CLASSICAL TESSELLATIONS AND 3-MANIFOLDS, SPRING 2014, HOMEWORK 3

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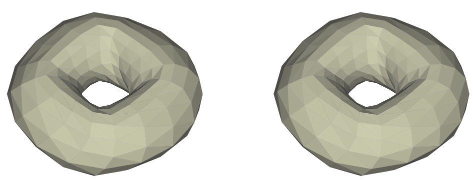
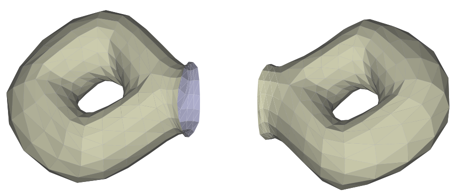
Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due April 25th.

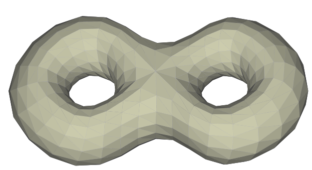
Problem 1. Given connected surfaces S_1 and S_2,



the *connect sum*, denoted $S_1\#S_2$, is the surface you get by cutting a small disk out of S_1 and S_2



and gluing the two boundary circles that result together.



(1) Show that the connect sum operation is associative and commutative; i.e. show that $(S_1\#S_2)\#S_3$ is the same topological surface as $S_1\#(S_2\#S_3)$, and similarly show that $S_1\#S_2$ is the same as $S_2\#S_1$, for any connected surfaces S_1, S_2, S_3.
(2) Show that the 2-sphere S^2 is an identity element for connect sum — i.e. that $S\#S^2 = S^2\#S = S$ for any connected surface S.
(3) Show that $P\#P = K$ where P is the projective plane, and K is the Klein bottle.
(4) Show that $T\#P = K\#P$ where T is the torus, K is the Klein bottle, and P is the projective plane. Show that this identification can be made by "sliding" the handle of T around a loop on P that reverses orientation.
(5) If $S_1\#S_2 = S^2$, show that S_1 and S_2 are both S^2 (Hint: what is the effect of connect sum on Euler characteristic?)

Problem 2. Suppose S is a closed surface (i.e. without boundary) which is made by gluing rigid Euclidean triangles in such a way that the lengths match along edges that are glued, and the angles at each vertex add up to 360°. Show that the Euler characteristic satisfies $\chi(S) = 0$.

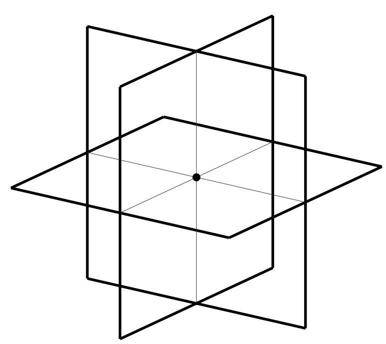
Problem 3. Let P be a polygon with $2n$ sides where $n \geq 2$, which are labeled in pairs with distinct labels $e_1, \cdots, e_n$ (with either orientation) so that the result of gluing edges with the same labels identifies all the vertices of P to a single vertex.

(1) Instead of gluing edges of P together, take *infinitely many copies* of P, and show that you can glue them together respecting edge labels, $2n$ around every corner, so that the result is (topologically) a plane tiled by copies of P.
(2) If $n = 2$, so that P has 4 sides, show that you can realize this tiling with Euclidean squares of side length 1. Find an example for $n = 3$ and draw a good picture of the tiling (remember to choose the edge labels so there is exactly one vertex after identification!) (Hint: as you add more and more hexagons to the picture, draw them smaller so that there is room for them)
(3) Think of the graph in the plane that you get made up from the edges of all the copies of P, together with its labels by the e_i, as the Cayley graph of some group. Deduce that a presentation for this group is
$$G := \langle e_1, e_2, \cdots, e_n \mid R \rangle$$
where R is a word of length $2n$ in which each e_i appears exactly twice, either as e_i or as e_i^{-1}, depending on how the edges appear (with orientation) in the boundary of P.

Note: if the surface you get by identifying edges of P is denoted S, then the group G above is called the *fundamental group* of S, and is denoted $\pi_1(S)$.

Problem 4. One way to embed a torus T in 3-dimensional space is to take a knotted circle K, thicken it slightly, and let T be the boundary of such a thickened neighborhood; one says that such a T bounds a "solid torus" — i.e. a space which is topologically a thickened circle. Can you embed a torus in $\mathbb{R}^3$ in such a way that it *doesn't* bound a solid torus?

Problem 5. It is impossible to embed a projective plane P in 3-dimensional space without making it intersect itself. Generically, a surface in 3-dimensional space intersects itself transversely in 1-dimensional arcs, like two coordinate planes crossing. But there might be isolated points where three sheets of the surface all cross transversely like three coordinate planes, in a "triple point", as in the Figure.



Find a way to put the projective plane P into 3-dimensional space in such a way that there is exactly one triple point of self-intersection.

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