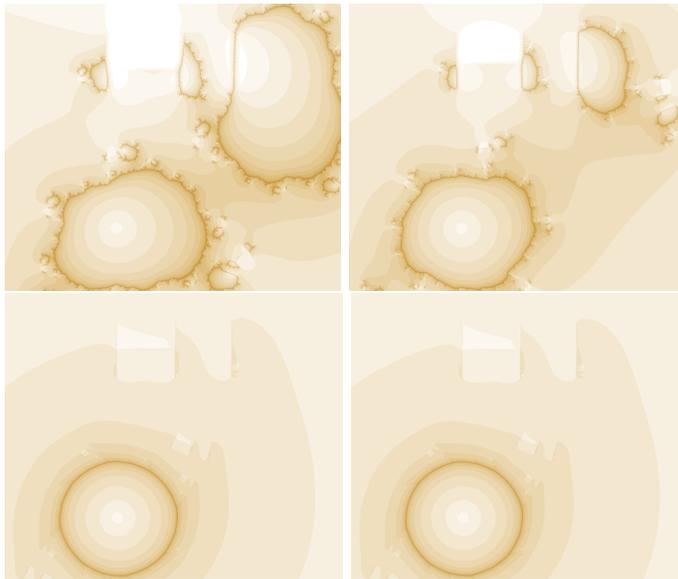


FIGURE 1. 80 dipoles and $N = 2$ for $\epsilon = 0.2, 0.1, 0.05, 0.02$. The convergence of the Julia sets to \hat{Y} is evident. Y is a pixelated “HI” at the top of each picture.

2. ACKNOWLEDGMENTS

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REFERENCES

- [1] J. Curtiss, *Riemann sums and the fundamental polynomials of lagrange interpolation*, Duke Math. J. **8** (1941), 525–532
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