NOTES ON FUNDAMENTAL GROUPS OF KÄHLER MANIFOLDS

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Abstract. These are notes for expository talks given in the Geometric Langlands Seminar at the University of Chicago on November 11 and November 18 2013.

1. Kähler manifolds

A basic reference for the geometry of Kähler manifolds is [15].

1.1. **Linear algebra.** We describe three kinds of structure on a (real) vector space: a positive inner product, a complex structure, and a symplectic structure; and what it means for these structures to be compatible.

1.1.1. *Positive inner product.* If V is a real vector space, a (symmetric) inner product is a symmetric bilinear map $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{R}$. If we choose a basis e_i for V, there is a unique symmetric matrix S so that if $v = \sum v_i e_i$ and $w = \sum w_i e_i$ then

$$\langle v, w \rangle = \sum v_i S_{ij} w_j$$

Since S is symmetric, it is diagonalizable, and its eigenvalues are real. The *signature* of $\langle \cdot,\cdot \rangle$ is p, q where S has p positive and q negative eigenvalues. The inner product is *positive definite* (or just *positive*) if $p = \dim(V)$ and $q = 0$. This is equivalent to $\langle v, v \rangle > 0$ for all $v \neq 0$. It is also equivalent to the existence of a choice of basis (an *orthonormal basis*) for which S becomes the identity matrix.

1.1.2. *Complex structure.* If V is a real vector space, and $J : V \to V$ has $J^2 = -1$, then the eigenvalues of J are i and $-i$, each occurring with multiplicity $\dim(V)/2$ (in particular, $\dim(V)$ is even). So J induces an endomorphism (which we also denote by J) on the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ which decomposes into eigenspaces $V_{\mathbb{C}} = V' \oplus V''$ where V' is the i eigenspace, and V'' is the $-i$ eigenspace. Given any $v \in V$, there is a canonical decomposition

$$v = \frac{1}{2}(v - iJv) + \frac{1}{2}(v + iJv)$$

where by inspection, we see that the first piece is in V' and the second in V''. We denote these two pieces v' and v''.

The map $V \to V'$ taking v to v' takes the action of J to multiplication by i; thus it gives V the structure of a complex vector space (in which J becomes multiplication by i).

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1.1.3. *Symplectic structure.* If V is a real vector space, a symplectic form is a nondegenerate antisymmetric bilinear map $\omega : V \times V \to \mathbb{R}$. Antisymmetry means $\omega(v,w) = -\omega(w,v)$, and nondegeneracy means that for any nonzero v there is a nonzero w with $\omega(v,w) \neq 0$. The existence of such a structure implies that $\dim(V)$ is even.

For any symplectic form there is a choice of basis $x_1, \cdots, x_n, y_1, \cdots, y_n$ (where $\dim(V) = 2n$) so that

$$\omega(x_i, y_j) = \delta_{ij}, \quad \omega(x_i, x_j) = \omega(y_i, y_j) = 0$$

Such a basis is called a *standard symplectic basis*.

1.1.4. *Compatibility.* Let V be a real vector space of dimension $2n$. A positive inner product $\langle \cdot, \cdot \rangle$, a symplectic form ω and a complex structure J is *compatible* if there is a standard symplectic basis $x_1, \cdots, x_n, y_1, \cdots, y_n$ which is at the same time an orthonormal basis, and satisfies $J(x_i) = y_i$. In terms of the basis $x_1, \cdots, x_n, y_1, \cdots, y_n$ we obtain in a natural way a complex basis $\frac{1}{2}(x_j - iy_j)$ for $V' \subset V_{\mathbb{C}}$. The compatibility of these structures is equivalent to the following:

(1) $\langle v, w \rangle = \langle Jv, Jw \rangle$
(2) $\omega(v,w) = \langle Jv, w \rangle$
(3) $\omega(v,w) = \omega(Jv, Jw)$

Notice that the first condition alone implies that Jv is perpendicular to v, for any v. For, $\langle v, Jv \rangle = \langle Jv, -v \rangle = -\langle v, Jv \rangle$.

Any two of these conditions implies the third, and therefore given any two compatible structures, a third compatible structure exists and is unique. At the level of Lie groups, this can be seen by intersecting the stabilizers of the three structures:

(1) $\mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R}) = \mathrm{U}(n)$;
(2) $\mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{U}(n)$; and
(3) $\mathrm{O}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{U}(n)$.

1.1.5. *Hermitian forms.* An inner product $\langle \cdot, \cdot \rangle$ on a complex vector space V complexifies to define a complex bilinear pairing $\langle \cdot, \cdot \rangle_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$. This is usually encoded by a *sesquilinear* map $H : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$, defined by

$$H(v,w) = \langle v, \bar{w} \rangle_{\mathbb{C}}$$

If we restrict H to $V' \subset V_{\mathbb{C}}$, and identify V' with V as above, then we claim that the real part is $\langle \cdot, \cdot \rangle$, and the imaginary part is the negative of a compatible symplectic form ω (up to a factor of 2):

$$H(v', w') = \frac{1}{4}\langle v - iJv, w + iJw \rangle = \frac{1}{2}\left(\langle v, w \rangle + i\langle v, Jw \rangle\right) = \frac{1}{2}\left(\langle v, w \rangle - i\omega(v,w)\right)$$

1.2. **Kähler manifolds.**

Definition 1.2.1. An *almost complex structure* on a real $2n$-dimensional manifold M is a smooth section $J \in \mathrm{End}(TM)$ with $J^2 = -1$. An almost complex structure is *integrable* if the field of i eigenspaces $T'M \subset T_{\mathbb{C}}M$ is integrable.

Integrability of an almost complex structure is equivalent to the existence of local charts modeled on open subsets of $\mathbb{C}^n$ with holomorphic transition functions; thus it is equivalent to the existence of a complex structure (in the usual sense).

Definition 1.2.2. An *almost symplectic structure* on a real $2n$-dimensional manifold M is a smooth section $\omega \in \Omega^2(M)$ of fiberwise symplectic forms (i.e. satisfying $\omega^n > 0$ pointwise). An almost symplectic structure is *integrable* if $d\omega = 0$.

Integrability of an almost symplectic structure is equivalent to the existence of local charts modeled on open subsets of $\mathbb{R}^{2n}$ (with its standard symplectic structure) with transition functions whose differentials preserve the pointwise symplectic structure.

Definition 1.2.3. A closed (real) $2n$-dimensional manifold M is *almost Kähler* if it admits a compatible Riemannian metric, almost complex structure, and almost symplectic structure. M is *Kähler* if the structures are integrable; i.e. it admits compatible Riemannian metric, complex structure, and symplectic structure.

Definition 1.2.4. A (finitely presented) group Γ is a *Kähler group* if it is isomorphic to π_1 of some compact Kähler manifold.

Example 1.2.5. Since the property of being a Kähler manifold is local, every finite cover of a Kähler manifold is Kähler; thus every subgroup of a Kähler group of finite index is a Kähler group.

On a complex manifold we can choose coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$ locally so that the vector fields $\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ are sections of $T'M$. The dual 1-forms $dz_j := dx_j + idy_j$ and $d\bar{z}_j := dx_j - idy_j$ are a local basis for the smooth complex-valued 1-forms $\Omega^1_{\mathbb{C}}M$, and any field of complex pairings can be expressed in the form

$$h := \sum h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta$$

A Hermitian metric H determines such an h by $H(v, w) = h(v, \bar{w})$; the Hermitian condition is equivalent to the symmetry of h (i.e. that $h_{\alpha\bar{\beta}} = \overline{h_{\beta\bar{\alpha}}}$) plus the positivity ($h(v, \bar{v})$ is real and positive for all nonzero v) pointwise. Any Riemannian metric on a complex manifold can be averaged under the action of J pointwise and then complexified to produce a Hermitian metric. Taking imaginary parts gives rise to an alternating 2-form $\omega \in \Omega^2_{\mathbb{C}}M$

$$\omega := \frac{i}{2} \sum h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

The Hermitian condition implies that ω is a *real nondegenerate* form (i.e. $\omega^n \neq 0$ pointwise); the manifold is Kähler iff $d\omega = 0$.

It is well-known in Riemannian geometry that one may always choose *geodesic normal coordinates*, so that a metric g, expressed in local coordinates as $g := \sum g_{ij} dx_i \otimes dy_i$ can always be made to satisfy $g_{ij} = \delta_{ij} + o(2)$ (for suitable smooth coordinates). For a Hermitian metric on a complex manifold, one can choose *holomorphic* local coordinates with this property *if and only if the metric is Kähler*. That is,

Proposition 1.2.6. *A Hermitian metric h on a complex manifold M is Kähler if and only if there are local holomorphic coordinates at any point for which $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta} + o(2)$.*

One direction of this proposition is easy: for such a choice of coordinates, the form ω is constant up to first order, and therefore $d\omega = 0$ at the given point. But this formula is coordinate free, and therefore holds at every point.

1.3. **Cohomology.** On any almost complex manifold M, the decomposition $T_{\mathbb{C}} = T' \oplus T''$ gives rise to $T^*_{\mathbb{C}} = (T')^* \oplus (T'')^*$ and we can decompose $\Omega^n_{\mathbb{C}}$ into $\oplus_{p+q=n}\Omega^{p,q}$ where $\Omega^{p,q}$ is (smooth) sections of the bundle $\Lambda^p(T')^* \wedge \Lambda^q(T'')^*$. If the almost complex structure is integrable, $\Omega^{p,q}$ is spanned locally by forms

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

thus (by differentiating), $d\Omega^{p,q} \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}$, and we can write $d = \partial + \bar{\partial}$ as the $(1,0)$ and $(0,1)$ parts respectively. These operators satisfy

$$\partial^2 = \bar{\partial}^2 = d^2 = 0,\ \partial\bar{\partial} = -\bar{\partial}\partial$$

Example 1.3.1. On a Kähler manifold, the symplectic form ω is both real (in ordinary Ω^2), and of type $\Omega^{1,1}$ in $\Omega^2_{\mathbb{C}}$.

Since $\bar{\partial}^2 = 0$, they make $\Omega^{p,*}$ into a complex, with *Dolbeault cohomology* groups $H^{p,q}_{\bar{\partial}}$. Analagous to the Poincaré Lemma (which proves the vanishing of ordinary cohomology locally), we have the Dolbeault Lemma:

Proposition 1.3.2 (Dolbeault Lemma). *If $\bar{\partial}\alpha = 0$ then locally we can write $\alpha = \bar{\partial}\beta$.*

This lets us take resolutions locally, and compute cohomology; if we abuse notation by writing Ω^p_h for the *holomorphic p-forms* (those locally of the form $\sum f_I dz_I$ with each f_I holomorphic and each multi-index $|I| = p$; equivalently, those forms in $\Omega^{p,0}$ in the kernel of $\bar{\partial}$) then we have the

Theorem 1.3.3 (Dolbeault Theorem).

$$H^q(M, \Omega^p_h) = H^{p,q}_{\bar{\partial}}(M)$$

From the Dolbeault Lemma one can deduce the

Proposition 1.3.4 (Local $i\partial\bar{\partial}$ lemma). *Suppose ω is a real 2-form of type $\Omega^{1,1}$. Then $d\omega = 0$ if and only if it can be written locally in the form $i\partial\bar{\partial}u$ for some real function u.*

Proof. We compute

$$d(i\partial\bar{\partial}u) = i(\partial + \bar{\partial})(\partial\bar{\partial}u) = i\partial^2\bar{\partial}u - i\partial\bar{\partial}^2 u = 0$$

proving one direction. Conversely, suppose $d\omega = 0$ so $\partial\omega = 0$ and $\bar{\partial}\omega = 0$. By the Poincaré Lemma we can write ω locally as $\omega = d\tau$. If we decompose $\tau = \tau^{1,0} + \tau^{0,1}$ we can choose τ with $\tau^{1,0} = \overline{\tau^{0,1}}$. So

$$\omega = d\tau = \bar{\partial}\tau^{0,1} + (\partial\tau^{0,1} + \bar{\partial}\tau^{1,0}) + \partial\tau^{1,0}$$

and comparing types we get $\bar{\partial}\tau^{0,1} = 0$. By the Dolbeault Lemma, we can write $\tau^{0,1}$ locally as $\bar{\partial}f$, and also locally $\tau^{1,0} = \partial\bar{f}$ (for the same f). But then $\omega = \partial\bar{\partial}f + \bar{\partial}\partial\bar{f} = i\partial\bar{\partial}(2\mathrm{im}\,(f))$ locally. $\square$

The *global* $i\partial\bar{\partial}$ Lemma says that if ω is exact, the function u can be found *globally*. The proof is essentially the same.

1.4. **Hodge theory.** A Riemannian metric induces an inner product on all natural bundles, including the cotangent bundle and its tensor and exterior powers. On a Riemannian metric of dimension n there is a Hodge star $* : \Omega^k M \to \Omega^{n-k} M$ defined pointwise by $\alpha \wedge *\beta := \langle \alpha, \beta \rangle d\text{vol}$. This satisfies $*^2 = (-1)^{k(n-k)}$ on k-forms. If we express d in terms of an orthonormal frame e_i by $d = \sum e_i \wedge \nabla_{e_i}$ then the adjoint $\delta := -(-1)^{nk} * d*$ has an expression in local coordinates as $\delta = -\sum e_i \lrcorner \nabla_{e_i}$. The *Laplacian* is defined by

$$\Delta := d\delta + \delta d$$

A form α is *harmonic* if $\Delta\alpha = 0$; the harmonic p-forms are denoted $\mathcal{H}^p$. On any compact manifold there is a *Hodge decomposition*

$$\Omega^p = \mathcal{H}^p \oplus d\Omega^{p-1} \oplus \delta\Omega^{p+1}$$

and one deduces that $H^p = \mathcal{H}^p$ (with the left side denoting de Rham cohomology), and every cohomology class contains a *unique* harmonic representative.

On a Kähler manifold of real dimension $2n$, Hodge star satisfies $* : \Omega^{p,q} \to \Omega^{n-q,n-p}$. We can define formal adjoints

$$\partial^* := - * \bar{\partial} *, \ \bar{\partial}^* := - * \partial *$$

and Laplace operators

$$\Delta^\partial := \partial\partial^* + \partial^*\partial, \ \Delta^{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

A surprisingly difficult local calculation gives the crucial identity

$$\Delta = 2\Delta^\partial = 2\Delta^{\bar{\partial}}$$

If we define the $\bar{\partial}$ harmonic forms α to be those with $\Delta^{\bar{\partial}}\alpha = 0$ and denote those of type (p,q) by $\mathcal{H}^{p,q}$, the analogue of the Hodge decomposition (proved more or less the same way) is

$$\Omega^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\partial}\Omega^{p,q-1} \oplus \bar{\partial}^*\Omega^{p,q+1}$$

One deduces the *Dolbeault isomorphism* $H^{p,q}_{\bar{\partial}} = \mathcal{H}^{p,q}$. On the other hand, since the three Laplacians agree (up to scale), the type decomposition of a harmonic form has harmonic components, and therefore $\mathcal{H}^k = \oplus_{p+q=k}\mathcal{H}^{p,q}$ for every k.

Note that $\bar{\partial}^* : \Omega^{p,q} \to \Omega^{p,q-1}$ and therefore vanishes identically on $\Omega^{p,0}$. The kernel of $\bar{\partial}$ on $\Omega^{p,0}$ are exactly the holomorphic p-forms, and therefore $\mathcal{H}^{p,0} = \Omega^p_h$. But wedge product of holomorphic forms is holomorphic, and so:

Proposition 1.4.1. *The holomorphic p-forms are harmonic, and therefore wedge product descends to* $\wedge : \mathcal{H}^{r,0} \times \mathcal{H}^{s,0} \to \mathcal{H}^{r+s,0}$.

It is *not* true in general that wedge product of harmonic forms is harmonic, even on a Kähler manifold.

Example 1.4.2. On a Riemann surface of genus at least two, one can take two harmonic 1-forms α and β which are not proportional, so that $\alpha \wedge \beta$ is not identically zero. On the other hand, any 1-form (harmonic or not) must vanish somewhere, by reason of Euler characteristic, so $\alpha \wedge \beta$ does vanish at some points. But the only harmonic 2-forms on a Riemann surface are constant multiples of the area form, so it follows that $\alpha \wedge \beta$ is not harmonic.

Since the ordinary Laplacian is a real operator, it commutes with complex conjugation, which therefore takes $\mathcal{H}^{p,q}$ isomorphically to $\mathcal{H}^{q,p}$. Finally, if we denote by $\bar{*}$ the composition of Hodge star with complex conjugation, then it turns out that $\bar{*}$ commutes with Δ, and therefore induces a (anti-linear) isomorphism from $\mathcal{H}^{p,q}$ to $\mathcal{H}^{n-p,n-q}$.

Theorem 1.4.3. *Let M be a compact Kähler manifold of complex dimension n. Then if we denote the dimension of $\mathcal{H}^{p,q}$ by $h^{p,q}$, we have*

$$b^k = \sum_{p+q=k} h^{p,q},\ h^{p,q} = h^{q,p},\ h^{p,q} = h^{n-p,n-q},\ h^{p,p} \geq 1 \text{ for all } 0 \leq p \leq n$$

The last fact follows from the fact that the symplectic form ω and all its powers are real of type (p,p), and (because ω^n is nowhere zero) are all nontrivial in cohomology. In particular, notice that for a Kähler manifold one has b^k even for k odd, and b^k positive for k even between 0 and n.

Example 1.4.4. Since $H_1(M) = H_1(\pi_1(M))$ is just the abelianization of $\pi_1(M)$, any group whose abelianization has odd rank is not a Kähler group. Since the property of being a Kähler manifold is inherited by finite covers, any group with a finite index subgroup whose abelianization has odd rank is not a Kähler group. For example, nonabelian free groups of every rank are not Kähler groups, since they all contain nonabelian free groups of odd rank.

The complex structure also lets us define a *twisted d operator* d^c (and its adjoint δ^c) by the formulas

$$d^c = J^{-1}dJ = -i(\partial - \bar{\partial}),\ \delta^c = - * d^c * = i(\partial^* - \bar{\partial}^*)$$

Note that d^c is a real operator, and squares to zero; thus there is a natural "twisted" de Rham cohomology $H^*_{d^c}$. On the other hand, one can check that $\Delta = \Delta_{d^c} := d^c\delta^c + \delta^c d^c$, so by Hodge theory there is an isomorphism $H^*_{d^c} = H^*$.

Furthermore, observe that $dd^c = 2i(\partial\bar{\partial})$, and we have the following counterpart to the (global) $\partial\bar{\partial}$ Lemma:

Proposition 1.4.5 (Global dd^c Lemma)**.** *Let M be a compact Kähler manifold and let α be d^c-exact and d-closed. Then there is a form β with $\alpha = dd^c\beta$.*

Proof. Write $\alpha = d^c\gamma$. By the Hodge theorem we can write $\gamma = \gamma_{\mathcal{H}} + d\beta + \delta\mu$ where $\gamma_{\mathcal{H}}$ is harmonic. Since M is Kähler, $\Delta = \Delta_{d^c}$ and therefore $d^c\gamma = d^c d\beta + d^c\delta\mu$. So it suffices to show that $d^c\delta\mu = 0$.

Since α is closed by hypothesis, and since $dd^c = -d^c d$ and $d^c\delta = -\delta d^c$, we have

$$0 = d\alpha = dd^c\delta\mu = -d\delta d^c\mu$$

But then

$$0 = \int \langle d\delta d^c\mu, d^c\mu\rangle \mathrm{dvol} = \int \langle \delta d^c\mu, \delta d^c\mu\rangle \mathrm{dvol}$$

(by the definition of adjoint) and therefore we must have pointwise $0 = \delta d^c\mu = -d^c\delta\mu$ as required. □

We can also define operators $L : \Omega^k \to \Omega^{k+2}$ by $L = \omega\wedge$ where ω is the symplectic form, and its adjoint $\Lambda : \Omega^{k+2} \to \Omega^k$ (in terms of an orthonormal basis e_i, we have a formula $\Lambda = \frac{1}{2}\sum Je_i \lrcorner e_i \lrcorner$). These satisfy the Kähler identities

$$[L, \delta] = d^c,\ [\Lambda, d] = -\delta^c,\ [L, d] = 0,\ [\Lambda, \delta] = 0$$

From the last identity we deduce

$$[L, \Delta] = [L, d]\delta + [L, \delta]d + d[L, \delta] + \delta[L, d] = d^c d + dd^c = 0$$

and similarly $[\Lambda, \Delta] = 0$. In other words, the operators L and Λ preserve the harmonic forms, and therefore act on $\oplus \mathcal{H}^{p,q}$. In other words,

Proposition 1.4.6. *On a Kähler manifold, the symplectic form ω is harmonic. Furthermore, for any other harmonic form α, the product $\omega \wedge \alpha$ is harmonic.*

Note that L takes $\Omega^{p,q}$ to $\Omega^{p+1,q+1}$ whereas Λ takes it to $\Omega^{p-1,q-1}$. Furthermore, the commutator $[L, \Lambda]$ acts on $\Omega^{p,q}$ as *multiplication* by $p + q - n$. So if we define $h := [L, \Lambda]$, and identify $\oplus \mathcal{H}^{p,q} = H^*$ we see that H^* of a Kähler manifold is in a natural way a module for the copy of $\mathfrak{sl}_2$ generated by L, Λ (the identities $[h, L] = -2L$ and $[h, \Lambda] = 2\Lambda$ are elementary). From the classification of finite dimensional $\mathfrak{sl}_2$ modules, we deduce the

Theorem 1.4.7 (Hard Lefschetz Theorem)**.** *The map $L^k : H^{n-k} \to H^{n+k}$ is an isomorphism, and if we denote by P^{n-k} the kernel of L^{k+1} then $H^m = \oplus_k L^k P^{m-2k}$. Furthermore, if we write $P^{p,q}$ for the intersection of P^k with $H^{p,q}$, then $P^m = \oplus_{p+q=m} P^{p,q}$.*

We shall see that a projective variety V in $\mathbb{P}^N$ is Kähler, in such a way that the cohomology class $[\omega]$ is Poincaré dual to the hyperplane class (the intersection of the variety with a generic hyperplane). So dualizing the Hard Lefschetz Theorem we deduce that intersection with a $\mathbb{P}^{N-k}$ in $\mathbb{P}^N$ gives an isomorphism from $H_{n+k}(V)$ to $H_{n-k}(V)$

Ordinary Poincaré duality on a closed oriented $2n$-manifold says that the pairing $\int : H^k \times H^{2n-k} \to \mathbb{C}$ defined by

$$\alpha, \beta \to \int_M \alpha \wedge \beta$$

at the level of forms is nondegenerate. Combining this with the Hard Lefschetz Theorem one deduces the following corollary:

Corollary 1.4.8. *For all $k \le n$ the pairing $H^k \times H^k \to \mathbb{C}$ defined by*

$$\alpha, \beta \to \int_M \alpha \wedge \beta \wedge \omega^{n-k}$$

is nondegenerate.

It follows that on a compact Kähler manifold, the composition $H^1(M) \times H^1(M) \to H^2(M) \to \mathbb{C}$ is nondegenerate, a fact which imposes strong nondegeneracy conditions on the ordinary cup product on H^1. If $\Gamma = \pi_1(M)$ then $H^1(M) = H^1(\Gamma)$ and $H^2(\Gamma) \to H^2(M)$ is injective, so for a Kähler group Γ the cup product pairing $H^1(\Gamma) \times H^1(\Gamma) \to H^2(\Gamma)$ must admit a nondegenerate pairing as a factor.

Example 1.4.9. If $b_1(\Gamma) \neq 0$ but $b_2(\Gamma) = 0$ then Γ is not a Kähler group. More generally, if $b_1(\Gamma) \neq 0$ but some subspace of $H^1(\Gamma)$ does not pair nontrivially with anything, then Γ is not a Kähler group. For example, let Σ_g denote the closed oriented (Riemann) surface of genus $g \ge 1$, whose fundamental group $\pi_1(\Sigma_g)$ is certainly Kähler, and let Γ denote the universal central extension

$$0 \to \mathbb{Z} \to \Gamma \to \pi_1(\Sigma_g) \to 0$$

Then the cup product vanishes identically on H^1 and Γ is not Kähler. In the case $g = 1$ the group Γ is the (3-dimensional) integral Heisenberg group.

1.5. **Holonomy.** A compatible complex structure, inner product, and symplectic form on a vector space V of dimension $2n$ has stabilizer isomorphic to $\mathrm{U}(n)$.

On any Riemannian manifold, there is a unique torsion-free connection preserving the metric, called the *Levi-Civita* connection ∇. If M is a complex manifold, and $E \to M$ is a holomorphic bundle with a Hermitian metric, any metric connection ∇ extends to connections $\nabla : \Omega^n(E) \to \Omega^{n+1}(E)$ (first one defines $\nabla : \Omega^n(E) \to \Omega^1 \otimes \Omega^n(E)$ and then antisymmetrizes the form). Decomposing forms into types, there is a unique metric connection ∇ called the *Chern* connection, for which $\nabla^{0,1} = \bar\partial$ when expressed in any local (holomorphic) coordinates (the point is that a holomorphic change of coordinates only changes the $(1,0)$-part of the expression of the connection).

A Kähler metric can be characterized by the fact that the Levi-Civita connection and the Chern connection on the tangent bundle (with its Hermitian structure coming from the metric) are equal. Consequently, the curvature of the Chern connection, and the curvature of the Levi-Civita connection coincide (as 2-forms with values in $\mathrm{End}(TM)$).

This is equivalent to the conditions that the almost complex structure J and the form ω are *parallel* under transport by the Levi-Civita connection ∇ associated to the metric. Equivalently, the holonomy group of the metric is isomorphic to a subgroup of $\mathrm{U}(n)$.

The fact that the Levi-Civita connection and the Chern connection on the tangent bundle of a Kähler manifold coincide can be generalized to give connections between the geometry of M and the curvature of other natural (holomorphic) bundles. The most important relation is the following fact. Let $K := \Lambda^{n,0}M$ denote the canonical bundle (i.e. the line bundle whose harmonic sections are the holomorphic n-forms, locally). And let ρ denote the Ricci form on M; i.e. the real, alternating 2-form on M defined by the formula $\rho(X,Y) := \mathrm{Ric}(JX,Y)$. Then the curvature of the canonical bundle (with its Hermitian form coming from the Kähler metric) is equal to $i\rho$.

Some further remarks are in order.

(1) The Kähler condition already implies that ρ is a real alternating form of type $(1,1)$. So the local $\partial\bar\partial$ Lemma implies that it can be expressed locally in the form $i\partial\bar\partial u$ for some real u. In fact, if the coefficients of the Hermitian metric are $h_{\alpha\bar\beta}$, then

$$\rho = -\partial\bar\partial \log\det(h_{\alpha\bar\beta})$$

(2) Since the canonical bundle is independent of the choice of metric, the form $-\rho/2\pi$ represents the first Chern class of K. Conversely, it is a famous theorem of Yau that on a Kähler manifold, for *every* 2-form σ representing the first Chern class of K (which is the negative of $c_1(M)$) there is a *unique* Kähler metric for which $-\rho/2\pi = \sigma$. As a corollary one deduces the Calabi-Yau theorem, that a Kähler manifold admits a Ricci-flat Kähler metric if and only if $c_1 = 0$.

(3) A Kähler metric is Ricci flat if and only if the holonomy is contained in $\mathrm{SU}(n)$. By the classification of Ricci flat manifolds, such a manifold is the product of a flat manifold with a product of pieces of complex dimension n_i each with irreducible holonomy exactly equal to $\mathrm{SU}(n_i)$. These irreducible

factors are called *Calabi-Yau* manifolds. A Calabi-Yau manifold has a compact universal cover, and its fundamental group is therefore finite.

1.6. **Weitzenböck formulae.** Suppose Δ is a "natural" second order elliptic operator on sections of a metric bundle E over a Riemannian manifold M. Naturality should mean that its symbol is invariant under the action of the orthogonal group on whatever bundle the symbol lies in. In many cases it is possible to take the square root of the symbol, and identify the square root as the symbol of a natural first order operator D, so that D^*D and Δ are second order operators with the same symbol. *A priori* one might expect the difference to be first order; but in many cases, the condition of naturality implies that the first order term is forced to vanish (because of the lack of an $O(n)$-equivariant bundle map between $\Omega^0(E)$ and $\Omega^1(E)$). Thus, the difference is a 0th order operator — i.e. a tensor. The only natural tensor fields on Riemannian manifolds are curvature fields, so we obtain a formula of the form

$$\Delta = D^*D + \mathcal{R}$$

for some D and $\mathcal{R}$. If α is in the kernel of Δ, then by integrating we get

$$0 = \int \langle D\alpha, D\alpha\rangle + \langle \mathcal{R}\alpha, \alpha\rangle d\text{vol}$$

The integral of the first term is non-negative, and strictly positive unless $D\alpha$ vanishes identically. So if $\mathcal{R}$ is a *positive* operator, the kernel of Δ must be trivial; and if $\mathcal{R}$ is *non-negative*, then $D\alpha = 0$. Such formulae are called (in this generality) *Weitzenböck formulae*, and the use of such formulae to prove triviality of the kernel of natural elliptic operators is called the *Bochner technique*.

Depending on the context, the operators $\mathcal{R}$ might be more or less complicated. The simpler $\mathcal{R}$ is, the more useful the formula.

Theorem 1.6.1. *Let E be a holomorphic Hermitian bundle over a Kähler manifold M. Then there is a formula*

$$2\Delta_{\bar\partial} = \nabla^*\nabla + \mathcal{R}$$

as operators on $\Omega^{p,q}(E)$, for a certain curvature operator $\mathcal{R}$. When $q = 0$ the formula for $\mathcal{R}$ specializes to

$$\mathcal{R}(\alpha \otimes \xi) = i(\rho^{(p)}\alpha) \otimes \xi + \sum_j \frac{i}{2}\alpha \otimes R^E(Je_j, e_j)\xi$$

where the sum is taken over an orthonormal basis e_j, where R^E is the curvature of E (with its induced Hermitian metric), and where $\rho^{(p)}\alpha := \sum_j \rho(e_j) \wedge e_j \lrcorner \alpha$ where ρ is the Ricci form of M.

Note that on $\Omega^{p,0}$ the operator $2\Delta_{\bar\partial}$ ($= \Delta$) specializes to $2\bar\partial^*\bar\partial$. Note too that harmonic $(p,0)$-forms are precisely the *holomorphic* $(p,0)$-forms; i.e. $H^0(M, \Omega^p_h)$. Taking E to be a trivial line bundle, we see that on a Ricci flat Kähler manifold, *every holomorphic $(p,0)$-form is parallel* (i.e. satisfies $\nabla\alpha = 0$). But $\text{SU}(n)$ fixes a nonzero vector on $\Lambda^{p,0}$ only for $p = 0$ and $p = n$. Thus $h^{p,0} = 0$ on a Calabi-Yau manifold except for $h^{0,0} = h^{n,0} = 1$. For n even this implies that the holomorphic Euler characteristic is 2, and therefore (since the universal cover of a Calabi-Yau manifold is compact) the fundamental group is either trivial or $\mathbb{Z}/2\mathbb{Z}$. For n odd there is no such restriction (since the holomorphic Euler characteristic is forced to be 0). In a similar vein, if the Ricci curvature of M is positive, then there are no holomorphic p-forms for $p > 0$.

To prove vanishing results for holomorphic sections of nontrivial line bundles, it is worth introducing some terminology:

Definition 1.6.2. A real $(1,1)$-form α on a complex manifold is *positive* (resp. *negative*) if $\alpha(\cdot, J\cdot)$ is positive definite (resp. negative definite). A cohomology class in $H^{1,1} \cap H^2_{\mathbb{R}}$ is positive (resp. negative) if it can be represented by a positive (resp. negative) $(1,1)$-form. A holomorphic line bundle L is positive (resp. negative) if there is a Hermitian structure on L for which iR^∇ is positive (resp. negative) where R^∇ is the curvature of the Chern connection ∇.

A line bundle is positive if and only if its first Chern class is positive. This can be proved by adjusting the curvature of the bundle by adjusting the metric, using the global $i\partial\bar{\partial}$ Lemma.

Example 1.6.3. The Kähler form of a Kähler manifold is positive. The Ricci form of a Kähler manifold with positive Ricci curvature is positive. The canonical bundle of a Kähler manifold has curvature $i\rho$, so if the manifold has positive Ricci curvature, the canonical bundle is negative. For example, $\mathbb{P}^n$ is Kähler, with positive Ricci curvature, so its canonical bundle is negative. The dual of a positive line bundle is negative and vice versa, so every projective variety admits a positive line bundle.

Applying the Weitzenböck formula, one obtains many vanishing results for holomorphic sections of positive/negative line bundles. More explicitly, one can show vanishing for cohomology of negative bundles for sufficiently *small* dimensions, and vanishing for cohomology of positive bundles for sufficiently *large* dimensions. Explicitly:

Proposition 1.6.4. *Let L be a negative holomorphic line bundle on a compact complex manifold M of complex dimension n. Then $H^q(M, L) = 0$ for $q < n$.*

Proposition 1.6.5. *Let L be a positive holomorphic line bundle on a compact complex manifold M of complex dimension n. Then there is a positive integer $k(L)$ so that $H^p(M, L^k)$ for all $p > 0$ and all $k \geq k(L)$.*

Theorem 1.6.6 (Kodaira vanishing)**.** *Let L be a positive holomorphic line bundle on a compact Kähler manifold M of complex dimension n. Then $H^{p,q}(M, L) = 0$ for all $p + q > n$.*

On the other hand, one has

Theorem 1.6.7 (Kodaira embedding)**.** *If L is positive, then $H^0(M, L^k)$ is* positive *for all sufficiently large positive k. Consequently a Kähler manifold is projective if and only if it admits a positive line bundle.*

Proof. For any holomorphic bundle E the *holomorphic Euler characteristic*

$$\Xi(E) := \sum (-1)^j \dim H^j(M, E)$$

can be computed from the Atiyah-Singer index theorem by the formula $\Xi(E) = \int \mathrm{Td}(M)\mathrm{ch}(E)$ where Td is the Todd class, and ch is the Chern character, both formal power series in Chern classes of TM and E respectively. All we need to know about the Todd class is that it starts with 1 in dimension 0. For a line bundle L, we have $\mathrm{ch}(L) = \sum_j c_1(L)^j/j!$ so $\mathrm{ch}(L^k) = \sum_j k^j c_1(L)^j/j!$. Since L is positive,

$c_1(L)^n$ is positive, and integrates over M to give a positive number. If k is big, this term dominates, and therefore $\Xi(L^k)$ is positive for all sufficiently big k. On the other hand, also by positivity, $H^p(M, L^k) = 0$ for all $p > 0$ and all sufficiently large k, so we deduce that $\dim H^0(M, L^k)$ is positive, and as big as we like when k is big. In other words, L is *ample*, and by taking enough independent sections when k is big, we get a projective embedding of M. □

1.7. Examples of Kähler manifolds.

Example 1.7.1 ($\mathbb{P}^n$). On $\mathbb{C}^{n+1}$ the function $f := \sum |z_i|^2$ is invariant under the unitary group, and homogeneous of order 2 under scaling. So the 2-form $i\partial\bar{\partial}(f)$ is invariant under the unitary group and under scaling. This 2-form is degenerate exactly along complex lines through the origin, and therefore descends to a closed nondegenerate 2-form ω on $\mathbb{P}^n$, also invariant under the (projective) unitary action. But this means that it is compatible with the complex structure on $\mathbb{P}^n$, and therefore gives $\mathbb{P}^n$ the structure of a Kähler manifold. The associated ($\mathrm{U}(n+1)$-invariant) Hermitian metric is called the *Fubini-Study* metric.

Example 1.7.2 (Finite covers and products). The product of two Kähler manifolds is Kähler. Any cover of a Kähler manifold is Kähler, and a finite cover of a compact Kähler manifold will be compact. So the class of Kähler groups is closed under taking finite products and passing to finite index subgroups. It is *not* closed under taking finite extensions; for example, the fundamental group of a Klein bottle contains $\mathbb{Z}^2$ (a Kähler group) with index two. However, it satisfies $b^1 = 1$ which is odd.

Example 1.7.3 (Nonsingular complex projective varieties). The Lagrangian subspaces for the symplectic structure are the totally real subspaces; by contrast, ω restricts to a symplectic form on complex subspaces, and thereby induces a Kähler structure on all holomorphic submanifolds of $\mathbb{P}^n$, which is to say, on all nonsingular complex projective varieties.

In the same vein, any holomorphic submanifold of a Kähler manifold is Kähler.

Example 1.7.4 (Bounded domains and their quotients). A bounded domain U in $\mathbb{C}^n$ carries a canonical Hermitian metric, called the *Bergman metric*, which is invariant under all biholomorphic self-mappings of U. This is a Kähler metric, and descends to a canonical Kähler metric on any quotient U/Γ, where Γ is discrete and properly continuous.

In fact, with respect to the Bergman metric, the canonical bundle is negative, and therefore (when Γ is cocompact) U/Γ is projective. Key examples of bounded domains with a lot of symmetry are Hermitian symmetric spaces, so that (for example) torsion free cocompact lattices in $\mathrm{SU}(p,q)$, $\mathrm{SO}(n,2)$, $\mathrm{Sp}(n)$ are Kähler groups.

Example 1.7.5 (Riemann surfaces). Riemann surfaces are Kähler manifolds, and so are their products. Atiyah–Kodaira found examples of nontrivial algebraic (Riemann) surface bundles over surfaces, which can be obtained as branched covers of products over certain sections.

Example 1.7.6 ($h^{2,0} = 0$). If M is any Kähler manifold with $h^{2,0} = 0$ then M is actually projective. For, we have also $h^{0,2} = 0$ and therefore $h^{1,1} = b^2$. The Kähler form ω can be approximated (by adding small harmonic forms) by real harmonic

2-forms with *rational* periods, and these nearby forms are necessarily of type $(1,1)$. On the other hand, sufficiently small perturbations are still nondegenerate, and therefore define a nearby Kähler metric for which the Kähler form has rational periods. After clearing denominators by scaling the metric, we can construct a line bundle whose first Chern class is represented by the Kähler form; in particular, this line bundle is positive, and therefore some power is ample, and M is projective.

This holds (for example) whenever M is Calabi-Yau of dimension at least 3. On the other hand, when M is Calabi-Yau of dimension 2 (i.e. a K3 surface) we have $h^{2,0} = 1$ and there are examples which are Kähler but not projective.

Example 1.7.7 (Voisin). Voisin found examples, in every complex dimension ≥ 4, of Kähler manifolds which are not *homotopic* to smooth projective varieties. The simplest examples are obtained by blowing up a (4 complex dimensional) torus twice. However, these examples have free abelian fundamental groups, and so their fundamental groups are Kähler groups.

1.8. **Lefschetz hyperplane theorem.** If M is a (complex) n dimensional smooth projective variety in $\mathbb{P}^N$, its intersection V with a generic hyperplane H is smooth. The inclusion $V \subset M$ induces $H^*(M) \to H^*(V)$, and the classical statement of the Lefschetz hyperplane theorem says that this map is an isomorphism in dimensions $\leq n-2$ and an injection in dimension $n-1$.

In fact this statement about homology has a refinement at the level of *homotopy*, which can be proved by Morse theory, as observed by Bott [6].

Theorem 1.8.1 (Lefschetz hyperplane). *Let M be a complex n dimensional smooth projective variety, and let V be its intersection with a generic hyperplane. Then $\pi_i(V) \to \pi_i(M)$ is an isomorphism for $i \leq n-2$ and is surjective for $i = n-1$.*

More generally, let M be a compact complex manifold, and $L \to M$ a positive line bundle with respect to some suitable Hermitian metric (for example, the inverse of the canonical bundle on $\mathbb{P}^N$ restricted to M under a projective embedding of M). Some power of L has sections, so after replacing L by a power if necessary we can find a holomorphic section α of L. Then the function $f := \log|\alpha|^2$ is Morse (at least outside the subvariety V where it vanishes) and at any critical point of f the Hessian has at least n negative eigenvalues (this follows from a local calculation). In particular, M is obtained from V by attached handles of dimension at least n, so that $\pi_i(M) = \pi_i(V)$ for $i \leq n-2$ and $\pi_i(V)$ surjects onto $\pi_i(M)$ in dimension $n-1$; the relative Hurewicz theorem gives the same conclusion for homology groups, and dualizing gives the classical Lefschetz hyperplane theorem.

It follows that any group which can arise as $\pi_1(M)$ for M a smooth projective variety can arise as $\pi_1(N)$ for N a smooth projective variety of complex dimension at most 3.

Example 1.8.2 (finite groups). Serre [**?**] showed that every finite group is a Kähler group. By passing to finite covers it suffices to show that S_n (the symmetric group on n letters) is a Kähler group for any n. First, S_n acts by permutation of factors on a product $\hat{\Pi} = \mathbb{P}^N \times \cdots \times \mathbb{P}^N$ (n factors). The quotient $\Pi := \hat{\Pi}/S_n$ is projective of dimension nN, and singular along a subspace of complex dimension N. Let $X = \Pi \cap H$ be the intersection with a generic linear subspace of codimension m. Then if $m > N$ we can choose H to be disjoint from the singular subspace, and then X is smooth. The preimage Y of X in $\hat{\Pi}$ is itself the intersection with a generic linear subspace $\hat{H}$ for some projective embedding, so providing $\dim(Y) = \dim(\Pi) - \operatorname{codim}(H) \geq 3$ we have that $\pi_1(Y) = \pi_1(\hat{\Pi}) = 1$. But S_n acts on Y as a deck group with quotient X, so $\pi_1(X) = S_n$.

Example 1.8.3 (Kollar). Suppose $\hat{\Gamma}$ acts properly discontinuously and cocompactly on a simply connected (typically noncompact) Kähler manifold X, and suppose $\hat{\Gamma}$ contains a normal subgroup Γ of finite index which acts freely (for example, $\hat{\Gamma}$ could be a cocompact lattice in $\mathrm{SU}(p,q)$, $\mathrm{SO}(n,2)$ or $\mathrm{Sp}(n)$ which is *not* assumed to be torsion free). Then we can take a compact Kähler manifold N with $\pi_1(N) = \hat{\Gamma}/\Gamma$ and then $\hat{\Gamma}$ acts *freely* and cocompactly on the product $X \times \tilde{N}$, and the quotient is Kähler with fundamental group $\hat{\Gamma}$.

1.9. **Calibrations.** A crucial geometric property of the Kähler form ω and its powers is that they are *calibrating* forms. This means that for any $2k$-plane π we have $\omega^k(\pi) \leq |\pi|$, where $|\pi|$ denotes the $2k$-dimensional volume of π. Moreover, this inequality is *strict* unless π is a complex subspace (i.e. invariant under J).

Proposition 1.9.1. *Let M be Kähler, and let V be a holomorphic submanifold of complex dimension k. Then V is volume minimizing in its homology class (among all compactly supported variations).*

Proof. The proof is elementary: if $V' = V - C \cup C'$ with C and C' compact and $[C' - C]$ trivial in homology, then

$$\mathrm{vol}(C') = \int_{C'} d\mathrm{vol}|_{TC'} \geq \int_{C'} \omega^k = \int_C \omega^k = \mathrm{vol}(C)$$

□

One significant corollary is a diameter–volume comparison theorem, for holomorphic submanifolds of Kähler manifolds of bounded geometry.

First recall that a complete Riemannian manifold M (not assumed to be compact) has *bounded geometry* if it satisfies two-sided curvature bounds $|K| \leq C$, and if the injectivity radius is bounded below by some uniform $\epsilon > 0$. For example, any cover of a compact Riemannian manifold (infinite or not) has bounded geometry.

Proposition 1.9.2. *Let M be a complete Kähler manifold of bounded geometry. Then for each k there is a constant C so that for any k-dimensional holomorphic submanifold V of M there is an estimate* $\mathrm{diameter}(V) \leq C \cdot \mathrm{vol}(V)$.

Proof. Because M has bounded geometry, it is uniformly holomorphically bilipschitz to $\mathbb{C}^n$ with its ordinary Kähler metric. Since the injectivity radius is bounded below by some uniform $\epsilon > 0$, it suffices to show that a k-dimensional holomorphic submanifold V in $\mathbb{C}^n$ passing through the origin intersects the ball of radius ϵ in a submanifold of volume at least C', for some uniform constant C'.

This is actually a property enjoyed by minimal surfaces in Euclidean space, and follows from the *monotonicity formula.* To wit, if $\Sigma^k \subset \mathbb{R}^n$ is minimal and passes through 0, for any $R > r > 0$ there is an inequality

$$R^{-k}\mathrm{vol}(B_R(0) \cap \Sigma) - r^{-k}\mathrm{vol}(B_r(0) \cap \Sigma) \geq 0$$

For calibrated manifolds this is obvious, since if it failed to hold one could make the volume smaller by replacing the intersection with $B_R(0)$ by a cone on $\partial B_R(0) \cap \Sigma$. It is proved in general by applying the divergence property of minimal submanifolds (i.e. $\int_\Sigma \mathrm{div}_\Sigma(X) = 0$ for any vector field X with compact support) to the radial vector field $r\partial_r$. □

2. 1-forms

For any reasonable space X with fundamental group Γ there is a map $X \to K(\Gamma, 1)$ inducing an isomorphism $H^1(\Gamma) \to H^1(X)$ and an injection $H^2(\Gamma) \to H^2(X)$. Furthermore, $H^1(X) = H^1(\Gamma) = \mathrm{Hom}(\Gamma, \mathbb{C})$.

2.1. **Isotropic subspaces of 1-forms.**

For a Kähler manifold we have $H^1(M) = H^{1,0} \oplus H^{0,1} = H^0(\Omega^1_h) \oplus \overline{H^0(\Omega^1_h)}$ so a basis for H^1 is given by holomorphic and anti-holomorphic 1-forms. More generally, $H^{p,0} = H^0(\Omega^p_h)$ so the holomorphic p-forms inject into cohomology. Since the wedge product of holomorphic forms is holomorphic, this means that for holomorphic forms α, β we have $[\alpha] \cup [\beta] = 0$ in cohomology if and only if $\alpha \wedge \beta$ vanishes identically (as a *form*). On the other hand, for any (p, q) there is a nondegenerate pairing between $H^{p,q}$ and $H^{q,p}$ given (at the level of forms) by $(\alpha, \beta) = \int \alpha \wedge \beta \wedge \omega^{n-p-q}$ (this follows from the Hard Lefschetz Theorem).

Theorem 2.1.1 (Castelnuovo-de Franchis)**.** *Let M be a compact Kähler manifold, and let $A \subset H^0(M, \Omega^1_h)$ be a maximal isotropic subspace of dimension $g \geq 2$. Then there exists a surjective holomorphic map $f : M \to C$ with connected fibers to a compact Riemann surface C of genus g so that V is contained in the image of f^*.*

Proof. Let $\alpha_1, \cdots, \alpha_g$ be a basis of A consisting of holomorphic 1-forms, and let $U \subset M$ be the open (and dense) subset of M where they do not all vanish. At any point where α_i and α_j do not both vanish, we must have $\ker(\alpha_i) = \ker(\alpha_j)$, since otherwise $\alpha_i \wedge \alpha_j$ would be nonzero at that point, contrary to the hypothesis that A is isotropic. So there is a (complex) codimension 1 distribution ξ equal to $\ker(\alpha_i)$ for any i at each point where α_i is nonsingular. Since the forms are closed, ξ is integrable and defines a (singular) foliation $\mathcal{F}$.

Since the α_i all have the same kernels, any two of them differ locally by multiplication by a unique holomorphic function. Therefore by taking ratios we get a holomorphic map $\phi : U \to \mathbb{P}^{g-1}$. If we write $\alpha_j = \phi_j \alpha_1$ locally, then $d\phi_j \wedge \alpha_1 = d\alpha_j = 0$, so the ϕ_j (which are inhomogeneous coordinates for ϕ) are constant on leaves of $\mathcal{F}$. It follows that the image of ϕ is 1-dimensional.

A priori we can extend ϕ to $\phi : M' \to D$ for some Riemann surface D and some blow-up M' of M, with Stein factorization $M' \to C \to D$ for some Riemann surface C. By construction, the α_i are pulled back from C, and therefore the genus of C is at least g. The exceptional locus of M' is a $\mathbb{P}^1$ bundle, and the image of each $\mathbb{P}^1$ in C is therefore constant. Thus we actually have $M \to C$ with connected fibers, and the genus of C is exactly g after all. □

The point of the argument is that the holomorphic map ϕ, which *a priori* is only defined on an open subset, actually has no indeterminacy, and is really defined on all of M. A generalization to the noncompact context is due to Napier, and is relevant in the next section.

Corollary 2.1.2 (Catanese)**.** *Let M be a compact Kähler manifold, and let $A \subset H^1(M)$ be a maximal isotropic subspace of dimension $g \geq 2$. Then there is a holomorphic map $f : M \to C$ with connected fibers, where C is a compact Riemann surface of genus g, and so that A is in the image of f^*.*

Proof. Choose (real) harmonic forms β_i a basis for A, and write $\beta_i = \alpha_i + \bar{\alpha}_i$ where each α_i is a harmonic $(1,0)$-form (and therefore holomorphic). Then because $\alpha_i \wedge \alpha_j$ is the $(2,0)$-part of $\beta_i \wedge \beta_j$ and is therefore holomorphic (and injects into cohomology), the fact that $0 = [\beta_i \wedge \beta_j]$ in cohomology implies that $\alpha_i \wedge \alpha_j = 0$ as *forms*.

Since the β_i are real and linearly independent over $\mathbb{R}$, the α_i are linearly independent over $\mathbb{C}$, and we can apply the Castelnuovo-de Franchis Theorem. □

We deduce the following:

Theorem 2.1.3 (Siu, Beauville)**.** *Let M be a compact Kähler manifold, and $g \geq 2$. Then there is a holomorphic map $f : M \to C$ with connected fibers and C a compact Riemann surface of genus at least g if and only if there is a surjective homomorphism $\pi_1(M) \to \pi_1(C)$.*

Proof. Since C is a $K(\pi, 1)$, a map on fundamental groups induces a map on spaces, which pulls back an isotropic subspace in $H^1(C)$ to an isotropic subspace in $H^1(M)$. Then apply the corollary of Catanese. The converse direction is easy, since a surjective map with connected fibers is surjective in π_1. □

Because of this theorem, we introduce the following definition:

Definition 2.1.4. A Kähler group Γ is *fibered* if it surjects onto $\pi_1(C)$ for some Riemann surface of genus at least 2; equivalently, if every compact Kähler manifold with fundamental group Γ fibers over some Riemann surface C with connected fibers.

Since $H^2(\Gamma)$ injects in $H^2(M)$ for any M with $\pi_1(M) = \Gamma$, we see that a maximal isotropic subspace of $H^1(\Gamma)$ corresponds to a maximal isotropic subspace of $H^1(M)$, and the condition of being fibered can be checked just by looking at $\wedge : H^1(\Gamma) \times H^1(\Gamma) \to H^2(\Gamma)$.

Note that the condition of being fibered implies $b_1 \geq 4$.

2.2. L_2 **cohomology.** Let X be a simplicial polyhedron (not necessarily finite), and let $\ell_2 C^k$ denote the space of simplicial k-cochains whose coefficients are square summable. If X is *uniformly locally bounded* — i.e. if there is a constant $c_j k$ so that each k-simplex is contained in the boundary of at most c_k $k+1$-simplices — then the coboundary map d_k takes $d_k : \ell_2 C^k \to \ell_2 C^{k+1}$.

Definition 2.2.1. For X a uniformly locally bounded simplicial polyhedron, we have the ℓ_2*-cohomology*

$$\ell_2 H^k(X) := \ker d_k / \text{im } d_{k+1}$$

and the *reduced* ℓ_2*-cohomology*

$$\overline{\ell_2 H}^k(X) := \ker d_k / \overline{\text{im } d_{k+1}}$$

where $\overline{\text{im } d_{k+1}}$ is the closure in the ℓ_2 topology.

These chain complexes and groups behave functorially under *uniformly* proper simplicial maps and homotopies.

If M is a smooth Riemannian manifold with uniformly bounded geometry (i.e. with 2-sided curvature bounds and injectivity radius bounded away from 0) we can also define the spaces $L_2\Omega^k$ of L_2 k-forms α for which $d\alpha$ is also in L_2, and then define L_2-cohomology

$$L_2H^k(M) := \frac{\ker d|L_2\Omega^k(M)}{dL_2\Omega^{k-1}}$$

and the reduced L_2 cohomology

$$\overline{L_2H}^k(M) := \frac{\ker d|L_2\Omega^k(M)}{\overline{dL_2\Omega^{k-1}}}$$

Under the assumptions on M, there is a uniformly locally bounded triangulation of M by simplices uniformly bilipschitz to the standard (Euclidean) simplex in each dimension, and integration of forms over simplices defines a chain map from $L_2\Omega^k$ to ℓ_2C^k inducing isomorphisms $L_2H^k \to \ell_2H^k$ and $\overline{L_2H}^k \to \overline{\ell_2H}^k$.

The advantage of working with reduced ℓ_2 or L_2 cohomology is that we have an analogue of the Hodge decomposition:

Theorem 2.2.2 (L_2 Hodge theorem). *Under the assumptions of uniformly bounded geometry for polyhedra or manifolds, every cohomology class in $\overline{\ell_2H}^k$ (resp. $\overline{L_2H}^k$) contains a unique cocycle minimizing the norm. Such a cocycle α is characterized by the fact that it satisfies an equation $\Delta\alpha = 0$, where in the case of $\overline{L_2H}^k$, the operator Δ is the ordinary Laplacian.*

In fact, if $\mathcal{H}^p_{(2)}$ denotes the space of L_2 harmonic p-forms (in the usual sense), there is a Hodge decomposition

$$L_2\Omega^p = \mathcal{H}^p_{(2)} \oplus \overline{d(L_2\Omega^{p-1})} \oplus \overline{\delta(L_2\Omega^{p+1})}$$

One subtlety is that δ defined by the formula $-*d*$ is no longer a formal adjoint to d on forms on a noncompact manifold, since integration by parts gives rise to a boundary term "at infinity". But for an L_2 form α, we can show $\Delta\alpha = 0$ if and only if $d\alpha = 0$ and $\delta\alpha = 0$, since

$$\int \langle \Delta\alpha, \alpha\rangle = \int \langle d\alpha, d\alpha\rangle + \langle \delta\alpha, \delta\alpha\rangle$$

(since $d\alpha$ and $\delta\alpha$ are not *a priori* L_2 one interprets this first by using cutoff functions).

On a Kähler manifold, since the equality $\Delta = 2\Delta_\partial$ still holds irrespective of compactness, there is a further decomposition of $\mathcal{H}^k_{(2)}$ into $\mathcal{H}^{p,q}_{(2)}$, and as above, an L_2 form α of (p,q)-type satisfies $\Delta\alpha = 0$ if and only if $\bar\partial\alpha = 0$ and $\bar\partial^*\alpha = 0$. Thus $\mathcal{H}^{p,0}_{(2)}$ consists of *holomorphic* L_2 p-forms.

Remark 2.2.3. A harmonic form which is not L_2 does *not* necessarily have to be in the kernel of d. In fact, suppose $\alpha \in \mathcal{H}^1_{(2),\mathrm{ex}}(M)$ so that $\alpha = df$ for some smooth f. We claim that actually f is harmonic (and therefore definitely not in L_2), although it is evidently not closed.

Since Δ and d commute, $d\Delta f = 0$ so Δf is a constant c. If we let φ be a compactly supported bump function equal to 1 on some big set B. Then

$$c^2 \int_X \varphi \, \mathrm{dvol} \le \langle \Delta f, \varphi\Delta f\rangle = c\langle df, d\varphi\rangle \le c\|d\varphi\|_2\|df\|_2$$

Choosing φ with $|d\varphi|^2$ small compared to φ pointwise, we see that $c = 0$. Thus f is harmonic.

One can also define $\overline{\ell_2 H}_k$ by taking infinite chains with square summable coefficients, and there is a duality between $\overline{\ell_2 H}^k$ and $\overline{\ell_2 H}_k$.

Example 2.2.4. If X is noncompact, and admits an isometry ψ with unbounded orbits, then by pairing a nontrivial cocycle α with a cycle C we get $\psi^n(\alpha)(C) \to 0$ so that the dimension of $\overline{\ell_2 H}^k$ is infinite if it does not vanish. On the other hand, if X is contractible, and ψ moves points a uniformly bounded distance, it is uniformly homotopic to the identity and therefore induces the identity map on cohomology. Thus $\overline{\ell_2 H}^*(X)$ vanishes identically. For example, we could take $X = X' \times \mathbb{R}$ for X' arbitrary (in particular, $\overline{\ell_2 H}^*(\mathbb{E}^n)$ vanishes for Euclidean space $\mathbb{E}^n$).

Example 2.2.5. Let X be a 3-regular tree, and let e be an oriented edge, and orient the other edges of X as in Figure 1, where e is the "middle" edge. Let C be the 1-

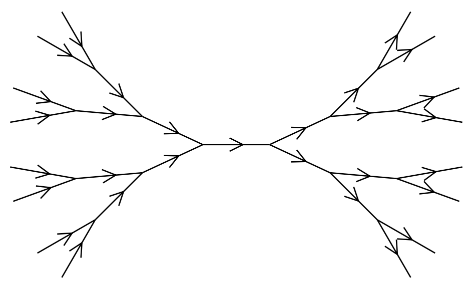

FIGURE 1. Orientation of edges determined by choice of e

cycle taking the value $1/2^k$ on each oriented edge which is distance k from e. There are 2^{k+1} edges at distance k, so $\|C\|_2 = 3$. This 1-cycle pairs nontrivially with any 1-cocycle α with support exactly equal to e. So $\overline{\ell_2 H}^1(X)$ is infinite dimensional.

Now embed X as a totally geodesic graph in $\mathbb{H}^2$ with each edge of length L and angle $2\pi/3$ between edges at each vertex. If L is big, this is a quasi-isometric embedding. Moreover there is an (approximate) nearest point projection $\pi : \mathbb{H}^2 \to X$ so that for each edge e, the subset of the preimage at distance t has length $\sim \sinh(t)$ and the projection map is (approximately) linear. Thus the L_2 norm of $\pi^*\alpha$ is finite, and we see that $\overline{L_2 H}^1(\mathbb{H}^2)$ is infinite dimensional.

Example 2.2.6 (Gromov, Kähler hyperbolic). A form α is *bounded* if $\|\alpha\|_\infty = \sup_{p\in M} |\alpha_p|$ is finite, where $|\alpha_p|$ is the operator norm at the point p.

Let M be a Kähler manifold (*not* assumed to be compact) with uniformly bounded geometry. Then the Kähler form ω is bounded, because the form is determined by the metric. Suppose furthermore that $\omega = d\alpha$ where α is a bounded 1-form. In this case we say that M is *Kähler hyperbolic*; in fact, more generally, we say that M is Kähler hyperbolic if the pullback of ω to the universal cover of M is d of a bounded form. If $\omega = d\alpha$ with α bounded, if β is any harmonic L_2 form, then $\omega \wedge \beta = d(\alpha \wedge \beta)$, and since α is bounded, $\alpha \wedge \beta$ is L_2, and therefore $\omega \wedge \beta$ is zero in $\overline{L_2 H}^*(M)$. On the other hand, since M is Kähler, the wedge product of any harmonic form with ω is harmonic, and harmonic L_2 forms are unique in their (reduced L_2) cohomology class. Thus $\omega \wedge \beta$ is identically zero. But wedge with ω

is injective on L_2 harmonic forms below the middle dimension, and Λ is injective above the middle dimension, so we deduce (by dualizing the argument above) that $\overline{L_2H}^k(M)$ vanishes for k different from the complex dimension of M.

It is a fact that on any complete simply-connected Riemannian manifold with strictly negative curvature (i.e. $K \le -C < 0$), any exact form is d of a bounded form. To see this, cone any submanifold to a point along geodesics (which are unique) and observe that the volume of a geodesic cone is bounded by a constant times the volume of its boundary.

Thus if M is a Kähler manifold with strict negative curvature, the universal cover of M has vanishing $\overline{L_2H}^*$ outside the middle dimension. More generally, if M is a locally symmetric Kähler manifold, then even though M is not necessarily strictly negatively curved, ω vanishes in flat directions, so M is Kähler hyperbolic in this case too.

Generalizing in another direction, if M has $\pi_2(M) = 0$, then the cohomology class of ω is pulled back from $H^2(\pi_1(M))$. If $\pi_1(M)$ is *word-hyperbolic*, then bounded cohomology surjects onto ordinary cohomology in dimensions ≥ 2, and therefore if M is compact Kähler with these properties, it is Kähler hyperbolic.

Remark 2.2.7. A Kähler hyperbolic compact manifold M is Kobayashi hyperbolic. For, M fails to be Kobayashi hyperbolic if and only if it admits a nonconstant holomorphic map of $\mathbb{C} \to M$ and the pullback of ω to $\mathbb{C}$ is not d of something bounded.

If a group Γ acts properly discontinuously and cocompactly on M, and M is acyclic in dimension k, then $\overline{L_2H}^k(M)$ depends only on Γ and not on the quotient. So in particular, if M is compact and $\tilde{M}$ denotes its universal cover, $\overline{L_2H}^1(\tilde{M})$ depends only on $\pi_1(M)$.

We have seen that $\overline{L_2H}^1(\tilde{M})$ vanishes for "most" higher dimensional Kähler manifolds M. In fact, it turns out that one (i.e. Gromov) can precisely classify Kähler manifolds for which $\overline{L_2H}^1(\tilde{M})$ does not vanish.

The first step is a Lemma, which shows that the space of L_2 holomorphic 1-forms obtained from real *exact* L_2 1-forms is isotropic.

Lemma 2.2.8 (Isotropy lemma)**.** *Let M be a complete Kähler manifold with uniformly bounded geometry. Let $\alpha_1, \alpha_2 \in \mathcal{H}^1_{(2),\mathrm{ex}}(M)$ be real, harmonic, and exact (as forms) 1-forms. Let φ_i be the $(1,0)$ component of α_i, so that φ_i is holomorphic and L_2. Then $\varphi_1 \wedge \varphi_2 = 0$ pointwise.*

Proof. The first remark to make is that on a complete Kähler manifold with uniformly bounded geometry, any L_2 harmonic form α is actually bounded. Equivalently, since harmonic forms are C^∞, there is no sequence of points p_i going off to infinity with $|\alpha|_{p_i} \to \infty$. Since the manifold has bounded geometry, we can integrate the L_2 norm of α on disjoint balls of some uniform positive radius ϵ about the p_i, and the claim will follow once we show that the square integral of α over a small ball is controlled (from below) by the value at the center. But this is obviously true for a linear form on $\mathbb{C}^n$, and since harmonic forms satisfy an elliptic PDE, the higher derivatives are controlled by the first derivative, and the claim follows.

Now let α be in $\mathcal{H}^1_{(2),\mathrm{ex}}(M)$, and write $df = \alpha$. Suppose ψ is an i-form which is closed and L_2. Then we claim that the class of $\alpha \wedge \psi$ is trivial in $\overline{L_2H}^*(M)$;

equivalently, it is in the closure of $dL_2\Omega^i(M)$. First, observe that since α is harmonic, it is bounded, and therefore $\alpha \wedge \psi$ is in L_2. If we define f_c to be equal to f where $|f(x)| < c$ and equal to c elsewhere, then $f_c\psi$ is in $\Omega^i_{(2)}$, and satisfies $d(f_c\psi) = df_c \wedge \psi = df \wedge \psi \cdot \chi_{V_c}$ where V_c is the subset where $|f| \le c$. But then $d(f_c\psi) \to df \wedge \psi$ in L_2, so $\alpha \wedge \psi = df \wedge \psi \in \overline{d(L_2\Omega^i)}$.

Finally, let $\alpha_j = df_j$ and $\varphi_j = \frac{1}{2}(df_j + i\beta_j)$ be the decomposition of the $(1,0)$ component of α_j into real and imaginary parts. Note that each β_j is harmonic, and L_2 because φ_j is. We compute

$$4\varphi_1 \wedge \varphi_2 = (df_1 \wedge df_2 - \beta_1 \wedge \beta_2) + i(df_1 \wedge \beta_2 + \beta_1 \wedge df_2)$$

But this implies that $df_1 \wedge \beta_2 + \beta_1 \wedge df_2 \in \overline{d(L_2\Omega^1)}$. On the other hand, $\varphi_1 \wedge \varphi_2$ is holomorphic, so $\mathrm{im}\,(\varphi_1 \wedge \varphi_2)$ is harmonic, and is orthogonal to $\overline{d(L_2\Omega^1)}$. So $\mathrm{im}\,(\varphi_1 \wedge \varphi_2) = 0$ pointwise, and therefore $\varphi_1 \wedge \varphi_2 = 0$ pointwise (the real part is constant, and since L_2, equal to 0). $\square$

Now, if α is any closed nonzero holomorphic 1-form on a complex manifold M, its kernel is tangent to the leaves of a holomorphic foliation $\mathcal{F}_\alpha$ (possibly singular where ω vanishes). Every two nonsingular leaves are homologous, and therefore if M is Kähler, they are calibrated and therefore have the same volume. If α is *exact* (for instance, if $H^1(M;\mathbb{R}) = 0$) we can write $\alpha = df$ for some holomorphic function f then

$$\int_{\mathbb{C}} \mathrm{vol}(f^{-1}(z))d\mathrm{area}(z) = \|df\|_2^2 < \infty$$

so the volume of the leaves of $\mathcal{F}_\alpha$ (which are contained in level sets of f) are *finite*, and equal to some finite constant c.

Because holomorphic submanifolds of a Kähler manifold are calibrated, if M has bounded geometry, there is some constant $C(c, M)$ so that any holomorphic submanifold of volume at most c has diameter bounded by C. Thus the leaves of $\mathcal{F}_\alpha$ have *uniformly bounded diameter*, and are therefore all *compact* (including the singular leaves). There is thus a (global) Stein factorization $\pi : M \to S$ for some Riemann surface S, whose fibers are the leaves of $\mathcal{F}_\alpha$.

If there are *two* linearly independent exact harmonic L_2 1-forms $\alpha_1 = df_1$, $\alpha_2 = df_2$, their holomorphic $(1,0)$ parts φ_1, φ_2 are not necessarily exact, but they define the *same* (singular) foliations $\mathcal{F}$ as above, and therefore their ratio is a holomorphic function (where defined), and defines a holomorphic map to $\mathbb{P}^1$ on an open set, constant on the leaves of $\mathcal{F}$. A lemma of Napier (a noncompact version of the Castelnuovo-de Franchis theorem) shows that this map has no indeterminacy, and then the same area argument then shows that some leaves of $\mathcal{F}$ are compact, and therefore all leaves are compact, and the Stein factorization maps M to a Riemann surface S in this case too.

Alternately, we can look at the map $M \to \mathbb{R}^2$ whose coordinates are f_1 and f_2. Since the two forms are linearly independent, the fibers are (generically) of real codimension 2, and in fact by using the fact that the f_i are harmonic, they are locally the real part of holomorphic functions, and we can identify the connected components of the fibers with the leaves of $\mathcal{F}$. This argument uses the fact that harmonic functions on Kähler manifolds are *pluriharmonic* — i.e. harmonic on every complex subvariety, a key observation due to Siu that we will pursue further in § 3. Since the coordinate functions are L_2, yet again we deduce that most fibers have finite volume and therefore diameter, and we get the desired Stein factorization.

Now, if ϕ is any holomorphic isometry of M, then ϕ^* acts by isometries on $\mathcal{H}^1_{(2),\mathrm{ex}}(M)$. Each α in $\mathcal{H}^1_{(2),\mathrm{ex}}(M)$ determines a closed nonzero L_2 holomorphic 1-form and therefore a foliation as above, and by the Isotropy Lemma 2.2.8 the foliations are independent of α. Thus ϕ descends to an automorphism of S, compatible with the projection π. Since the fibers are compact of constant volume, L_2 holomorphic forms on S pull back to L_2 holomorphic forms on X, so $\pi^* : \mathcal{H}^1_{(2)}(S) \to \mathcal{H}^1_{(2)}(M)$ is injective. Conversely, if $\alpha \in \mathcal{H}^1_{(2),\mathrm{ex}}(M)$ is dg for some $g : M \to \mathbb{C}$ then g is harmonic, and hence (because M is Kähler) is pluriharmonic, as we shall show in § 3. Its restriction to every compact holomorphic subvariety is therefore constant, so g descends to S, and we deduce that S is hyperbolic and noncompact. From this we can readily conclude:

Theorem 2.2.9 (Gromov)**.** *Let M be Kähler with bounded geometry, and suppose $H^1(M;\mathbb{R}) = 0$ but $\mathcal{H}^1_{(2)}(M) \neq 0$. Then there is a proper holomorphic map $h : M \to \mathbb{D}^2$ with connected fibers. Moreover, the fibers are permuted by* $\mathrm{Aut}(M)$ *and h induces an isomorphism $h^* : \mathcal{H}^1_{(2)}(\mathbb{D}^2) \to \mathcal{H}^1_{(2)}(M)$.*

Proof. Since $H^1(M;\mathbb{R}) = 0$, every closed 1-form is exact so $\mathcal{H}^1_{(2)}(M) = \mathcal{H}^1_{(2),\mathrm{ex}}(M)$. We construct $h : M \to S$ as above, and observe that S is hyperbolic and noncompact. This implies that its fundamental group is free, and since h is surjective with connected fibers, $\pi_1(M)$ surjects onto a free group. But $H^1(M;\mathbb{R}) = 0$, so this free group is actually trivial, and $S = \mathbb{D}^2$. □

Remark 2.2.10. Actually, when $H^1(M)$ is zero, we don't need the isotropy lemma to deduce that the foliations determined by different L^2 holomorphic 1-forms are the same. For, by exactness, either foliation has compact connected leaves which are the fibers of a proper holomorphic map to a noncompact Riemann surface. If a leaf of one foliation were not contained in a leaf of the other, the proper map determined by the second foliation would be *nonconstant*, thus giving a nonconstant holomorphic map from a compact complex variety to a noncompact Riemann surface, which is absurd.

Now let M be a *compact* Kähler manifold, with fundamental group Γ. The universal cover $\tilde{M}$ certainly satisfies $H^1(\tilde{M};\mathbb{R}) = 0$ and has bounded geometry, and therefore $\mathcal{H}^1_{(2)}(\tilde{M}) = \overline{\ell_2 H}^1(\pi_1(M))$. Hence

Corollary 2.2.11. *Let Γ be a Kähler group with $\overline{\ell_2 H}^1(\Gamma) \neq 0$. Then Γ is commensurable with the fundamental group of a closed Riemann surface of genus ≥ 2.*

If Γ has infinitely many ends, then $\mathcal{H}^1_{(2)}(\tilde{M})$ is nonzero (and therefore infinite dimensional). For instance, we can let X be a compact set which separates two ends, and let f take values 0 and 1 on the components of $\tilde{M} - X$, so that df has compact support and is in L_2. Evidently df is not d of an L_2 function. On the other hand, since $\tilde{M}$ has at least 3 ends, it satsifies a *linear isoperimetric inequality*: every compact submanifold Y of $\tilde{M}$ satisfies $\mathrm{vol}(\partial Y) > C \cdot \mathrm{vol}(T)$ for some positive constant C. Such a linear isoperimetric inequality is dual to the existence of a *spectral gap* at 0 for the operator d on $L_2\Omega^0$, i.e. $\|df\|_2 \geq C' \cdot \|f\|_2$. In particular, this implies that $d(L_2\Omega^0)$ is actually *closed*, and therefore

$$L_2H^1(\tilde{M}) = \overline{L_2H}^1(\tilde{M}) = \mathcal{H}^1_{(2)}(\tilde{M})$$

is nontrivial, and in fact infinite dimensional.

By a theorem of Stallings, a finitely presented group has infinitely many ends (equivalently, it is not virtually $\mathbb{Z}$ and has more than 1 end) if and only if it splits as an amalgam over a finite group. The group $\mathbb{Z}$ is not a Kähler group, because it has $b_1 = 1$ odd. Thus:

Corollary 2.2.12. *A Kähler group is either finite or 1-ended.*

If Γ is a Kähler group for which there is a short exact sequence $0 \to K \to \Gamma \to H \to 0$ with $H^1(K;\mathbb{R})$ finite dimensional, then if we define M_K to be the cover with fundamental group K, the quotient H acts on M_K by deck transformations. If $\mathcal{H}^1_{(2)}(M_K)$ is nontrivial, we have already seen that it is infinite dimensional, so $\mathcal{H}^1_{(2),\mathrm{ex}}(M_K)$ is infinite dimensional and invariant under H. Hence as before, M_K fibers over S with compact connected fibers. Since $H_1(K)$ surjects onto $H_1(S)$, we must have that $H_1(S)$ is finitely generated; but H is infinite, so $H_1(S)$ is trivial, and $S = \mathbb{D}^2$. Thus:

Corollary 2.2.13 (Arapura–Bressler–Ramachandran [3])**.** *If a Kähler group Γ fits in an exact sequence $0 \to K \to \Gamma \to H \to 0$ where $\overline{L_2H}^1(H)$ is nontrivial and $H^1(K;\mathbb{R})$ is finite dimensional, then H is commensurable with the fundamental group of a closed Riemann surface of genus ≥ 2. In particular, no Kähler group is an extension of a group with infinitely many ends by a finitely generated group.*

2.3. **Cuts.** The situation above can be generalized considerably. The following discussion is taken largely from [12]. We restrict attention in what follows to hyperbolic groups and their quasiconvex subgroups, for simplicity.

Definition 2.3.1. Let G be a word-hyperbolic group. A quasiconvex finitely generated subgroup H is of *codimension one* if X/H has at least two ends, where X is a cocompact G space (for example, X could be the Cayley graph of G).

If e is an end of X/H, the *capacity* of e is the infimum of $\int_{X/H} |dg|^2$ over all functions g which are 0 on the complement of e, and 1 on the complement of a compact subset of e. The quotient X/H is *stable* if each end has positive *capacity.* For X the Cayley graph of G (as above), stability follows from nonamenability of the (Schreier) coset graph of H in G. For, nonamenability of this coset graph implies a linear isoperimetric inequality (by the Folner criterion), which implies as before that $d(L_2\Omega^0)$ is actually closed, and any function g with finite energy can be relaxed to a *nonconstant* proper harmonic function, whose energy gives a lower bound on the capacity.

Now let A be an arbitrary finitely generated subgroup of G with H as above. Let Y be the Cayley graph of A. It is a fact that $Y/(A \cap H)$ is stable unless A is (virtually) cyclic. This follows from:

Proposition 2.3.2. *Suppose A is a subgroup of a hyperbolic group G, and H a quasiconvex subgroup of G. If $A/A \cap H$ and $A \cap H$ are infinite, or if A is nonelementary and $A \cap H$ is finite, then A contains a (nonabelian) free group F such that F meets no A-conjugate of $A \cap H$*

The free group F witnesses the positive capacity of the ends of $Y/(A \cap H)$. This proposition is easy to prove when A is quasiconvex, since then $H \cap A$ is quasiconvex in H of infinite index. An arbitrary finitely generated group A is not necessarily quasiconvex, but it is "randomly" quasiconvex, in the sense that random walk in A behaves like random walk in a quasiconvex subgroup.

Now suppose G hyperbolic is the fundamental group of a compact Kähler manifold, and suppose G contains a codimension one subgroup H. By replacing H by a free product $\langle H, a\rangle$ where a moves the limit set of H off itself if necessary, we can assume that the limit set of H separates the limit set of G into at least three components. Since G has a codimension one subgroup, it is cubulated, and by Agol's theorem ([1]), is linear. So in particular, H is residually finite, and there is some finite index subgroup H' for which $\tilde{M}/H'$ has at least three ends; by abuse of notation we denote H' by H in what follows. Since this quotient space is stable, there is a *finite energy* proper harmonic map f to an interval separating any end from any other, where finite energy and harmonic implies precisely that $df \in \mathcal{H}^1_{(2),\mathrm{ex}}(\tilde{M}/H)$. Since there are at least three ends, we can find at least two independent such forms df_1, df_2 and therefore a proper holomorphic map $\tilde{M}/H \to S$ as above.

On the other hand, every fiber has uniformly bounded volume and therefore uniformly bounded *diameter*, and so the fundamental groups of the fibers project to only *finitely many* possible subgroups of G under the covering projection $\tilde{M}/H \to M$. But this means that the normalizer of the fundamental group of a fiber has *finite index* in G.

Now, the fundamental group of a fiber is contained in H; but a normal subgroup contained in an infinite index quasiconvex subgroup of a hyperbolic group must actually be finite. Thus the fundamental group of the fibers is finite, and G is virtually a surface group after all. This proves:

Theorem 2.3.3 (Delzant–Gromov)**.** *If a Kähler group is hyperbolic and contains a quasiconvex subgroup of codimension one, then it is commensurable with a surface group.*

In the same vein, one can show that if Γ is Kähler and admits a homomorphism to a hyperbolic group G with *sufficiently many codimension one subgroups* (enough to separate the points of $\partial_\infty G$) then the homomorphism virtually factors through a surjection to a surface group $\Gamma \to \pi_1(S)$, which is necessarily induced by a proper holomorphic map with connected fibers.

Corollary 2.3.4. *Let Γ be a cocompact lattice in $U(n,1)$ for $n \geq 2$, and let H be a convex cocompact subgroup of Γ. Then the limit set of H does not disconnect $\partial\mathbb{H}^n_{\mathbb{C}} := \partial_\infty\Gamma$.*

This is in stark contrast to the situation for lattices in $O(n,1)$, which often contain convex cocompact subgroups whose limit set is a topological S^{n-2} in $S^{n-1} = \partial\mathbb{H}^n_{\mathbb{R}}$.

Remark 2.3.5. The components of the preimages in $\tilde{M}$ of the ends of X/H are called *filtered ends* by Geoghegan. The appeal to Agol's theorem to find a finite index subgroup $H' \subset H$ for which $\tilde{M}/H'$ has at least three ends is a shortcut, at the cost of invoking a big theorem that was not available to Delzant-Gromov. Rather, they worked directly with the filtered ends, constructing an H-equivariant harmonic map f from $\tilde{M}$ to a tree T (with leaves corresponding to the filtered ends) such that the quotient $\tilde{M}/H \to T/H$ has finite energy (they call this *finite H-energy*).

The holomorphic 1-form α on $\tilde{M}$ with real part df does *not a priori* have finite energy unless H is finite, but it is nevertheless exact on $\tilde{M}$, and the integral leaves tangent to $\ker(\alpha)$ are closed and define an H-equivariant holomorphic map from $\tilde{M}$ to $\mathbb{C}$ and therefore $\tilde{M}/H \to \mathbb{C}/H$ complexifying $\tilde{M}/H \to T/H$. Since T has at least 3 ends, this map is "branched" over the vertex, and the branched leaf is H-invariant, and therefore its quotient is compact (this is similar to an argument of Gromov-Schoen). But this implies (by calibration) that every leaf of the quotient foliation on $\tilde{M}/H$ is compact, and we obtain the desired structure in this case too.

2.4. **BNS invariants.** Let G be a finitely generated group, with Cayley graph $C(G)$ (with respect to some finite generating set). A homomorphism $\phi : G \to \mathbb{R}$ determines a function (which by abuse of notation we also denote ϕ) on $C(G)$ taking the given value on the vertices (which are identified with G) and linear on edges.

Associated to ϕ we form the set $\phi^{-1}[0,\infty) \subset C(G)$. This set is *coarsely connected* if there is a constant K so that the K-neighborhood in $\subset C(G)$ is connected.

Definition 2.4.1. A nontrivial homomorphism $\phi : G \to \mathbb{R}$ is *regular* if $\phi^{-1}[0,\infty)$ is coarsely connected, and *exceptional* otherwise.

This property evidently depends only on the projective class of ϕ; if we denote the "unit" sphere in $H^1(G;\mathbb{R})$ by $S(G)$, then we can denote by $\Sigma(G)$ the regular homomorphisms, and $E(G)$ the exceptional ones.

The definition of the sets $E(G)$ and $\Sigma(G)$ are due to Bieri–Neumann–Strebel [5], and are sometimes called *BNS invariants*. They also proved the following (easy) proposition:

Proposition 2.4.2. *The subset $E(G) \subset S(G)$ is* closed.

Example 2.4.3. If $\phi : G \to \mathbb{Z}$ has kernel K, then ϕ is regular if and only if K is finitely generated.

Example 2.4.4. If $G = \pi_1(M)$ for M a 3-manifold, then $E(G)$ are the open *fibered faces* of the unit ball in the Thurston norm (i.e. the faces whose rational points correspond to fibrations over S^1).

Example 2.4.5. Let G be free, or π_1 of a closed surface of genus ≥ 2. Then $S(G) = E(G)$. In fact, every Q is the largest metabelian quotient of a surface group of genus ≥ 2, then $E(Q) = S(Q)$. To see this, observe that for a nontrivial homomorphism $Q \to \mathbb{Z}$ the kernel is the first homology of an infinite abelian cover of $\pi_1(S)$, which is infinitely generated. Thus every rational homomorphism is exceptional, and since rational homomorphisms are dense and E is closed, $E(Q) = S(Q)$.

In this context, Delzant proved (by methods similar to those above) the following theorem:

Theorem 2.4.6 (Delzant)**.** *Let M be a compact Kähler manifold, and let $\alpha \in H^1(M;\mathbb{R})$ be thought of as a homomorphism from $\pi_1(M)$ to $\mathbb{R}$, or as a harmonic 1-form. Then α is in $E(G)$ if and only if M virtually admits a holomorphic map to a Riemann surface with a closed harmonic 1-form which pulls back to f.*

The structure of the proof is as follows. First represent α by a harmonic 1-form (by abuse of notation with the same name, and let φ be its $(1,0)$-part. Let

$\hat{M} \to M$ be an abelian cover (e.g. the universal abelian cover) on which the pullback $\hat{\varphi}$ is exact, so that $\hat{\varphi} = df$ for some holomorphic map $f : \hat{M} \to \mathbb{C}$, and let $g : \hat{M} \to \mathbb{R}$ be the real part. Note that $H_1(M;\mathbb{Z})$ acts on $\hat{M}$ cocompactly by deck transformations, and we can identify $H_1(M;\mathbb{Z})$ with its orbit; having done this, (and chosen a basepoint in $g^{-1}(0)$) the function g restricted to $H_1(M;\mathbb{Z})$ agrees with α.

Thus under the hypothesis that α is exceptional, if we denote by $N = g^{-1}[0,\infty)$, the preimage $\tilde{N}$ of N in the universal cover $\tilde{M}$ has infinitely many components; equivalently, for some connected component N' of N the inclusion $N' \to \hat{M}$ is not surjective on π_1 (this observation is due to Bieri–Neumann–Strebel, and is one of the main uses of their invariant).

It follows that $\partial\tilde{N}$ disconnects $\tilde{M}$ into infinitely many components (in a $\pi_1(M)$-equivariant way) and collapsing connected components of the level sets of g to points defines a $\pi_1(M)$-equivariant harmonic map to an $\mathbb{R}$-*tree*. Then the method of Gromov-Schoen gives the desired map to a Riemann surface.

Now let's suppose G is a Kähler group, and that $E(G)$ is empty. Then the commutator subgroup G^1 has finite rank, and the conjugation action of $H_1(G)$ on $G^1 \otimes \mathbb{R}$ has image in $\mathrm{GL}(m,\mathbb{Z})$ for some m. But by a theorem of Beauville, this action is actually unitary, and therefore has finite image; in particular, G/G^2 is *virtually nilpotent*. Now it is a theorem of John Groves that any solvable group of finite type which is not virtually nilpotent admits a subgroup of finite index which surjects onto a solvable but not virtually nilportent *metabelian* group (i.e. one for which G^2 is trivial).

So suppose G is a Kähler group which is solvable but not virtually nilpotent. Then G virtually has a metabelian quotient K for which K^1 is infinite dimensional. But then $E(K)$ is nonempty, and some finite index subgroup G' of G has $E(G')$ nonempty, and therefore G' virtually surjects onto a surface group of genus at least 2, contrary to the fact that G is solvable. Thus one deduces the following corollary:

Corollary 2.4.7 (Delzant)**.** *A Kähler group which is solvable is virtually nilpotent.*

The corresponding theorem for fundamental groups of projective varieties is due to Arapura and Nori [4].

2.5. **Formality and rational homotopy type.** If X is a space, information about the nilpotent quotients of $\pi_1(X)$ (at least over $\mathbb{R}$) can be derived from the *rational homotopy type* of X; i.e. the localization of X (in the sense of homotopy theory) at $\mathbb{Q}$ (for simply connected spaces this rational homotopy type recovers the higher homotopy groups of X, modulo torsion).

If M is a smooth manifold, the de Rham complex $\Omega^*(M)$ is a (skew-commutative) differential graded algebra (hereafter dga) with $\wedge$ as product, and d as differential. If A and B are dgas, a morphism of dgas $f : A \to B$ is a *quasi-isomorphism* if it induces an isomorphism in homology. Quasi-isomorphisms do not necessarily have inverses, so the relation of quasi-isomorphism is not by itself an equivalence relation on dgas; the equivalence relation it *generates* is called *weak equivalence*.

If A is a dga, a 1-*minimal model* for X is a a certain limit $\mathcal{M}(2,0)$ of dgas

$$\mathcal{M}(1,1) \to \mathcal{M}(1,2) \to \cdots$$

where each $\mathcal{M}(1,n) \to \mathcal{M}(1,n+1)$ is an inclusion, together with a morphism $f : \mathcal{M}(2,0) \to A$ which induces isomorphisms in H^0 and H^1, and an injection in H^2. We are exclusively interested in *connected* dgas; those with $H^0 = \mathbb{R}$.

Sullivan shows that 1-minimal models exist, and that weakly equivalent dgas have isomorphic 1-minimal models. In fact, the terms $\mathcal{M}(1,n)$ in the minimal model can be computed in a precise way from the nipotent part of $\pi_1(M)$.

Let $\Gamma = \pi_1(M)$, and let Γ_i be the ith term in the lower central series, so that $\Gamma_1 := \Gamma$, $\Gamma_{n+1} := [\Gamma_n, \Gamma]$, and therefore $N_n(\Gamma) := \Gamma/\Gamma_n$ is the maximal n-step nilpotent quotient of Γ. There are central extensions

$$0 \to \Gamma_{n-1}/\Gamma_n \to N_n(\Gamma) \to N_{n-1}(\Gamma) \to 0$$

and we can define $N_n(\Gamma) \otimes \mathbb{R}$ inductively by

$$0 \to (\Gamma_{n-1}/\Gamma_n) \otimes \mathbb{R} \to N_n(\Gamma) \otimes \mathbb{R} \to N_{n-1}(\Gamma) \otimes \mathbb{R} \to 0$$

(at the level of homotopy, the central extensions correspond to fibrations in which the fundamental group of the base acts trivially on the fundamental group of the fiber; localizing the base and the fiber inductively defines a space with fundamental group $N_n(\Gamma)\otimes\mathbb{R}$). Note that $N_n(\Gamma)$/torsion embeds as a lattice in the real nilpotent Lie group $N_n(\Gamma) \otimes \mathbb{R}$ (this is called the *Malcev completion*). If we denote by $\mathcal{L}_n$ the $\mathbb{R}$ Lie algebra of $N_n(\Gamma) \otimes \mathbb{R}$ then we get a tower of real, nilpotent Lie algebras

$$\cdots \mathcal{L}_n \to \mathcal{L}_{n-1} \to \cdots \to \mathcal{L}_1$$

which are *dual* to the dgas $\mathcal{M}(1,n)$. Thus, the 1-minimal model of M determines the torsion-free nilpotent completion of $\pi_1(M)$.

We describe the construction of $\mathcal{M}(2,0)$ associated to a dga A. First, take $\mathcal{M}(1,1)$ to be the free dga on $H^1(A)$, and $\mathcal{M}(1,1) \to A^1$ to be any section $H^1(A) \to A^1$ on generators, extended as a dga morphism. Then $H^2(\mathcal{M}(1,1)) \to H^2(A)$ is not necessarily injective, and we define $\mathcal{M}(1,2)$ to be the dga generated in degree 1 by $\mathcal{M}(1,1)$ and by a copy of $\ker : H^2(\mathcal{M}(1,1)) \to H^2(A)$, with differential defined as follows: if $v \in \mathcal{M}(1,2)$ in degree 1 corresponds to $v' \in \ker : H^2(\mathcal{M}(1,1)) \to H^2(A)$ we choose some cycle v'' in $\mathcal{M}(1,1)^2$ representing v', and define $dv = v''$. In fact, since $H^2(\mathcal{M}(1,1)) = \wedge^2\mathcal{M}(1,1)^1 = \wedge^2 H^1(A)$ we can choose v'' of the form $\sum a_i \wedge b_i$ with $a_i, b_i \in H^1(A)$.

We continue inductively. If $H^2(\mathcal{M}(1,n)) \to H^2(A)$ is ever injective, we take $\mathcal{M}(2,0) = \mathcal{M}(1,n)$. Otherwise, we add new generators to $\mathcal{M}(1,n)$ in dimension 1 as primitives for $\ker : H^2(\mathcal{M}(1,n)) \to H^2(A)$ to form $\mathcal{M}(1,n+1)$, and continue. Evidently the direct limit $\mathcal{M}(2,0)$ has the desired properties.

The biggest obstacle to computing $\mathcal{M}(2,0)$ from this algorithm is to get a good description of $\Omega^*(M)$, or a dga weakly equivalent to it. If M is homotopy equivalent to a finite simplicial complex, one can take the *PL de Rham complex*, i.e. the algebra of forms which on each simplex are given by the restriction of polynomial forms with rational coefficients. But certain spaces enjoy a property which makes the computation much easier:

Definition 2.5.1. A smooth manifold M is *formal* if $\Omega^*(M)$ and $H^*(M)$ are weakly equivalent.

Theorem 2.5.2 (Deligne–Griffiths–Morgan–Sullivan [10])**.** *Compact Kähler manifolds are formal.*

Proof. This is an easy consequence of the dd^c lemma (and the d^cd lemma, which has the same proof), applied several times.

First consider the subcomplex of Ω^* consisting of forms which are d^c-closed; denote this Ω^c. Exterior d restricts to a boundary operator on Ω^c, since $d^cd = -dd^c$ so $d^cd(\alpha) = 0$ for $\alpha \in \Omega^c$. Next consider the quotient complex $H^*_{d^c} = \Omega^c/d^c\Omega^*$ with d as differential. In fact, d restricts to the zero operator on $\Omega^c/d^c\Omega^*$, since a d^c closed d-exact form is d^c-exact (it is even of the form $d^cd\beta$).

Thus there are morphisms $i : (\Omega^c, d) \to (\Omega^*, d)$ induced by inclusion and $\rho : (\Omega^c, d) \to (H^*_{d^c}, d) = (H^*_{d^c}, 0) \cong (H^*, 0)$ induced by quotient. So the proof will follow if we verify that these morphisms are quasi-equivalences.

The map i^* is injective, since if $y \in \Omega^c$ (and therefore d^c-closed) is dz, it is d^cdw. The map i^* is surjective, since if a is closed, d^ca is closed and d^c-exact, so $d^ca = dd^cw$. But a is homologous to $a + dw$, which is in Ω^c since $d^ca + d^cdw = 0$.

The map ρ^* is injective, since if $y \in \Omega^c$ is d-closed and in the image of d^c then $y = dd^cz$ and is therefore homologically trivial in Ω^c. The map ρ^* is surjective, since it is a quotient map at the level of chains. This completes the proof. □

It follows that the torsion-free nilpotent completion of $\pi_1(M)$ for M a compact Kähler manifold depends *only* on the pairing $\wedge^2 H^1(M) \to H^2(M)$. This leads to many examples of groups which can't arise as fundamental groups of Kähler manifolds.

Example 2.5.3. If $[u], [v], [w] \in H^*$ satisfy $[u] \cup [v] = [v] \cup [w] = 0$ then the *Massey product* $\langle [u], [v], [w] \rangle$ is the set of classes $[sw + ut]$ for which $ds = uv$ and $dt = vw$ for representative cycles u, v, w. It is well-defined on the quotient $H^*/([u]H^* + [w]H^*)$. It turns out that Massey triple products are preserved under weak equivalence of dgas. So on a compact Kähler manifold, all Massey products are zero, since we can compute them in the dga $(H^*, 0)$ where the differentials are all trivial.

3. Harmonic maps

3.1. **Energy.**

Definition 3.1.1. Let $f : M \to N$ be a smooth map between compact Riemannian manifolds. For each $x \in M$ we think of df as a section of $T^*M \otimes f^*TN$. The Riemannian metrics on M and N induce a fiberwise inner product on this bundle, and we can define the *energy density* $e(f)(x) := \frac{1}{2}\|df(x)\|^2$, and the *energy*

$$E(f) := \int_M e(f)\, d\mathrm{vol}$$

Now let f_t be a one parameter family of maps deforming f. We can think of this as a map $F : M \times \mathbb{R} \to N$ whose restriction to $M \times t$ is f_t. We can use the Levi-Civita connetion on N to infinitesimally identify $T_{f_tx}N$ with $T_{fx}N$ for small t; together with the trivial connection on the $\mathbb{R}$ factor, this gives rise to a "Levi-Civita" connection ∇ on the bundle F^*TN.

The df_t for various t are sections of the bundle $F^*TN|_{M\times t}$, so to compate their lengths we need to parallel transport them back to f^*TN using ∇ in the $\partial/\partial t$ direction. If we write $f_t(x) = \exp_{f(x)}(t\psi(x)) + o(t)$ for some vector field $\psi \in \Gamma(f^*TN)$ then we can compute

$$\nabla_{\partial/\partial t} df_t = \sum \nabla_{\partial/\partial t} \frac{\partial f_t^i}{\partial x^\alpha} \frac{\partial}{\partial f^i} \otimes dx^\alpha = \sum \nabla_{\partial/\partial x^\alpha} \frac{\partial f_t^i}{\partial t} \frac{\partial}{\partial f^i} \otimes dx^\alpha = \nabla\psi$$

Thus

$$\begin{aligned}\frac{d}{dt}E(f_t)|_{t=0} &= \frac{1}{2}\frac{d}{dt}\left(\int_{M\times t}\langle df_t, df_t\rangle \mathrm{dvol}\right)|_{t=0} = \int_M \langle df, \nabla_{\partial/\partial t} df_t|_{t=0}\rangle \mathrm{dvol} \\ &= \int_M \langle df, \sum \nabla_{\partial/\partial x^\alpha}\psi \otimes dx^\alpha\rangle \mathrm{dvol} \\ &= -\int_M \sum \langle \nabla_{\partial/\partial x^\alpha} df, \psi\otimes dx^\alpha\rangle \mathrm{dvol} = -\int_M \langle \mathrm{tr}\nabla df, \psi\rangle \mathrm{dvol}\end{aligned}$$

So f is a critical point for energy if and only if $\Delta f := \mathrm{tr}\nabla df = \pm * \nabla * df = 0$ as a section of f^*TN.

Example 3.1.2. If $N = \mathbb{R}$ or S^1 then f is harmonic as a *map* if and only if df is harmonic as a 1-form (in the usual sense), and the energy of f is the L_2 norm squared of df.

Example 3.1.3. If Σ is a Riemann surface, the energy of $f : \Sigma \to N$ is conformally invariant and is therefore well-defined at the level of a holomorphic structure on Σ. Given $f : \Sigma \to N$ we can compute

$$E(f) = \int_\Sigma \left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \overline{z}}\right\rangle dz\wedge d\overline{z}$$

and furthermore, f is harmonic if and only if the quadratic differential $\varphi(z)dz^2 := \langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}\rangle dz^2$ is holomorphic.

More generally, ∇df is a symmetric 2-form with values in f^*TN. Complexifying and taking the imaginary part gives $\nabla d^c f$, an alternating 2-form with values in f^*TN. Taking traces thus corresponds to taking a wedge product with a suitable power of ω, and then applying Hodge star; in particular, $f : M \to N$ is harmonic (where M is a Kähler manifold) if and only if $\omega^{n-1}\wedge \nabla d^c f = 0$. Another way to see this is to use the identity $*\alpha = \omega^{n-1}\wedge J\alpha$ valid for 2-forms (up to a nonzero multiplicative constant) so that

$$\Delta f = *\nabla * df = *\nabla\omega^{n-1}\wedge Jdf = *\omega^{n-1}\wedge \nabla Jdf$$

(all equalities up to nonzero multiplicative constants).

Definition 3.1.4. A map $f : M \to N$ where M is Kähler is *pluriharmonic* if its restriction to (the germ of) each complex curve C in M is harmonic.

Since C is Kähler with $\omega|_C$ as its symplectic form, a map $f : M \to N$ is pluriharmonic if and only if $\nabla d^c f$ is zero on every complex line; but since $\nabla d^c f$ is the imaginary part of the complexification of a real symmetric form, it vanishes on every totally real subspace; thus f is pluriharmonic if and only if $\nabla d^c f = 0$.

3.2. **Eells–Sampson and generalizations.** The fundamental existence theorem for harmonic maps is the following:

Theorem 3.2.1 (Eells–Sampson)**.** *Let M and N be compact Riemannian manifolds, and suppose N has non-positive sectional curvature. Then every homotopy class of map $f : M \to N$ contains a harmonic representative.*

The harmonic representative is essentially unique, up to composition with isometric motions of the image. Eells–Sampson actually prove more, namely that the gradient flow of energy (i.e. a *nonlinear heat flow* equation $\partial f/\partial t = \Delta f$) applied to the initial map f converges (as time goes to infinity) to a harmonic map.

Note that one can say more: a harmonic map to a non-positively curved manifold is a *stable* critical point for energy. This follows from the second variation formula; if f_t is a smooth variation of f with $\Delta f = 0$ and $W := \partial f/\partial t|_{t=0}$, then

$$\frac{d^2}{dt^2}E(f_t)|_{t-0} = -\int_M \sum_i \langle R^N(df(e_i), W)W, df(e_i)\rangle d\text{vol}$$

where R^N denotes the curvature operator on N, and the sum is taken over an orthonormal basis e_i for M, pointwise (one can think of this as an example of a Weitzenböck formula). Thus if N has non-positive sectional curvature, so that $\langle R(X,Y)Y,X\rangle \le 0$ for all X, Y, we get $\frac{d^2}{dt^2}E(f_t)|_{t=0} \ge 0$. So when the target space is nonpositively curved, the inclusion of the minimal energy maps into all the maps in a homotopy class is a weak homotopy equivalence.

If N is a non-positively curved locally symmetric space, a homotopy class of map $f : M \to N$ arises from a homomorphism $\pi_1(M) \to \pi_1(N) \subset G$ for some Lie group G with $\tilde{N} = G/K$, and lifts to an *equivariant* map $\tilde{M} \to \tilde{N}$. More generally, one can ask when an equivariant map $\tilde{M} \to G/K$ is homotopic (through equivariant maps) to an equivariant harmonic map. Corlette [9] showed that there is a positive answer providing the image of $\pi_1(M)$ in G is reductive — i.e. its Zariski closure is a real reductive subgroup of G.

3.3. **Hermitian sectional curvature and the Siu–Sampson–Bochner formula.** If M is a Riemannian manifold, the curvature tensor R may be complexified and extended to the complexified tangent bundle, and we define the *Hermitian sectional curvature* to be the Hermitian form on $\Lambda^2 TM \otimes \mathbb{C}$ sending $X \wedge Y$ to

$$K_{\mathbb{C}}(X \wedge Y) := \frac{\langle R(X,Y)\bar{Y}, \bar{X}\rangle}{\|X \wedge Y\|^2} \in \mathbb{R}$$

Example 3.3.1*.* If X and Y are real and orthonomal, $K_{\mathbb{C}}(X \wedge Y) = K(X \wedge Y)$, the ordinary sectional curvature along the 2-plane they span.

Remark 3.3.2*.* We use the convention $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ in the formula above. This convention agrees with that of Cheeger–Ebin and Kobayashi–Nomizu (for example) but disagrees with Milnor.

Definition 3.3.3. A Riemannian manifold has *non-positive Hermitian sectional curvature* of $K_{\mathbb{C}}(X \wedge Y) \le 0$ for all X, Y.

We also write $K_{\mathbb{C}} \le 0$ and $K_{\mathbb{C}} < 0$ if the inequality is strict.

Example 3.3.4 (Pinched negative curvature)*.* The strict inequality $K_{\mathbb{C}} < 0$ holds whenever ordinary K is negative and sufficiently pinched. In fact, since $K_{\mathbb{C}}$ is homogeneous, one only needs *local* pinching. The optimal pinching constant of 4 was established by Hernandez [14]; i.e. if there is a strictly positive function $f : M \to \mathbb{R}$ such that $-4f(x) \le K_x \le -f(x)$ pointwise, then $K_{\mathbb{C}} \le 0$; and if $-4f(x) < K_x \le -f(x)$ then $K_{\mathbb{C}} < 0$.

Example 3.3.5 (Locally symmetric spaces)*.* By the previous example, locally symmetric spaces of noncompact type have nonpositive complex sectional curvatures (but this follows more immediately by a local calculation in each case).

Example 3.3.6. If M is Kähler and $Z, W \in T^{1,0}$ then $K_{\mathbb{C}}(Z \wedge W) = 0$. This is because J is parallel with respect to the Levi-Civita connection ∇, and therefore

$$R(X,Y)JZ = JR(X,Y)Z$$

and

$$R(X,Y,JZ,JW) = R(X,Y,Z,W) = R(JX,JY,Z,W)$$

on TM. But then complexifying and using $J = i$ on $T^{1,0}$, we get

$$R(Z,W,\bar{W},\bar{Z}) = R(X - iJX, Y - iJY, Y + iJY, X + iJX) = 0$$

on $T^{1,0}M$, by collecting terms, and using the identities above.

Definition 3.3.7. Let M be compact Kähler. A map $f : M \to N$ to a Riemannian manifold N is *pluriharmonic* if its restriction to the germ of every complex curve C in M is harmonic.

It can be shown by direct calculation that f is pluriharmonic if and only if $\nabla d^c f = 0$ where $d^c f = -Jdf$ as before. The map f is harmonic if and only if $\nabla d^c f \wedge \omega^{n-1}$ is zero. Now, the operation of wedging with ω^{n-1} pointwise is a $\mathrm{U}(n)$ invariant linear form, so $\Lambda^{1,1}$ decomposes pointwise into $\mathbb{R}\omega \oplus \Lambda_0^{1,1}$. It turns out that $\Lambda_0^{1,1}$ is an *irreducible* $\mathrm{U}(n)$-module. Thus the pointwise pairing

$$\alpha, \beta \to \alpha \wedge \beta \wedge \omega^{n-2}$$

which is not identically zero (as can be checked) is *definite*, and in fact *negative definite*. In particular, for any harmonic map f we have $\nabla d^c f \wedge \nabla d^c f \wedge \omega^{n-2}$ in $f^*(TN \otimes TN)$ is a *nonpositive* (symmetric) operator on TN, and is strictly negative unless f is pluriharmonic.

On the other hand,

$$d(\nabla d^c f \wedge d^c f \wedge \omega^{n-2}) = -R^N \, d^c f \wedge d^c f \wedge \omega^{n-2} + \nabla d^c f \wedge \nabla d^c f \wedge \omega^{n-2}$$

Integrating this exact form over a compact Kähler manifold M one obtains zero. On the other hand, the second term is strictly negative unless f is pluriharmonic (in which case it vanishes), while $R^N \, d^c f \wedge d^c f$ is the Hermitian sectional curvature pulled back from N. Thus we obtain a (very sketchy) proof of:

Theorem 3.3.8 (Siu [17], Sampson [16])**.** *If M is compact Kähler and N is a Riemannian manifold with non-positive Hermitian sectional curvature, then every harmonic map $f : M \to N$ is pluriharmonic, and the Hermitian sectional curvature of N along every complex 2-plane $X \wedge Y$ where X and Y are in the image of vectors in $T^{1,0}M$ (under the complexification of df) must vanish.*

One of the main applications is to locally symmetric targets N. Suppose $N = \Gamma\backslash G/K$ for some real semisimple Lie group G without compact factors and with maximal compact subgroup K, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G. Then the complexification of df must map $T^{1,0}M$ pointwise into an *abelian* subalgebra $\mathfrak{a} \subset \mathfrak{p}^{\mathbb{C}}$, where we identify $\mathfrak{p}$ with TN pointwise. It is then an algebraic problem to translate this into further constraints. For example,

Theorem 3.3.9 (Carlson–Toledo [8])**.** *Let $\tilde{N} = G/K$ be a symmetric space of noncompact type with no $\mathbb{H}^2$ factor. If $\mathfrak{a} \subset \mathfrak{p}^{\mathbb{C}}$ is an abelian subalgebra, then*

$$\dim_{\mathbb{C}} \mathfrak{a} \leq \frac{1}{2} \dim_{\mathbb{C}} \mathfrak{p}^{\mathbb{C}}$$

with equality if and only if G/K is Hermitian symmetric, and $\mathfrak{a} = \mathfrak{p}^{1,0}$ for some invariant complex structure on G/K.

Theorem 3.3.10 (Siu)**.** *Let M be compact Kähler, $N = \Gamma\backslash G/K$ a locally Hermitian symmetric space with no $\mathbb{H}^2$ factors, and $f : M \to N$ a harmonic map with rank equal to* $\dim(N)$*. Then f is holomorphic for some invariant complex structure on G/K. Hence (because N has nonpositive sectional curvature), and homotopy equivalence $f : M \to N$ is homotopy to a biholomorphic map.*

The restriction that there should be no $\mathbb{H}^2$ factors is essential, since a topological surface admits nontrivial holomorphic moduli, and a diffeomorphism between two Riemann surfaces is homotopic to a harmonic map which is typically *not* holomorphic.

Another application concerns maps to real hyperbolic manifolds; in this case, any abelian subalgebra $\mathfrak{a}$ is 1-dimensional, and therefore df has real rank at most 2. There are two cases: either the real rank is at most 1 in which case $f : M \to N$ factors through a map to a closed geodesic, or the real rank is 2, in which case f factorizes as $M \to S \to N$ where S is a compact Riemann surface, $M \to S$ is holomorphic, and $S \to N$ is harmonic.

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