CLASSICAL TESSELLATIONS AND 3-MANIFOLDS, SPRING 2014, HOMEWORK 2

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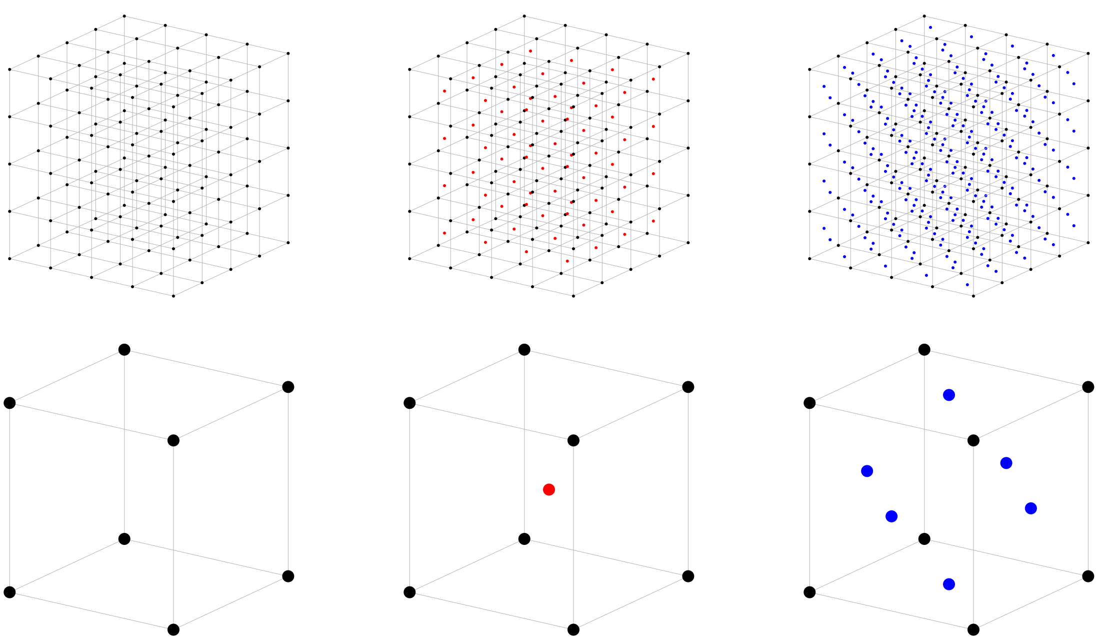
Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due April 18th.

Problem 1. Thinking of 3-dimensional Euclidean space $\mathbb{E}^3$ with its usual tiling by unit cubes, consider the following three lattices:

(1) the *primitive cubic lattice*, with one lattice point at every vertex of every cube;
(2) the *body-centered cubic lattice*, with one lattice point at every vertex *and* one lattice point in the center of each cube; and
(3) the *face-centered cubic lattice*, with one lattice point at every vertex *and* one lattice point in the center of each *face* of each cube.

These lattices are illustrated in the figure.



For each of these three lattices,

(1) determine how many lattice points there are per unit of volume;
(2) determine the minimum distance between any pair of lattice points; and
(3) determine the shape of the subset of $\mathbb{E}^3$ which is closer to one lattice point than to any other.

Which lattice corresponds to the most efficient way to pack oranges (among these three possibilities)? Are ripe pomegranate seeds *really* shaped like rhombic dodecahedra? (open one up and find out!)

Problem 2. This problem has three parts.
(i): Let f be a polynomial of degree n with real coefficients. If n is odd, show that f has a real root.
(ii): Let M be an $n \times n$ matrix with real entries. If n is odd, show that M has a real eigenvalue.

(iii): Let φ be an orientation-preserving isometry of the n-dimensional sphere (i.e. the round unit sphere in $\mathbb{E}^{n+1}$). If n is even, show that φ has a fixed point.

Problem 3. Give an example of an orientation-preserving isometry f of the 3-sphere such that f^k has no fixed point for every nonzero integer k. Thinking of the 3-sphere as $\mathbb{R}^3$ together with a point "at infinity", draw a picture of the 3-sphere and the dynamics of your isometry on it. What do the orbits of points look like? i.e. for a point p, what does the set of points $\{f^i(p) \text{ for } i \in \mathbb{Z}\}$ look like?

Problem 4. Give an example of a discrete group of orientation-preserving isometries of $\mathbb{E}^4$ that is abstractly isomorphic to $\mathbb{Z}^2$ but does not contain any translations (other than the identity element).

Problem 5. Give an example of a discrete group of isometries of $\mathbb{E}^2$ that contains $\mathbb{Z}^2$ as a subgroup, and in which every element has infinite order, but which contains some elements that are not translations. Can you draw a Cayley graph for your example?

Problem 6. A (real or complex) number α is an *algebraic integer* if it is a root of some *monic* polynomial p with integer coefficients; i.e. a polynomial of the form

$$p(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

where the a_i are all (ordinary) integers.

Let $\mathbb{Q}(\sqrt{5})$ denote the *field* consisting of all real numbers of the form $a+b\sqrt{5}$ where a and b are (ordinary) rational numbers, and let $\mathcal{O}$ denote the set of elements of $\mathbb{Q}(\sqrt{5})$ which are algebraic integers. Show that the elements of $\mathcal{O}$ are all roots of monic polynomials of degree (at most) 2. If $\alpha \in \mathcal{O}$ is an algebraic integer which is a root of a degree 2 monic polynomial p, what is the relationship between the coefficients of p and the element α?

Let $\phi : \mathbb{Q}(\sqrt{5}) \to \mathbb{R}^2$ send the number $a + b\sqrt{5}$ to $(a + b\sqrt{5}, a - b\sqrt{5})$. Show that ϕ is an injective homomorphism from $\mathbb{Q}(\sqrt{5})$ (thought of as an abelian group with addition as the group law) into $\mathbb{R}^2$; i.e. that $\phi(x + y) = \phi(x) + \phi(y)$ (note that this map is *not* surjective). Show further that the image of $\mathcal{O}$ is a lattice in $\mathbb{R}^2$, and deduce that $\mathcal{O}$ is abstractly isomorphic to the free abelian group $\mathbb{Z}^2$. How many lattice points of $\phi(\mathcal{O})$ are there per unit of area?

An algebraic integer α is a *unit* if $1/\alpha$ is also an algebraic integer. Which algebraic integers in $\mathcal{O}$ are units? Where do their images under ϕ lie in $\mathbb{R}^2$?

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